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A DIRECTIONAL SHADOW ELASTICITY OF SUBSTITUTION

BY

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ABSTRACT

This paper defines and analyzes the properties of a directional shadow elasticity of substitution, i.e. an elasticity of substitution defined for an arbitrary price change. The concept generalizes various measures of the elasticity of substitution such as the shadow elasticity of substitution, the "own" Allen-Uzawa (partial) elasticity of substitution, and the elasticity of substitution between factor groups. It permits a generalization of the traditional factor share lemma to production functions involving more than two inputs, and provides a logical relationship between the elasticities of substitution defined on the production and on the cost side.

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INTRODUCTION

This paper defines and analyzes the properties of a directional shadow elasticity of substitution, i.e. an elasticity of substitution defined for an arbitrary price change. The concept generalizes various measures of the elasticity of substitution such as the shadow elasticity of substitution, the "own" Allen-Uzawa (partial) elasticities of substitution, and the elasticity of substitution between factor groups. It permits a generalization of the traditional factor share lemma to production functions involving more than two inputs, and provides a logical relationship between the elasticities of substitution defined on the production and the cost side.

The elasticity of substitution was originally defined by Hicks (1932) for the case of only two inputs, and generalizations to an arbitrary number of inputs have been presented by Allen (1938), Uzawa (1962) and McFadden (1963). The definition of the directional shadow elasticity of substitution (DSES) is due to Frenger (1978), and the present paper extends that work by providing alternative definitions of, and justifications for, the use of the DSES.

Most of this paper was written in 1975/1976. Recent work on the estimation of Generalized Leontief cost functions lead to the idea of using the DSES to test for the concavity of the cost function, and this test is presented in the new section 1.4. The DSES gives us, in contrast to the eigenvalues of the Hessian of the cost function, an economically meaningful measure of whether the cost function is concave or not, and the extent of variations of its curvature at an arbitrary point in the price space. An emprical application of this procedure is presented in Frenger (1985).

Sec. 1 A DIRECTIONAL SHADOW ELASTICITY OF SUBSTITUTION

1.1 The Cost Function

Let P^n be the positive orthant of the n-dimensional Euclidian space R^n, i.e.

$$P^n = \{p | p = (p_1,\ldots,p_n); \quad p_i > 0, \; i=1,\ldots,n\}$$

and let

$$C(y,p) \tag{1}$$

be a cost function defined on an open subset $D \subset P^n$ for each level of output y. P^n will be called the price space. Since $C(y,p)$ is a cost function, we know that it is[1]:

C1 - nondecreasing function of y and p

C2 - linearly homogeneous in p

C3 - concave in p

In the following it will additionally be assumed that $C(y,p)$ is:

C4 - twice continuously differentiable[2]

C5 - strictly increasing in p

The last two conditions are necessary in order to be able to define both the usual and the directional shadow elasticities of substitution. Whenever we talk about a cost function in the following, we will assume that it possesses properties C1 through C5.

From Shephard's Lemma (Shephard 1953) we know that:

$$\frac{\partial C(y,p)}{\partial p_i} = x_i(y,p) > 0 \tag{2}$$

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1) Standard references are Shephard (1953), Uzawa (1964), Diewert (1971), and McFadden (1978). These are also standard references on duality theory. For a survey of duality theory see Diewert (1974,1982).

2) Differentiability removes the possibility of having "corners" on the factor price frontier, or "flats" on the isoquants (see McFadden, 1978). It does not exclude the Leontief (fixed coefficients) production function.

i.e. that the derivative of the cost function with respect to the price of the i'th input gives the cost minimizing demand for that input. Condition C4 insures that the derivative exists, and condition C5 insures that it will be strictly positive.

Let C_i and C_{ij} be the first and second order derivatives of the cost function with respect to p_i, and p_i and p_j, respectively. Then the Shadow Elasticity of Substitution (SES) was defined by McFadden (1963) as

$$\sigma_{ij} = \frac{-\dfrac{C_{ii}}{C_i^2} + 2\dfrac{C_{ij}}{C_i C_j} - \dfrac{C_{jj}}{C_j^2}}{\dfrac{1}{p_i C_i} + \dfrac{1}{p_j C_j}}, \qquad i,j=1,\ldots,n, \tag{3}$$

holding the output rate, total cost, and p_k, $k \neq i,j$ constant.[3] σ_{ij} measures the possibility of substituting the i'th input for the j'th as their relative prices change, and is a measure of the curvature of the factor price frontier in a plane through p parallel to the i'th and the j'th coordinate axes. The factor price frontier, a concept introduced by Samuelson (1962), is defined as the set M of all price vectors, given which a given output rate y^o can be produced at the same cost C^o, i.e.

$$M = M(y^o,C^o) = \{p \mid p \in D,\ C(y^o,p) = C^o\} \,. \tag{4}$$

$M(y^o,C^o)$ is an (n-1) dimensional manifold in the price space. When $p \in M(y^o,C^o)$, we will call $M=M(y^o,C^o)$ the factor price frontier through p, and will denote it by M(p) (the output rate y^o will generally be a constant, and will be ignored notationally as an argument).

For each point $p \in M$, the gradient vector $\nabla C(y,p) = x(y,p)$ will be perpendicular to M and to the tangent plane T(p)[4] at p, where

$$T(p) = \{v \mid vx=0,\ x=x(y,p) \text{ and } v \in R^n,\ v \neq 0\} \,. \tag{5}$$

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3) σ_{ij} will in general be a function of the output rate y and the prices p, even though these arguments have not been written explicitly.

4) $T(p) \cup \{0\}$ is an affine subspace of dimension (n-1). The exclusion of $\{0\}$ from T(p) facilitates the exposition below.

Assume that only p_i and p_j are allowed to change. The condition on v implied by (5), becomes

$$v_i x_i + v_j x_j = 0, \qquad v_k=0,\ k\neq i,j \tag{6}$$

and there is only one direction vector v (up to a factor of proportionality) which satisfies condition (6): this is the direction in which σ_{ij} is defined. In the next subsection, we will define a measure of the curvature of M(p) in an arbitrary direction $v\varepsilon T(p)$.

1.2 Definition of Directional Shadow Elasticity of Substitution (DSES)

Assume that we are given a production process producing a fixed level of output with only two inputs, and that the input prices change such as to keep total cost constant[1], i.e. $x_1\Delta p_1 + x_2\Delta p_2 = 0$. We can then determine the shadow elasticity of substitution between inputs 1 and 2 from:

$$\sigma_{12} = -\frac{\frac{\Delta x_1}{x_1} - \frac{\Delta x_2}{x_2}}{\frac{\Delta p_1}{p_1} - \frac{\Delta p_2}{p_2}} \qquad (1)$$

This is in fact one of the definitions of the SES.

However if there are more than two inputs, and prices change, while keeping total cost constant, what can we then say about the ease or degree of substitution? It becomes impossible to say something about the individual SES, but we can say something rather specific about the curvature of the factor price frontier <u>in the direction of the observed price change</u>.

Four alternative expressions for this directional shadow elasticity of substitution (DSES) are given in this subsection, the first of which emphasizes the analogy with (1) above:

DEFINITION: Let $v \varepsilon T(p)$, then the Directional Shadow Elasticity of Substitution at p in the direction v, written DSES(v), is defined as:

$$\text{DSES}(v) = -\frac{\sum_i x_i v_i \frac{dx_i}{x_i}}{\sum_i x_i v_i \frac{v_i}{p_i}} \qquad (2)$$

holding the output rate y constant.

The denominator will be positive as long as v is not identically zero, in which case the DSES is not defined. All summations are from 1 to n unless otherwise specified.

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1) Because of the linear homogeneity of the cost function, there always is such a generalization.

The DSES(v) is just the ratio of a weighted average of the percentage change in the cost minimizing inputs to the weighted average of the percentage change in the input prices, the weight of each input being the change in value of the input. Since total cost and the output rate were assumed constant, the DSES(v) represents a measure of the curvature of the factor price frontier in an arbitrary direction v tangent to the frontier.

We can express the change in the input vector dx as a function of the second derivatives of the cost function since

$$dx_i = \sum_j \frac{\partial x_i}{\partial p_j} v_j = \sum_{j=1}^{n} C_{ij} v_j \quad , \tag{3}$$

and we can therefore rewrite the definition of the DSES as

$$\mathrm{DSES}(v) = - \frac{\sum\sum_{ij} C_{ij} v_i v_j}{\sum_i x_i v_i \frac{v_i}{p_i}} \quad , \qquad v \varepsilon T(p) \, . \tag{4}$$

Since the cost function is linearly homogeneous in prices, we can express the second derivatives C_{ij} as functions of the shadow elasticities of substitution σ_{ij}, $i,j=1,\ldots,n$. Let

$$\alpha_i = \frac{p_i x_i}{\sum_k p_k x_k}, \; i=1,\ldots,n; \quad \text{and} \quad C = \sum_k p_k x_k$$

then we can write (by definition $\sigma_{ii}=0$, $i=1,\ldots,n$):[2)]

$$\frac{C_{ij}}{x_i x_j} = \frac{1}{2C} \left[\frac{\alpha_i+\alpha_j}{\alpha_i \alpha_j} \sigma_{ij} - \frac{1}{\alpha_i} \sum_k (\alpha_i+\alpha_k)\sigma_{ik} - \frac{1}{\alpha_j} \sum_k (\alpha_j+\alpha_k)\sigma_{jk} \right. \tag{5}$$

$$\left. + \sum\sum_{k\ell} (\alpha_k+\alpha_\ell)\sigma_{k\ell} \right]$$

and the numerator of (4) becomes

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{C_{ij}}{x_i x_j} \alpha_i \alpha_j \frac{v_i}{p_i} \frac{v_j}{p_j} C^2 =$$

$$= \frac{C}{2} \left\{ \sum\sum_{ij} (\alpha_i+\alpha_j)\sigma_{ij} \frac{v_i v_j}{p_i p_j} - \sum\sum_{ik} (\alpha_i+\alpha_k)\sigma_{ik} \frac{v_i}{p_i} (\sum_j \alpha_j \frac{v_j}{p_j}) \right.$$

2) See Frenger (1976), sec. 1.2.

$$- \sum_{jk}\sum(\alpha_j+\alpha_k)\sigma_{jk}\frac{v_j}{p_j}(\sum_i\alpha_i\frac{v_i}{p_i}) + \sum_{k\ell}\sum(\alpha_k+\alpha_\ell)\sigma_{k\ell}\sum_{ij}\sum\alpha_i\alpha_j\frac{v_i}{p_i}\frac{v_j}{p_j}\Big\} =$$

$$= \frac{1}{2}\sum_{ij}\sum(p_ix_i+p_jx_j)\sigma_{ij}\Big[\frac{v_i}{p_i}\frac{v_j}{p_j} - \frac{v_i}{p_i}(\sum_k\alpha_k\frac{v_k}{p_k}) - \frac{v_j}{p_j}(\sum_k\alpha_k\frac{v_k}{p_k})$$

$$+ (\sum_k\alpha_k\frac{v_k}{p_k})(\sum_k\alpha_k\frac{v_k}{p_k})\Big] =$$

$$= \frac{1}{2}\sum_{ij}\sum(p_ix_i+p_jx_j)\sigma_{ij}(\frac{v_i}{p_i} - \sum_k\alpha_k\frac{v_k}{p_k})(\frac{v_j}{p_j} - \sum_k\alpha_k\frac{v_k}{p_k}) \qquad (6)$$

and since $v\epsilon T(p)$ implies that $\sum_k\alpha_k\frac{v_k}{p_k} = 0$, we have that

$$\sum_{ij}\sum C_{ij}v_iv_j = \frac{1}{2}\sum_{ij}\sum(p_ix_i+p_jx_j)\sigma_{ij}\frac{v_i\,v_j}{p_i\,p_j} \qquad v\epsilon T(p) \qquad (7)$$

Thus we have a third expression for the DSES(v) this time expressed as a function of the shadow elasticities of substitution, since (4) and (7) give,

$$\mathrm{DSES}(v) = -\frac{1}{2}\frac{\sum_{ij}\sum(p_ix_i+p_jx_j)\sigma_{ij}\frac{v_iv_j}{p_ip_j}}{\sum_i x_iv_i\frac{v_i}{p_i}}, \qquad v\epsilon T(p) \qquad (8)$$

A more intuitive measure of the DSES may be provided by the following argument. In differential geometry the curvature of a surface in a given direction $v\epsilon T(p)$ is measured by the "normal curvature".[3] In the case of the factor price frontier M (see eq. 1.1.4) at p it is given by

$$k(u) = \sum_{ij}\sum C_{ij}\,u_iu_j = \frac{1}{\|v\|^2}\sum_{ij}\sum C_{ij}\,v_iv_j \qquad u_i = \frac{v_i}{\|v\|} \qquad (9)$$

evaluated at p. It measures the curvature of the curve generated by the intersection of M and the plane formed by x and v. u is the unit vector in the direction v.

The Cobb-Douglas cost function is characterized by the fact that its shadow elasticities of substitution are everywhere equal to unity, and in

3) See e.g. O'Neill (1966), p. 196. I have yet to find a reference which gives such a definition for spaces of dimension greater than 3.

fact its DSES(v) is also equal to unity in every direction v in the tangent plane.[4] We can use this fact as a yardstick by which to measure the DSES of arbitrary cost functions.

DEFINITION: LET $C(y,p)$ be any cost function defined on $D \subset P^n$.

Let $x^o = \nabla C(y,p)\Big|_{p=p^o}$, and $\alpha_i^o = \dfrac{p_i^o x_i^o}{\sum_k p_k^o x_k^o}$,

then

$$C^{CD}(y^o,p) = y^o \prod_i p_i^{\alpha_i^o} \tag{10}$$

is the <u>best approximating</u> Cobb Douglas cost function to $C(y,p)$ at (y^o,p^o)[5].

$C^{CD}(y,p)$ best approximates $C(y,p)$ at (y^o,p^o) in the sense that $\ln C^{CD}(y,p)$ is the first order Taylor expansion of $\ln C(y,p)$ in terms of $\ln y$ and $\ln p_i$, $i=1,\ldots,n$, i.e.

$$\ln C(y^o,p) \simeq \ln C(y^o,p^o) + \sum_i \frac{p_i^o x_i^o}{C(y^o,p^o)}(\ln p_i - \ln p_i^o)$$

or

$$C(y^o,p) \simeq \frac{C(y^o,p^o)}{\prod_i (p_i^o)^{\alpha_i^o}} \prod_i p_i^{\alpha_i^o} = y^o \prod_i p_i^{\alpha_i^o}$$

The normal curvature at (y^o,p^o) of the best approximating Cobb Douglas cost function in the direction $v \in T(p^o)$ is given by (see eq. 9):

$$k^{CD}(u) = \frac{1}{\|v\|^2} \sum_{ij}\sum C_{ij}^{CD} v_i v_j =$$

$$= \frac{C(y^o,p^o)}{\|v\|^2}\left[\sum_i \sum_{j\neq 1} \frac{\alpha_i^o \alpha_j^o}{p_i^o p_j^o} v_i v_j + \sum_i \left(1-\frac{1}{\alpha_i^o}\right)\left(\frac{\alpha_i^o}{p_i^o}\right)^2 v_i^2\right]$$

$$= \frac{1}{\|v\|^2}\left[\frac{1}{C(y^o,p^o)} \sum_{ij}\sum x_i^o v_i x_j^o v_j - \sum_i \frac{x_i^o}{p_i^o} v_i^2\right]$$

$$= -\frac{1}{\|v\|^2} \sum_i x_i^o v_i \frac{v_i}{p_i^o} \tag{11}$$

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4) See lemma 4 of sec. 1.3.

5) The output rate y^o is assumed constant in taking this approximation.

LEMMA: The Directional Shadow elasticity of Substitution of the cost function $C(y,p)$ at a point p^o in the direction $v \in T(p^o)$, is given by the ratio of the normal curvature of $C(y,p)$ at p^o to the normal curvature of the best approximating Cobb Douglas cost function to $C(y,p)$ at p^o, i.e.

$$\mathrm{DSES}(v) = \frac{k(u)}{k^{CD}(u)} \tag{12}$$

This section has given us four alternative ways of defining the directional shadow elasticity of substitution (see eqs. 2, 4, 8, and 12)[6]. They will all be used in the following sections when studying the properties and possible uses of the DSES.

It may be noted that definition (12) is the most specific in that it specifies the curve ("the normal section")[7] along which the curvature is to be measured. For any direction v, this is the curve formed by the intersection of the factor price frontier and the affine plane generated by x and v (both x and v are "attached" at p). The intersection of the factor price frontier and any other affine plane containing v would generate another curve through p.[8] The shadow elasticity of substitution, for example, is generally defined in terms of the curve generated the intersection of the factor price frontier and the (v_i, v_j) affine plane, a plane which will in general not contain x.

6) A fifth definition is given in Frenger (1978). This was the first definition to be proposed, and relies on a different argument than those which lead to the four expressions above.

7) See O'Neill (1966), p. 197.

8) Would these curves have the same curvature, but different torsion?

1.3 Properties of the DSES

We can now prove the following properties of the Directional Shadow Elasticity of Substitution

LEMMA 1. For every λ, $\text{DSES}(\lambda v) = \text{DSES}(v)$.

The result follows directly from (1.2.4), and implies that DSES is homogeneous of degree zero in v and that $\text{DSES}(v) = \text{DSES}(-v)$. The homogeneity property implies that the values of DSES(v) are determined by its values for $\|v\| = 1$.

LEMMA 2. Assume that conditions C1, C2, C4 and C5 hold. Then the cost function $C(y,p)$ is concave in p if and only if $\text{DSES}(v) \geqq 0$ for every $v \in T(p)$, $v \neq 0$.

Proof: Let $C(y,p)$ be concave, then the quadratic form in the numerator of (1.2.4) is negative semidefinite, while the denominator is strictly positive, and $\text{DSES} \geqq 0$. Conversely, $\text{DSES}(v) \geqq 0$ for all $v \neq 0$ implies that the quadratic form in (1.2.4) is negative semidefinite and thus that $C(y,p)$ is concave.

The lemma is useful since it gives us an alternative way of determining whether the cost function is concave, since concavity will follow if we can show that $\text{DSES}(v) \geqq 0$ for all v, or equivalently that the minimum of DSES(v) over $v \in T(p)$ is non-negative. Such a test will be developed in the next subsection.

Let v_{ij} be a vector in T(p) with only the i'th and j'th component different from 0, i.e. the price change is limited to the i'th and the j'th prices, all other prices being constant. Then the definition of the DSES reduces to the definition of the shadow elasticity of substitution σ_{ij}, i.e.

LEMMA 3: Let v_{ij} be defined as above, then $DSES(v_{ij}) = \sigma_{ij}$

Proof: Apply definition (1.2.8) remembering that $x_i v_i + x_j v_j = 0$ and $v_k=0$, $k\neq i,j$. ■

Let us define the coefficients[1)]

$$a_{ij}(v) = -\frac{1}{2}\frac{(x_i p_i + x_j p_j)\dfrac{v_i v_j}{p_i p_j}}{\sum_s x_s v_s \dfrac{v_s}{p_s}} \qquad i,j=1,\ldots,n$$

Then
$$\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(v) = 1 \tag{4}$$

and we can rewrite the DSES as an affine combination of the σ_{ij}

$$DSES(v) = \sum_{ij}\sum a_{ij}(v)\,\sigma_{ij} \tag{5}$$

where the "weights" are not necessarily between 0 and 1.

LEMMA 4: At any given point $p\varepsilon P$, the DSES(v) has the same value in every direction $v\varepsilon T(p)$ iff the shadow elasticities of substitution are all equal, i.e. $\sigma_{ij} = \sigma$ for every $i\neq j$.

Proof: If $\sigma_{ij} = \sigma$ for every $i\neq j$, then it follows from (4) and (5) that $DSES(v) = \sigma$. If $DSES(v) = \sigma$ for every $v\varepsilon T(p)$ then, from lemma 3, $DSES(v_{ij}) = \sigma_{ij} = \sigma$ since $v_{ij}\varepsilon T(p)$. ■

McFadden (1963) analyzed the family of functions that have constant shadow elasticities of substitution for every p in the price space. A function which is to have all the DSES(v) constant, must have the SES constant and hence must belong to a subset of the functions defined in McFadden (see 1963, p. 76).

1) Should one define $a_{ii}=0$, $i=1,\ldots,n$, or use above definition? Note that $\sum_{ij}\sum a_{ij} = 0$ if the above definition is used.

1.4 Using the DSES to test for concavity

This section will utilize the fact that the cost function is concave if and only if the DSES(v) is non-negative for every v[1] to formalize a test for the concavity of the cost function. The test procedure is closely related to finding the eigenvalues of the Hessian of the cost function, but the use of the DSES gives the test a more intuitive economic interpretation.

Partial tests for concavity are provided by the own first derivatives of the cost function which should be non-positive and by the shadow elasticities of substitution, which should be non-negative. Complete tests may be obtained by computing all higher order principal minors of the Hessian of the cost function, or by computing its eigenvalues. These tests, as well as the test developed below, may be called deterministic since they determine whether or not the estimated, or fitted, cost function is concave. Statistical tests for the concavity of the true cost functions are developed by Lau (1978) using the Cholesky factorization of a real symmetric matrix.

The basic idea of the test is to determine the minimum value of DSES(v) for v in T(p). If this minimum is non-negative, then the cost function is concave. As a biproduct we will also determine the maximum value of DSES(v), and the directions in which the maxima and the minima are obtained.

Define the variables

$$G_{ij} = -\left(\frac{p_i}{x_i}\right)^{1/2} C_{ij} \left(\frac{p_j}{x_j}\right)^{1/2} , \qquad (1)$$

$$r_i = \left(\frac{x_i}{p_i}\right)^{1/2} v_i . \qquad (2)$$

1) See lemma 2 above.

The definition (1.2.4) of the directional shadow elasticity of substitution can now be written

$$\mathrm{DSES}(r) = \frac{\sum_i \sum_j r_i G_{ij} r_j}{\sum_i r_i^2}, \tag{3}$$

while the requirement that $v \varepsilon T(p)$ becomes the condition

$$r \;\varepsilon\; R(p) = \{ r \mid r\, r^n = 0,\ r \neq 0 \}. , \tag{4}$$

where $r^n = ((p_1 x_1)^{1/2}, \ldots, (p_n x_n)^{1/2})$. Finding the extreme values of DSES(v) over T(p) is equivalent to finding the extreme values of DSES(r) over R(p). But equation (3) represents the Rayleigh quotient of the matrix G, and its critical values over R^n are given by the eigenvalues $\lambda_1, \ldots, \lambda_n$ of G.[2)] And r^n is an eigenvector of G associated with the eigenvalue $\lambda_n = 0$ since $G\, r^n = 0$.

This implies that R(p) is spanned by the remaining n-1 eigenvectors $r^1, \ldots r^{n-1}$ of G[3)], and that the critical values of DSES(r) restricted to R(p) are given by the n-1 associated eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$. Using (2) we see that the critical points of DSES(v) are given by

$$v^i = \hat{d}\, r^i , \qquad i=1,\ldots n-1, \tag{5}$$

where $\hat{d}$ is a diagonal matrix with $d = ((p_1/x_1)^{1/2}, \ldots, (p_n/x_n)^{1/2})$ on the diagonal.

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2) See Hestenes (1975, p.73).
3) Or in the case of multiple roots, by a set of n-1 orthogonal eigenvectors of G, which are also normal to r^n.

Finding the extreme values of DSES(r) on R(p), or equivalently the extreme values of DSES(v) on T(p), means finding the smallest and the largest among the n-l eigenvalues $\{\lambda_1,\ldots,\lambda_{n-1}\}$. Denote these by $\underline{\lambda}$ and $\bar{\lambda}$ respectively. We have shown that

<u>LEMMA 1:</u> The cost function C(y,p) is concave at p if and only if $\underline{\lambda} \geqq 0$, and $\underline{\lambda} \leqq \mathrm{DSES}(v) \leqq \bar{\lambda}$ for every $v \in T(p)$.

It follows in particular that $\underline{\lambda} \leqq \sigma_{ij} \leqq \bar{\lambda}$ for $i,j = 1,\ldots,n$, $i \neq j$, i.e. that all the shadow elasticities of substitution lie in the interval $[\underline{\lambda},\bar{\lambda}]$. The directions in which the extreme values are obtained can be determined by using (5) to find the v^i vectors associated with $\underline{\lambda}$ and $\bar{\lambda}$.[4)]

4) See Frenger (1985) for an empirical application of the proposed procedure.

Sec. 2 OTHER ELASTICITIES OF SUBSTITUTION

2.1 Shadow Elasticity of Substitution between Input Groups

We can use the definition of DSES(v) to define the elasticity of substitution between two inputs groups. Let A and B be two disjoined sets of inputs, i.e. $A \subset N$, $B \subset N$, $A \cap B = \emptyset$, $N = \{1,2,\ldots,n\}$. Assume that the prices change proportionately within each input group, and that all other prices remain unchanged such as to leave us on the same factor price frontier, i.e.

$$\text{i)} \quad \frac{v_i}{p_i} = \gamma_A \qquad \text{for every } i\varepsilon A$$

$$\text{ii)} \quad \frac{v_i}{p_i} = \gamma_B \qquad \text{for every } i\varepsilon B \tag{1}$$

$$\text{iii)} \quad v_i = 0 \qquad \text{for every } i\varepsilon A\cup B$$

$$\text{iv)} \quad \gamma_A \sum_{i\varepsilon A} x_i p_i + \gamma_B \sum_{i\varepsilon B} x_i p_i = \gamma_A \alpha_A + \gamma_B \alpha_B = 0$$

where γ_A and γ_B are the factors of proportionality and

$$\alpha_A = \frac{\sum_{i\varepsilon A} x_i p_i}{\sum_{i\varepsilon N} x_i p_i} \qquad \alpha_B = \frac{\sum_{i\varepsilon B} x_i p_i}{\sum_{i\varepsilon N} x_i p_i} \tag{2}$$

Let v_{AB} be the direction vector given by (1), then:

DEFINITION: The Elasticity of substitution between two disjoined input groups A and B at a point p is given by:

$$SES_{AB} = DSES(v_{AB}) \tag{3}$$

Given the two input groups A and B, there is for each point in the price space a unique (except for sign) direction vector v_{AB} wich satisfies condition (1). We can therefore remove any reference to the direction vector v from the expression for the SES_{AB}[1], by rewriting the definition (1.2.8) with $v = v_{AB}$.

1) This is what is done in the definition (1.1.3) of the shadow elasticity of substitution. In fact, $SES_{AB} = \sigma_{ij}$ when $A = \{i\}$, and $B = \{j\}$.

The numerator of (1.2.8) becomes:

$$\sum_{i\varepsilon N}\sum_{j\varepsilon N}(x_ip_i+x_jp_j)\sigma_{ij}\frac{v_i}{p_i}\frac{v_j}{p_j} =$$

$$= C\gamma_A^2\sum_{i\varepsilon A}\sum_{j\varepsilon A}(\alpha_i+\alpha_j)\sigma_{ij} + 2C\gamma_A\gamma_B\sum_{i\varepsilon A}\sum_{j\varepsilon B}(\alpha_i+\alpha_j)\sigma_{ij} + C\gamma_B^2\sum_{i\varepsilon B}\sum_{j\varepsilon B}(\alpha_i+\alpha_j)\sigma_{ij}$$

$$= C\gamma_A\gamma_B\left\{-\frac{\alpha_B}{\alpha_A}\sum_{i\varepsilon A}\sum_{j\varepsilon A}(\alpha_i+\alpha_j)\sigma_{ij} + 2\sum_{i\varepsilon A}\sum_{j\varepsilon B}(\alpha_i+\alpha_j)\sigma_{ij}\right.$$

$$\left. - \sum_{i\varepsilon B}\sum_{j\varepsilon B}(\alpha_i+\alpha_j)\sigma_{ij}\right\}$$

where by definition $\sigma_{ii}=0$, $i=1,\dots,n$. And the denumerator becomes:

$$\sum_{i\varepsilon N}x_iv_i\frac{v_i}{p_i} = C\sum_{i\varepsilon A}\alpha_i\gamma_A^2 + C\sum_{i\varepsilon B}\alpha_i\gamma_B^2 = -C\gamma_A\gamma_B\left[\alpha_A+\alpha_B\right]$$

The shadow elasticity between input groups A and B can now be written

$$SES_{AB} = \frac{1}{2}\frac{\alpha_A\alpha_B}{\alpha_A+\alpha_B}\left\{-\frac{1}{\alpha_A^2}\sum_{i\varepsilon A}\sum_{j\varepsilon A}(\alpha_i+\alpha_j)\sigma_{ij} + \frac{2}{\alpha_A\alpha_B}\sum_{i\varepsilon A}\sum_{j\varepsilon B}(\alpha_i+\alpha_j)\sigma_{ij}\right.$$

$$\left. - \frac{1}{\alpha_B^2}\sum_{i\varepsilon B}\sum_{j\varepsilon B}(\alpha_i+\alpha_j)\sigma_{ij}\right\} \quad (4)$$

It will be shown in sec. 4 that this definition of the elasticity of substitution between two input groups, coincides with the natural definition in terms of price aggregates when the function is separable.

Assume that $A = \{i\}$ and that $B = N-\{i\}$, i.e. that the i'th price increases (decreases) while all the other prices change proportionately so as to keep the price change in the tangent plane. We will call this the shadow elasticity of substitution of the i'th input (SES_i). Applying (4) directly gives

$$SES_i = \frac{1}{2}\alpha_i(1-\alpha_i)\left[\frac{2}{\alpha_i(1-\alpha_i)}\sum_k(\alpha_i+\alpha_k)\sigma_{ik} - \frac{1}{(1-\alpha_i)^2}\sum_{k\ell}\sum(\alpha_k+\alpha_\ell)\sigma_{k\ell}\right]$$

$$= \frac{\alpha_i}{1-\alpha_i}\left[\frac{1}{\alpha_i}\sum_k(\alpha_i+\alpha_k)\sigma_{ik} - \frac{1}{2}\sum_{k\ell}\sum(\alpha_k+\alpha_\ell)\sigma_{k\ell}\right] \quad (5)$$

2.2 DSES for arbitrary Price Change

The shadow elasticity of substitution as defined by McFadden (1963) and the more general directional shadow elasticity of substitution defined in sec. 1.2 above required that any change in the prices be such as to keep total cost constant: if some prices increased there had to be offsetting reduction(s) in some other prices.

But since the cost function is linearly homogeneous in prices, or, by the old adage, since "only relative prices matter", we can always deflate any price change so as to keep us on the same factor price frontier. In this section, we will use this procedure to define the directional shadow elasticity of substitution for an arbitrary price change $v' = (v_1',\ldots,v_n')$.

Let v' be any vector in R^n, and define the normalized price change variable v by

$$v_i = v_i' - p_i \sum_k \alpha_k \frac{v_k'}{p_k} \tag{1}$$

Then v has the following two properties

1) $v \varepsilon T(p)$, i.e. to the tangent space at p, since

$$\sum_i x_i v_i = \sum_i x_i v_i' - (\sum_i x_i p_i)(\sum_k \alpha_k \frac{v_k'}{p_k}) = 0$$

2) v and v' give rise to the same change in the demand for factors, since

$$dx_i(v') = \sum_j \frac{\partial x_i}{\partial p_j} v_j = \sum_j C_{ij} v_j'$$

and

$$dx_i(v) = \sum_j \frac{\partial x_i}{\partial p_j} v_j = \sum_j C_{ij} v_j' - (\sum_j C_{ij} p_j)(\sum_k \alpha_k \frac{v_k'}{p_k}) = \sum_j C_{ij} v_j'$$

Equation (1) shows that the vector v can be written as a linear combination of p and v', or alternatively that v' lies in the two-dimensional subspace generated by p and v.

We will define the DSES for an arbitrary price vector v', by associating with v' the economically equivalent vector v, which lies in the intersection of the plane generated by p and v', and the tangent plane T(p). Hence

$$\mathrm{DSES}(v') = -\frac{1}{2}\frac{\sum_{ij}\sum (p_i x_i + p_j y_j)\sigma_{ij}\left(\frac{v_i'}{p_i} - \sum_k \alpha_k \frac{v_k'}{p_k}\right)\left(\frac{v_j'}{p_j} - \sum_k \alpha_k \frac{v_k'}{p_k}\right)}{\sum_i p_i x_i \left(\frac{v_i'}{p_i} - \sum_k \alpha_k \frac{v_k'}{p_k}\right)^2} \qquad (2)$$

This implies that any price change vector v' in the plane generated by p and v will have the same DSES as the vector v, though there will be a change in cost for any $v' \notin T(p)$.

There is something artificial about the definition of the SES for $n \geq 3$: why should the situation when two prices change in opposite direction so as to keep cost constant be of special interest (the case for the DES is probably somewhat more intuitive) except that mathematically it has the great advantage of reducing everything to the two factor case?

It would seem that the following question gets closer to the intuitive idea of substitution: assume that the price of the i'th good changes (all other prices remaining constant), at what rate can we substitute good i for the other goods, while output remains constant?[1] But this is exactly what is measure by DSES(v') for

$$v' = \begin{cases} v_k' = 1 & k=i^{2)} \\ v_k' = 0 & k=1,\ldots,n;\quad k \neq i. \end{cases}$$

The reader may convince himself by applying (2) that:

$$\mathrm{DSES}(v') = \mathrm{SES}_i \qquad (3)$$

where SES_i is given by (2.1.5), i.e. the shadow elasticity of substitution between the input groups {i} and N-{i}.

---

1) This is obviously a case of designing a question to fit the answer, but does that make the question less important?

2) $v_i=1$ just represents an arbitrary normalization.

2.3 The Allen-Uzawa Elasticity of Substitution

The Allen-Uzawa (partial) elasticity of substitution (AUES)[1] assumes that only one price, e.g. p_j, changes, while all other prices and the output rate (though not cost) remain constant. Expressed in terms of the cost function (1.1.1), the AUES between the i'th and the j'th input becomes:

$$AUES_{ij} = \frac{\partial x_i(y,p)}{\partial p_j}\frac{C(y,p)}{x_i(y,p)x_j(y,p)} = \frac{C_{ij}C}{C_iC_j} \qquad i,j=1,\ldots,n \qquad (1)$$

The AUES has a very simple relationship to the price elasticity of demand of the factor inputs:

$$E_{ij} = \frac{\partial x_i(y,p)}{\partial p_j}\frac{p_j}{x_i} = \frac{C_{ij}}{C_iC_j}x_jp_j = \alpha_j AUES_{ij} \qquad i,j=1,\ldots,n \qquad (2)$$

It turns out that there is also a very simple relationship between the $AUES_{ii}$, the E_{ii}, and the SES_i as given in (2.1.5) and (2.2.3). Substituting the definition of the SES in terms of the second derivatives of the cost function (see 1.1.3) for $\sigma_{k\ell}$ in the expression for the SES_i gives[2,3]

$$SES_i = -\frac{\alpha_i}{1-\alpha_i}\frac{CC_{ii}}{x_i^2} = -\frac{\alpha_i}{1-\alpha_i}AUES_{ii} = -\frac{E_{ii}}{1-\alpha_i} \qquad (3)$$

---

1) The AUES was defined by Allen (1938, p. 504) in terms of the production function. The above formulation in terms of the cost function is due to Uzawa (1962). Neither Allen nor Uzawa give any economic justification for choosing the AUES as a measure of the degree of possible factor substitution. The best justification is probably that the AUES reduces to the SES (and to the direct elasticity of substitution (for definition, see McFadden, 1963)) when n=2, or when the function is separable.

2) Using the notation of Frenger (1976), and equations (1.2.4), (1.2.11), (1.2.12), and in the second step (1.2.9), of that paper give:

$$DSES_i = \frac{\alpha_i}{1-\alpha_i}(T_i-T) = -\frac{\alpha_i}{1-\alpha_i}Cg_{ii}$$

3) Since the AUES may be negative, and the DSES may not, there is in general no direction v such that the DSES(v) = $AUES_{ij}$, i,j=1,...,n. But an unresolved question is whether there exists a simple relationship between $AUES_{ij}$ and the DSES(v) for some v.

2.4 Factor Shares

When the concept of the elasticity of substitution was originally introduced by Hicks (1932), it was to study what happened to the relative share of a factor as its supply increased. The purpose of this section is to show that Hicks' conclusion (1932, p. 247) for the case n=2, has its natural extension to n≥3, when the elasticity of substitution is "properly" defined. We will look at the factor share α_i of the i'th input:

$$\frac{\partial}{\partial p_i}\alpha_i = \frac{\partial}{\partial p_i}\frac{p_i x_i}{\Sigma p_k x_k} =$$

$$= \left[\sum_k p_k x_k\right]^{-2}\left\{\left(\sum_k p_k x_k\right)(x_i + p_i C_{ii}) - p_i x_i\left(\sum_k p_k C_{ki} + x_i\right)\right\}$$

$$= \left[\sum_k p_k x_k\right]^{-1}\left\{x_i + p_i C_{ii} - \alpha_i x_i\right\}$$

since $\sum_k p_k C_{ki} = 0$ by linear homogeneity

$$= \frac{\alpha_i}{p_i}\left[1 + p_i x_i \frac{C_{ii}}{x_i^2} - \alpha_i\right] = \frac{\alpha_i}{p_i}\left[(1-\alpha_i) + \alpha_i \mathrm{AUES}_{ii}\right]$$

and applying (2.3.3)

$$= \frac{\alpha_i}{p_i}(1-\alpha_i)(1 - \mathrm{SES}_i) \qquad (1)$$

The share of the i'th factor will increase as its price decreases (its supply increases) if the $\mathrm{SES}_i > 1$, and it will decrease if the $\mathrm{SES}_i < 1$. When n=2, $\mathrm{SES}_i = \sigma$ and we have the traditional result.

Sec. 3 DUALITY OF DIRECTIONAL ELASTICITIES OF SUBSTITUTION

3.1 A Directional Direct Elasticity of Substitution

Thus far the discussion has been entirely in terms of the cost function and the shadow elasticities of substitution. But with the duality between cost and production functions, and between the shadow elasticity of substitution and the direct elasticity of substitution, it is a straightforeward matter to define a directional direct elasticity of substitution[1)] (DDES).

Let the production function

$$y = f(x) \tag{1}$$

be defined on P^n, now called the input space, and assume that:

P1 - f(x) is nondecreasing in x
P2 - f(x) is quasiconcave

In order to be able to define the direct elasticity of substitution, and to insure its existence everywhere we will further assume that:

P3 - f(x) is twice continuously differentiable
P4 - f(x) is strictly increasing in x

In the following f(x) will always be the production function dual to the cost function C(y,p) defined in sec. 1.1.[2)]

At any given point x in the input space, let S(x) be the hyperplane tangent to the isoquant at x, i.e.

$$S(x) = \{u \mid u=(u_1,\dots,u_n),\ \sum_i f_i u_i = 0,\ f_i = \frac{\partial f}{\partial x_i}\} \tag{2}$$

---

1) Perhaps not the most fortunate choice of name?

2) For references see footnote 1 of sec. 1.1. It is probable that twice continuous differentiability and quasiconcavity of both C(y,p) and f(x) imply that both functions are strictly quasiconcave, and that in this case C1 through C5 imply P1 through P4 and vice versa. See McFadden (1978).

DEFINITION: Let $u \varepsilon S(x)$, then the directional direct elasticity of substitution at x in the direction u is:

$$DDES(u) = \left[- \frac{\sum_i f_i u_i \frac{df_i}{f_i}}{\sum_i f_i u_i \frac{u_i}{x_i}} \right]^{-1} \tag{3}$$

This definition is dual to the definition of the directional shadow elasticity of substitution in the direction v of sec. 1.2. Analogously to that section we have the following two equivalent expressions for the DDES:

$$DDES(u) = \left[- \frac{\sum\sum_{ij} f_{ij} u_i u_j}{\sum_i f_i u_i \frac{u_i}{x_i}} \right]^{-1} \qquad u \varepsilon S(x) \tag{4}$$

and

$$DDES(u) = \left[- \frac{1}{2} \frac{\sum\sum_{ij} (x_i f_i + x_j f_j) \mu_{ij}^{-1} \frac{u_i u_j}{x_i x_j}}{\sum_i f_i u_i \frac{u_i}{x_i}} \right]^{-1} \qquad u \varepsilon S(x) \tag{5}$$

where μ_{ij} is the direct elasticity of substitution between the i'th and the j'th input[3,4,5]

---

3) Because of the inversion necessary to express the f_{ij} as functions of the μ_{ij}, it is believed that f must be homothetic for this inversion to be possible. Intuitively, when f is not homothetic the matrix of second derivatives has more than $n(n-1)/2$ degrees of freedom (see sec. 1.2 of Frenger, 1976).

4) The DDES has the same properties as those described in lemmas 1 through 4 of sec. 1.3, when these are suitably reinterpreted.

5) There is of course also an obvious analogue to (1.2.12).

3.2 Duality of DSES and DDES

We will now analyze some of the relationships that exist between the DSES evaluated at p, and the DDES evaluated at the dual point x(p).

For the special case of the SES and the DES we know that:

LEMMA 1: $DES_{ij}[x(p)] \leq SES_{ij}(p) \qquad i,j=1,\ldots,n$

with equality if and only if the i'th and j'th inputs are weakly separable[1] from all the other inputs, i.e.

$$\frac{\partial}{\partial p_k}\frac{x_i}{x_j} = 0 \qquad k=1,\ldots,n,\ k\neq i,j.^{2)}$$

Why is the $DES_{ij} = SES_{ij}$ when the functions are separable? The following breef argument will make the lemma more intuitive. Let us look upon the changes in dx_k, k=1,...,n as caused by a change in the i'th and the j'th price, which is such as to leave us on the same factor price frontier, i.e. $x_i dp_i + x_j dp_j = 0$. Changing prices will leave the level of output constant, and will therefore represent a movement along the isoquant.

---

1) $\{p_i,p_j\}$ will form a weakly separable input set in the cost function if and only if $\{x_i,x_j\}$ form a homogeneously separable input set in the production function (see Lau, 1969, p. 385).

2) The inequality, and the fact that separability implies equality is proved in Frenger (1975, pp. 68-73). On the other hand, if $DES_{ij} = SES_{ij}$, then the term

$$\left(\frac{H_i}{x_i} - \frac{H_j}{x_j}\right)'(H^{BB})^{-1}\left(\frac{H_i}{x_i} - \frac{H_j}{x_j}\right)$$

in eq. 9, p. 72 (Frenger, 1975) must vanish. But since H^{BB} is negative definite, this will only occur if $\left(\frac{H_i}{x_i} - \frac{H_j}{x_j}\right) = 0$, i.e. if the cost function is separable.

The inputs will change as a response to changing relative prices, and since $dp_k=0$, $k\neq i,j$, and $x_j dp_j = -x_i dp_i$ we have that

$$dx_k = \sum_{\ell} C_{k\ell} dp_\ell = x_k \left[\frac{C_{ki}}{x_k x_i} - \frac{C_{kj}}{x_k x_j}\right] x_i dp_i \qquad k=1,\ldots,n \qquad (1)$$

But for $k\neq i,j$ the expression inside the square brackets is 0 by separability and $dx_k=0$ for $k\neq i,j$. And since total output is unchanged

$$\sum_k f_k dx_k = f_i dx_i + f_j dx_j = 0 \qquad (2)$$

The induced cost minimizing input change is exactly a change in the direction in which the DES_{ij} is defined, and we must therefore have that $DES_{ij} = SES_{ij}$ or stated slightly differently the DES_{ij} will normally be less than the SES_{ij} because the DES_{ij} stipulates that all inputs levels, except x_i and x_j, remain fixed, while the SES_{ij} allows all inputs to respond to the new price structure. When the production structure is separable, however, this does not make any difference because the x_k, $k\neq i,j$, remain at their optimal (cost minimizing) level even after the price change.

In terms of the directional derivatives, this argument can be generalized to non-separable functions, and the following duality theorem for directional elasticities of substitution. Any price change vector $v\in T(p)$, i.e. tangent to the factor price frontier at p, will induce a change in the input vector. Call this change $dx(v)$, where v has been included as an argument to emphasize the dependence of dx upon the price change. Since there is no change in the output rate, we must have $dx(v)\in S[x(p)]$, the tangent hyperplane to the isoquant at $x(y,p)$.

THEOREM: Let $v\in T(p)$, then

$$DDES[dx(v)] = DSES(v)$$

where $DSES(v)$ is to be evaluated at p and $DDES[dx(v)]$ is to be evaluated at the dual point $x(y,p)$.

Proof: Incomplete (the theorem ought to be regarded at present as a conjecture).

Sec. 4 AGGREGATE ELASTICITY OF SUBSTITUTION

In sec. 2.1 we showed how the definition of a directional elasticity of substitution could be used to define the elasticity of substitution between two input groups. In this section we will show that this latter concept is closely related to the aggregate elasticity of substitution.

Let $C(y,p)$ be the cost function given in 1.1.1, and assume that it is weakly separable with respect to the partition $\{N_i : i=1,\ldots,r\}$ of the index set $N=\{1,2,3,\ldots,n\}$. It is further assumed that there exists linearly homogeneous (consistent) price aggregates $\rho^v = \rho^v(p^v)$, where $p^v = \{p_k | k \varepsilon N_v\}$[1]; so that the cost function may be written in terms of these aggregates only, i.e.

$$C(y,p) = \mathcal{C}[y;\rho^1(p^1), \rho^2(p^2),\ldots, \rho^r(p^r)] \tag{1}$$

Regarding $\mathcal{C}$ solely as a function of the price aggregates we define the aggregate shadow elasticity of substitution between the i'th and the j'th separable input groups ($i \neq j$) as:[2]

$$\Sigma_{ij} = \frac{-\dfrac{\mathcal{C}_{ii}}{\mathcal{C}_i^2} + 2\dfrac{\mathcal{C}_{ij}}{\mathcal{C}_i \mathcal{C}_j} - \dfrac{\mathcal{C}_{jj}}{\mathcal{C}_j^2}}{\dfrac{1}{\rho^i \mathcal{C}_j} + \dfrac{1}{\rho^j \mathcal{C}_j}} \tag{2}$$

where $\mathcal{C}_i$ and $\mathcal{C}_{ij}$ are the partial derivatives of $\mathcal{C}$ with respect to the consistent price aggregates.

Let us express the aggregate elasticity of substitution as a function of the unaggregated elasticities. Since C and $\mathcal{C}$ represent the same functions with respect to the p_i, $i=1,\ldots,n$, their derivatives with respect to these price variables must coincide, and therefore[3]

$$\mathcal{C}_i = \frac{\partial C}{\partial p_k} \Big/ \frac{\partial \rho^i}{\partial p_k} = \frac{x_k}{\rho_k^i} \qquad \text{for every } k \varepsilon N_i$$

---

1) This is often called homogeneous separability condition. For cost functions this is implied by the linear homogeneity of $\mathcal{C}$. See footnote 1, sec. 3.2.

2) See (1.1.3) for the definition of the (unaggregated) shadow elasticity of substitution.

3) See Frenger (1975) pp. 58-65 for more complete derivation.

$$\mathcal{C}_{ij} = \frac{\partial C}{\partial p_k \partial p_\ell} \Big/ \frac{\partial \rho^i}{\partial p_k} \frac{\partial \rho^j}{\partial p_\ell} = \frac{x_k x_\ell}{\rho_k^i \rho_\ell^j} \left(\frac{C_{k\ell}}{x_k x_\ell}\right) \qquad \text{for every } k \in N_i,\ \ell \in N_j,\ i \neq j$$

$$\mathcal{C}_{ii} = \left(\frac{x_k}{\rho_k^i}\right)^2 \frac{1}{x_k} \frac{\sum_{s \in N_i} p_s C_{ks}}{\sum_{s \in N_i} p_s x_s} \qquad \text{for every } k \in N_i \qquad (3)$$

Because of separability we can define

$$L_{ij} = \frac{\mathcal{C}_{ij}}{\mathcal{C}_i \mathcal{C}_j} = \frac{C_{k\ell}}{x_k x_\ell} \qquad \text{for every } k \in N_i,\ \ell \in N_j,\ i \neq j \qquad (4)$$

and

$$L_{vv} = \frac{\mathcal{C}_{vv}}{\mathcal{C}_v^2} = \frac{1}{x_k} \frac{\sum_{s \in N_v} p_s C_{ks}}{\sum_{s \in N_v} p_s x_s} = \sum_{s \in N_v} \alpha_s^v \frac{C_{ks}}{x_k x_s}$$

$$\text{for every } k \in N_v,\ v=1,\ldots,r \qquad (5)$$

where

$$\alpha_s^v = \frac{p_s x_s}{\sum_{s \in N_v} p_s x_s}$$

Bacause of the homogeneous separability condition, L_{vv} is independent of the index $k \in N_v$ appearing in (5). Since the variables ρ^v are linearly homogeneous:

$$\rho^v \mathcal{C}_v = \frac{x_k}{\rho_k^v} \sum_{k \in N_v} \rho_k^v p_k = \sum_{k \in N_v} x_k p_k$$

and the aggregate elasticity of substitution can be rewritten

$$\Sigma_{ij} = \left[\frac{1}{\sum_{k \in N_i} p_k x_k} + \frac{1}{\sum_{k \in N_j} p_k x_k}\right]^{-1} \left[-L_{ii} + 2L_{ij} - L_{jj}\right] \qquad (6)$$

It remains to express the L_{ij}'s as functions of the σ_{ij}'s. Let

$$\delta_k = p_k x_k \quad \text{and} \quad C^v = \sum_{k \in N_v} p_k x_k$$

and evaluate the following two double summations:

$$\sum_{k\epsilon N_i} \sum_{\ell\epsilon N_j} (\delta_k+\delta_\ell)\sigma_{k\ell} = \sum_{k\epsilon N_i} \sum_{\ell\epsilon N_j} \delta_k\delta_\ell\left(-\frac{C_{kk}}{x_k^2} + 2\frac{C_{k\ell}}{x_k x_\ell} - \frac{C_{\ell\ell}}{x_\ell^2}\right) \quad i\neq j$$

$$= -C^j \sum_{k\epsilon N_i} \delta_k \frac{C_{kk}}{x_k^2} + 2\sum_{k\epsilon N_i} \sum_{\ell\epsilon N_j} \delta_k\delta_\ell \frac{C_{k\ell}}{x_k x_\ell} - C^i \sum_{\ell\epsilon N_j} \delta_\ell \frac{C_{\ell\ell}}{x_\ell^2}.$$

$$= C^iC^j\left[-\sum_{k\epsilon N_i} \alpha_k^i \frac{C_{kk}}{x_k^2} + 2L_{ij} - \sum_{\ell\epsilon N_j} \alpha_\ell^j \frac{C_{\ell\ell}}{x_\ell^2}\right]$$

and solving the expression for L_{ij} gives

$$2L_{ij} = \frac{1}{C^iC^j} \sum_{k\epsilon N_i} \sum_{\ell\epsilon N_j} (\delta_k+\delta_\ell)\sigma_{k\ell} + \sum_{k\epsilon N_i} \alpha_k^i \frac{C_{kk}}{x_k^2} + \sum_{\ell\epsilon N_j} \alpha_\ell^j \frac{C_{\ell\ell}}{x_\ell^2} \qquad (7)$$

Similarly

$$\sum_{k\epsilon N_i} \sum_{\ell\epsilon N_i} (\delta_k+\delta_\ell)\sigma_{k\ell} = \sum_{k\epsilon N_i} \sum_{\ell\epsilon N_i} \delta_k\delta_\ell\left(-\frac{C_{kk}}{x_k^2} + 2\frac{C_{k\ell}}{x_k x_\ell} - \frac{C_{\ell\ell}}{x_\ell^2}\right)$$

$$= 2(C^i)^2\left[\sum_{k\epsilon N_i} \alpha_k^i \sum_{\ell\epsilon N_i} \alpha_\ell^i \frac{C_{k\ell}}{x_k x_\ell} - \left(\sum_{\ell\epsilon N_i} \alpha_\ell^i\right) \sum_{k\epsilon N_i} \alpha_k^i \frac{C_{kk}}{x_k^2}\right]$$

$$= 2(C^i)^2\left[L_{ii} - \sum_{k\epsilon N_i} \alpha_k^i \frac{C_{kk}}{x_k^2}\right]$$

which, when solved for L_{ii}, gives:

$$L_{ii} = \frac{1}{2(C^i)^2} \sum_{k\epsilon N_i} \sum_{\ell\epsilon N_i} (\delta_k+\delta_\ell)\sigma_{k\ell} + \sum_{k\epsilon N_i} \alpha_k^i \frac{C_{kk}}{x_k^2} \qquad (8)$$

Setting this into the expression for the aggregate elasticity of substitution (eq. 6) gives

$$\Sigma_{ij} = \frac{1}{2}C\frac{C^iC^j}{C^i+C^j}\left[-\frac{1}{(C^i)^2} \sum_{k\epsilon N_i} \sum_{\ell\epsilon N_i} (\alpha_k+\alpha_\ell)\sigma_{k\ell} + \frac{2}{C^iC^j} \sum_{k\epsilon N_i} \sum_{\ell\epsilon N_j} (\alpha_k+\alpha_\ell)\sigma_{k\ell} - \frac{1}{(C^j)^2} \sum_{k\epsilon N_j} \sum_{\ell\epsilon N_j} (\alpha_k+\alpha_\ell)\sigma_{k\ell}\right] \qquad (9)$$

which (except for multiplying numerator and denominator by C^2) is seen to be identical to eq. 4 of sec. 2.1.

The definition of an elasticity of substitution between input-groups (SES_{AB} of sec. 2.1) represents therefore a generalization of the concept of an aggregate elasticity of substitution, which can be defined unambiguously only for separable cost functions.

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