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Torfinn Harding and Haakon O. Aa. Solheim

Documents

Documentation of a model on the household portfolio

Statistics Norway/Research Department

Abstract

This note gives an introduction to a model on household portfolios developed in Unit for macroeconomics, Statistics Norway. The model simulates optimal portfolio allocation under assumptions of expected return on the four most important financial objects in the household portfolio. It is based on the theory of mean-variance portfolio frontier, but adjusts for short sale constraints in 3 objects (housing, interest bearings and stocks), and long-sale constraint in the last object (debt). The note is meant to document the programming of the model, and will be useful particulary for those interested in using the model.

1 Introduction

This note gives a description of a model of optimal allocation of households financial portfolio. The model is developed in Unit for macroeconomics, Statistics Norway. It is a standard algorithm for calculation of a mean-variance portfolio frontier with short- and long-sale constraints. The programming of the model is done in OX. Some algorithms for calculation of actual portfolios based on data from tax-return schemes are also documented.

The first part of the note works through the mean-variance framework, and how portfolio allocations on the portfolio frontier can be analyzed for a given level of risk aversion. We then discuss points connected to the calibration of the model. A short users guide for the different algorithms is further given. Finally we give some thoughts on how the model can be used in connection to the rest of the model framework used by the macroeconomic group in Statistics Norway.

2 Theoretical background

2.1 The mean-variance framework and the portfolio frontier

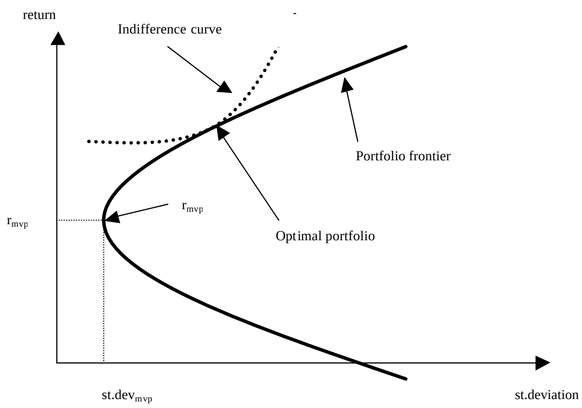
A standard assumption in finance is that an investor evaluate a portfolio on two criteria; the expected return of the portfolio and the expected variance of this return. Technically this assumption is only valid under rather strict restrictions on the utility function. However, the mean-variance assumption makes it possible to obtain testable implications of theory that can not be obtained under more general forms of utility.

The mean-variance model has been a central piece of financial economics since it was presented by Markowitz (1952). An important implication of this model is the portfolio frontier. The mathematics of the portfolio frontier was first shown in Merton (1972), but has since become a central working tool in both theoretical and applied finance.

Strictly defined, the mean-variance frontier is the boundary of the set of means and variances of the returns on all portfolios of a given set of assets. This boundary can be found by minimizing the variance of return for given mean return. There exists a mean-variance frontier as long as there are two returns not perfectly correlated and yielding different means. A typical illustration of the mean-variance frontier is displayed in figure 1. The mean-variance frontier of all risky assets is graphed as the hyperbolic region.

Figure 1: Mean-variance frontier

Illustration of the minimum-variance frontier.



If R is a vector of asset returns, E is the vector of mean returns, E=E(R), and Σ is the variance-covariance matrix, $\Sigma = E[(R-E)(R-E)']$, we can derive the mean-variance frontier using the Lagrangian approach. A portfolio is defined by its weights on the initial securities. Thus, $w'R$ is the portfolio return, where the weights sum to one, $w'1 = 1$, and mean return on the portfolio is given by $w'E = \mu$. Minimizing variance for a given mean leads to the following minimizing problem:

$$minimise\ w'\Sigma w, \tag{1}$$

s.t.

$$w'E = \mu \ \ and \ \ w'1 = 1. \tag{2}$$

For a given mean portfolio return, the variance of the minimum variance portfolio is then

$$var(R) = \frac{C\mu^2 - 2B\mu + A}{AC - B^2}, \tag{3}$$

and the weights are given by

$$w = \Sigma^{-1}\frac{E(C\mu - B) + 1(A - B\mu)}{AC - B^2}, \tag{4}$$

where

$$A = E'\Sigma^{-1}E;\ B = E'\Sigma^{-1}1;\ C = 1'\Sigma^{-1}1. \tag{5}$$

The marginal rate of transformation (between risk and return) is given by

$$MRT = \frac{D\sigma}{A\mu - B}, \tag{6}$$

where $D = BC - A^2$.

The analytical solution for the mean-variance frontier is based on the assumption that there are no constraints on the possible position one can take in any asset. For practical applications this will be a problem. Most investors will be constrained in their ability to sell an object short. This is especially relevant for households.

Imposing short sale constraints has the implications that no analytical solution for the mean-variance frontier can be derived. When we compute mean-variance frontiers below we use a programming sub-routine to calculate the optimal portfolios when the short sale constraints are binding.

2.2 Risk aversion

In the mean-variance framework, optimal portfolio allocation is defined by tangency between the indifference curve and the efficient frontier, as shown in figure 1. The point of tangency is where the marginal rate of substitution equals the marginal rate of transformation between risk and return.

Under the assumptions of constant relative risk aversion[1], normally distributed portfolio returns with mean μ and variance σ and investors maximizing expected utility, it can be shown that the marginal rate of substitution is given by (see e.g. Flavin and Yamashita, 2002, for a reference):

$$MRS = \rho\sigma, \tag{7}$$

where σ is the variance of the portfolio and the risk aversion, ρ, in optimum is then given by

$$\rho = \frac{D}{A\mu - B}. \tag{8}$$

Solving for the return, μ, gives:

$$\mu = \frac{1}{A}(\frac{D}{\rho} + B). \tag{9}$$

By making assumptions about the households utility function and the level of risk aversion, we can quantify the optimal portfolio, and thereby evaluate wether the household's actual portfolio is close to the optimal portfolio. Alternatively we can use the actual portfolios to calculate the level of risk aversion these imply.

[1]Utility function: $U(W) = \frac{1}{1-\rho}W^{1-\rho}$

2.3 Calibration

In practice households tend to invest in four objects: house, interest rate bearing assets, equity and insurance claims. In addition the household can increase exposure by borrowing to invest. We chose to ignore insurance claims, as such claims are not very liquid.

As we compare with actual Norwegian portfolios, we use Norwegian data. One should expect that the relevant mean-variance frontier is based on a rational forecast of returns. However, it is not clear what is a rational forecast of returns when making long term investments. Our data on actual portfolios cover the years 1998 to 2002. We therefore base our analysis on actual return over the period from 1993 to 2002. 1993 is a reasonable starting point, as Norway implemented a major tax reform in 1992. For equity we use the return on the Oslo Stock Exchange. For interest rates we use the average rate on debt and short term deposits respectively, collected from Norges Bank.

The pre tax return on housing consists of two components. This is the change in the housing price (capital gain) and the stream of service the house offers the owner (housing consumption). To measure capital gains we use the quarterly relative change in the housing price index for self owned houses. This measurement is imprecise, as we do not fully account for the fact that house investments are object specific, and that the risk of single investment is higher than the risk of investing in the general housing index.[2]

The consumption component is more problematic. Moum (1995) discusses four different approaches to calculate the stream of service received from a self owned house. The calculation can be based on: i) the costs of buying the service, ii) observed rental prices, iii) the costs a rental house owner is meeting or iv) the social costs of providing housing services. Under perfect competition and no public distortions the four methods should give the same results. Under a tax regime favoring self owned houses the methods may give different re-

[2]Englund, Hwang and Quigley (2002) find the object specific risk to be substantial. However, it is not clear how much of this variance is due to investments (or neglect of necessary maintenance) in the given object, and how much is due to factors outside the control of the owner (like changes in relative demand between different types of housing, different areas within the community et.c.). To the extent that the owner can control the relative value of a house, the index would be a relevant reference point.

sults. Moum's concludes that the best solution is to calculate the income stream based on rental prices. One weakness with this approach is the likely lack of representativeness of renters for homeowners. The other methods are however seen as more problematic because they demand calculations of capital returns. [3]

To find consumption return we use estimations from the Norwegian National Accounts. These are calculated according to Moum's preferred approach. To find the relative return, we divide the consumption stream on the total stock of housing capital.

Returns and a correlation matrix are given in table 1. To compare the model with actual data it is important to use post tax observations. All returns are therefore returns after tax, where taxes are based on the current (post 1992) tax regime. We focus on real returns, and use changes in the CPI as a measure of inflation.

Table 1: Real return and correlation matrix: four assets in the household portfolio

We assume that the household portfolio consists of housing, interest bearing assets, equity and debt. The table reports the expected return and standard deviation of each asset, and the correlation matrix.

Expected returns are based on actual returns, after tax, quarterly observations, from Norwegian data over the period 1993:1-2002:2. All numbers in per cent. Real returns are calculated using realised inflation measured by the CPI. Sources: Norges Bank and Statistics Norway

	Housing	**Interest**	**Equity**	**Debt**
Return	0.11	0.01	0.13	0.03
Standard deviation	0.11	0.01	0.31	0.01
Correlation				
Housing	1			
Interest	-0.19	1		
Equity	0.31	-0.09	1	
Debt	-0.05	0.93	0.04	1

Note that we ignore transaction costs. The transaction cost of a house is probably higher than the transaction cost in the other assets. This means that housing may have slightly too high return in our data. However, even transaction costs in financial markets are substantial for a non-professional investor.[4]

[3] An approximation to using data on actual rental cost, is to assume that the stream of consumption services is a fixed percentage of the house value.

[4] Buying and selling a share in a mutual fund easily takes up more than 3 % of the investment. As a comparison, the same transaction in the housing market probably will cost between 3 and 10 % of the housing value. For housing assets, the share will be higher for low cost properties, as many of the costs incurred are independent of asset price.

Table 2: Real return on housing

The table reports real, pre tax return on housing, and the two components of real return, implicit consumption and capital gains. Note that as all numbers are real returns, the return from capital gains and consumption do not add up to pre tax total return on housing.

	Housing	**Capital gains**	**Consumption**
Return	0.12	0.06	0.04
Standard deviation	0.11	0.11	0.02
Correlation			
Interest	-0.23	-0.26	0.27
Equity	0.31	0.26	0.27
Debt	-0.07	-0.17	0.63

In table 2 we report the two types of return on housing. As one can see, while the standard deviation of capital gains from housing is high, the standard deviation from housing consumption is low. Further, while capital gains are negatively correlated with the cost of debt, the consumption income from housing is strongly positively correlated with the cost of debt (0.63). This is due to a positive correlation between the cost of rental and the interest rate—as the cost of holding a home increases, the cost of renting the home will increase as well. A result is that the consumption return works as a hedge on borrowing costs. If the cost of borrowing rises, implicit return of holding a house, and thereby the consumption return, rises as well.

3 Model and model usage

The model needs four historical return series to run. It calculates portfolio allocations with short-sale constraints on the first three columns in the return matrix, and long-sale constraint on the last column.

The model works as follows: It calculates ten portfolios for a given return. That is calculating ten different combinations of objects that gives a particular return. Those ten portfolios are then ranked according to standard deviation. The portfolio with the lowest standard deviation, given that it suffices the short- and long-sale constraints, is picked. This is the one lying on the relevant efficient frontier for that particular return.

The program then calculates ten different combinations of objects that satisfies a given risk aversion. The return on these portfolios is calculated, and the portfolio with the same return and same weights as the one lying on the efficient frontier for that particular return, is picked. Then we know the optimal portfolio weights for a given return, and to which risk aversion this is an optimal allocation.

3.1 Functions

All functions are attached, and are also available electronically.[5] The following is a short users guide to each program-block of the model. Is is assumed that the reader has basic knowledge of OX.

Output from all functions is a vector of four elements. Each element represents a portfolio share and the sum of the shares is one.

The input variables to the model are:

- mX_org: A matrix of four columns, representing rates of return on the four objects. The time dimension is from the top to the bottom in the matrix, with the earliest observation at the top. From this matrix the program will calculate expected return and covariance. The three first objects will be short-sale constrained, while the last object will be long-sale constrained.

- a and b: Two scalars limiting the number of rows in mX_org used to calculate expected return and covariance. a determines the first row used, b the last.

- z: scalar defining expected return on the optimal portfolio

- rho: scalar defining risk aversion of the optimal portfolio

[5] See $X : /530/htfaak/Finansiellmodellering/Husholdningenesporteføljer$

3.1.1 f_omega_uncon(const mX_org, const a, const b, const z)

The function calculates, for given expected return z, the optimal portfolios without short-sale and long-sale constraints. Expected return is given in the matrix mX_org[a:b][].

3.1.2 f_omega_mvp_uncon(const mX_org, const a, const b)

The function calculates the minimum-variance portfolio without short-sale and long-sale constraints. Expected return is given in the matrix mX_org[a:b][].

3.1.3 f_omega_c(const mX_org, const a, const b, const z)

The function calculates, for given return z, the optimal portfolios with short-sale constraints on column 1-3 in matrix mX_org, and with long-sale constraint on column 4 in mX_org. Expected return is given in the matrix mX_org[a:b][].

3.1.4 f_omega_mvp_c(const mX_org, const a, const b)

The function calculates the minimum-variance portfolio with short-sale constraints on column 1-3 in matrix mX_org, and long-sale constraint on column 4 in mX_org. Expected return is given in the matrix mX_org[a:b][].

3.1.5 f_omega(const mX_org, const a, const b, const rho)

The function calculates optimal portfolios with short-sale constraints on column 1-3 in matrix mX_org, and long-sale constraints on column 4 in mX_org, for given risk aversion rho (ρ). Expected return is given in the matrix mX_org[a:b][].

The function uses the algorithm in f_omega_c, and f_omega_c must therefore be in the program to get output.

4 Implementation into the macro econometric model MODAG/KVARTS

The implementation into the rest of the model framework used by the Unit for macroeconomics, the macro econometric model KVARTS/MODAG (from now called KVARTS), will have to be considered in the light of what output from the portfolio model that can be useful in KVARTS. Here we will focus on relative portfolio weights.

When the portfolio model is used in interaction with KVARTS, return series from KVARTS can be picked via an input file in FAME. This file is available electronically and is only of interest for internal use in Statistics Norway.[6]

Suggested approach is:

- Run the portfolio model on historical figures, for example in a five or ten year window. Calibrate the risk aversion such that the debt share and housing share is about the last observed debt- and housing-share. It could be a good idea to use a programming procedure to adjust this a precise as possible.
- Run the model again for the given risk aversion, but use the input values from KVARTS for a year into the future.
- Replicate the procedure for each year, as far as there is data.

Note that in this analysis we will only change the matrix of expected returns when we update with model simulations. For the levels of the expected returns, this is fine. KVARTS says, however, little about the variance in the return series, and thereby little about the covariance matrix between the expected return series. To solve this, we use the covariance matrix of the historical returns. This is assuming that the risk and the correlation between the risks of each object stays the same in the future as in the historical period. A version of

[6] UNIX: /ssb/ovibos/fame/kvarts/kapitalmarkeder/DataDP395.inp

the model where it is possible to adjust the interval for the level of return and the interval for calculating the covariance matrix is available from the authors.

The result of this exercise is time series of return levels that gradually takes in new information contained in the prognosis from KVARTS.

To compare simulation results on asset holdings from the portfolio model with the corresponding prognosis from KVARTS, the following objects are comparable:

- Housing in the portfolio model: (0.83*PBOL+0.17*BORETTQ)*K83[T-1] in KVARTS
- Interest bearings in the portfolio model: BIF300 in KVARTS
- Stocks in the portfolio model: AMF300 in KVARTS
- Debt in the portfolio model: BG300 in KVARTS.

In the portfolio model we only look at net interest bearings (there are never holdings of both interest bearings and debt simultaneously), which corresponds to BIF300-BG300. When we compare, BIF300-BG300 is interpreted as interest bearings when it takes a positive value and debt when it takes a negative value.

All ratios is relative to net wealth in the portfolio model, which corresponds to the sum of housing, stocks and net interest bearings in both models.

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