Discussion Paper

Central Bureau of Statistics, P.B. 8131 Dep, 0033 Oslo 1, Norway

No. 10 24 May 1985

SPECIFICATION OF CONSUMER DEMAND MODELS WITH STOCHASTIC ELEMENTS IN THE UTILITY FUNCTION AND THE FIRST-ORDER CONDITIONS

BY

ERIK BIØRN

CONTENTS

Page
1. Introduction 1
2. The structural model 2
3. The reduced form 4
4. The consumption function 6
5. Some remarks on the error distribution 7
References 11

ABSTRACT

In this note, we derive a simultaneous system of static consumer demand functions from a model with stochastic elements explicitly specified in the utility function and the first-order conditions for constrained utility maximization. The utility function is of the Stone-Geary form, with saving included as a separate "commodity". Stochastic variation in the parameters between observation units, as well as errors of measurement in consumption, saving, and income, are also allowed for. Some remarks on the error distribution following from this specification are given.

SPECIFICATION OF CONSUMER DEMAND MODELS WITH STOCHASTIC ELEMENTS IN THE UTILITY FUNCTION AND THE FIRST-ORDER CONDITIONS*)

by

Erik Biørn
Central Bureau of Statistics, Oslo, Norway

1. Introduction

The specification of the stochastic elements of a complete system of consumer demand functions is an important problem in empirical demand analysis. However, some aspects of the problem seem to be undeservedly neglected in the literature. The strategy commonly chosen is a two-stage procedure; first, to specify a set of deterministic demand functions which conform to utility-maximizing behaviour, and second, to furnish these functions with (additive) stochastic disturbances.[1] Rarely, attempts are made to connect the two parts of the model formulation.[2]

On the other hand, when dealing with a formally similar problem within the context of producer's behaviour - i.e., when constructing the product supply and factor demand functions of a (typical) profit-maximizing firm with a parametrically specified production function - the standard approach is essentially different. The stochastic elements are introduced into the model from the outset, in the form of disturbances in the production function and the equations representing the first-order conditions for maximization of average (or expected) profit.[3] From this structural specification, the reduced form equations, i.e., the product supply and factor demand functions, can be derived, and their stochastic properties reflect the way in which the stochastic elements are introduced into the structural equations.

*) I wish to thank Jørgen Aasness for useful comments on a previous version of the paper.

1) See e.g. Deaton (1975), Chs. 3 and 4.
2) For an interesting exception, see, however, Theil (1975), Ch. 2.6.
3) See e.g. Marschak and Andrews (1944), and Nerlove (1965).

A pertinent question is: Why not follow the latter approach also when specifying the stochastic structure of consumer demand functions? One answer may be that the output level is an observable variable, whereas its counterpart in the consumer demand model, the utility level, is not; consequently, the utility function cannot be considered an econometric structural equation. An alternative (but related) way of explaining the current practice is to call attention to the fact that the econometrician is frequently interested in properties of the production function without being concerned with the product supply or factor demand functions, whereas the utility function is of limited interest in itself. Neither of these answers is, however, satisfactory.

The purpose of this note is to derive a simultaneous system of (static) consumer demand functions from a model with stochastic elements explicitly specified in the utility function and the first order conditions for (constrained) utility maximization. We assume that the average utility function is of the *Stone-Geary* form. Otherwise, the specification is fairly general: *Saving* is introduced as a separate argument ("commodity") in the utility function, as a summary way of representing the consumer's concern for future time periods; and we allow for (i) *stochastic variations in the parameters*, (ii) *errors of measurement* in consumption, saving, and income, and, as already declared, (iii) *disturbances* in the consumer's optimizing conditions.

2. The structural model

Assume the utility function has the Stone-Geary form

$$(1) \qquad U = \sum_{i=1}^{N} \beta_i^* \log(x_i^* - \gamma_i^*) + \beta^* \log(s^* - \gamma^*),$$

where x_i^* and s^* denote the quantity of the i'th commodity consumed and the volume of saving respectively (i=1,...,N). The coefficients β_i^*, γ_i^*, β^*, and γ^* are assumed to be *known by the consumer* in the process of optimization, but are, of course, *unknown to the econometrician*. Moreover, we shall assume that the coefficients differ between consumers, and that the differences appear to the econometrician as random variations. We thus have a specification with *random coefficients*,

$$(2) \qquad \begin{cases} \beta_i^* = \beta_i + \varepsilon_i & (i=1,\ldots,N), \\ \beta^* = \beta + \varepsilon, & \end{cases}$$

$$(3)\quad \begin{cases} \gamma_i^* = \gamma_i + v_i & (i=1,\ldots,N), \\ \gamma^* = \gamma + v, \end{cases}$$

where β_i, β, γ_i, and γ denote the common expected values of the coefficients, an ε_i, ε, v_i, and v are stochastic errors. Finally, we assume that the values of consumption and saving observed by the econometrician (e.g., the values reported by the consumer) deviate from their "true" values, i.e., those on which the actual decisions are made, by stochastic *errors of measurement* u_i and u. The values observed are thus

$$(4)\quad x_i = x_i^* + u_i \qquad (i=1,\ldots,N),$$

$$(5)\quad s = s^* + u.$$

Let p_i and P denote the price of the i'th commodity and the "price" of saving (i.e., the price index used to deflate nominal saving to get its real value, which is an argument in the utility function), respectively. The variables are exogenous and observed without error, and we shall, for simplicity, treat them as *non-stochastic* in the sequel. Furthermore, let y denote the income observed (i.e., the sum of observed consumption expenditure and observed saving), and y^* the true income, i.e.,

$$(6)\quad y^* = \sum_i p_i x_i^* + Ps^*,$$

$$(7)\quad y = \sum_i p_i x_i + Ps.$$

Eqs. (4) - (7) imply

$$(8)\quad y = y^* + \sum_i p_i u_i + Pu.$$

The problem of optimization as seen from the consumer's point of view is the following: Maximize the utility level U with respect to x_i^* $(i=1,\ldots,N)$ and s^*, subject to the budget constraint (6), taking p_i $(i=1,\ldots,N)$, P (a function of the p_i's), and y^* as given. We write the first-order conditions for this problem as

$$(9) \qquad \frac{\partial U}{\partial x_i^*} = \frac{\beta_i^*}{x_i^* - \gamma_i^* + w_i} = \omega p_i \qquad (i=1,\dots,N),$$

$$(10) \qquad \frac{\partial U}{\partial s^*} = \frac{\beta^*}{s^* - \gamma^* + w} = \omega P,$$

where ω denotes the marginal utility of income, and w_i and w are random *disturbances* intended to capture errors in maximization. We may, for instance, imagine that the consumer, for one reason or another, is unable to attain the maximizing utility level exactly, or that his target can be described only approximately as constrained maximization of the utility function (1).

3. The reduced form

From (6), (9), and (10) we get, after elimination of ω, the following system of expenditure functions

$$(11) \qquad p_i x_i^* = p_i(\gamma_i^* - w_i) + \frac{\beta_i^*}{\Sigma\beta_j^* + \beta^*}\{y_i^* - \Sigma p_j(\gamma_j^* - w_j) - P(\gamma^* - w)\} \qquad (i=1,\dots,N),$$

and the following saving function

$$(12) \qquad Ps^* = P(\gamma^* - w) + \frac{\beta^*}{\Sigma\beta_j^* + \beta^*}\{y_i^* - \Sigma p_j(\gamma_j^* - w_j) - P(\gamma^* - w)\}.$$

The left hand side of (11) and (12) represent the "true" expenditure on the i'th commodity, and the "true" value of saving, respectively. The first terms on the right hand side represent the corresponding values of 'minimum consumption' and 'minimum saving', when allowance is made for the disturbances in the consumer's optimization, whereas the expression in the curly brackets may be interpreted as the corresponding "true" value of the 'supernumerary income'. The fractional expressions before the curly brackets represent the marginal propensity to consume of the i'th commodity and the marginal propensity to save, respectively. Recalling (2) and (3), we note that 'supernumerary income' as well as the marginal propensities to consume and save are stochastic variables in this model.

The marginal propensities to consume and to save can be decomposed into a deterministic and a stochastic part. Let us assume, without loss of generality, that the parameters β_i^* and β^* in the utility function (1) have been normalized so as to add to unity,

$$(13) \qquad \Sigma\beta_i^* + \beta^* \quad = \quad 1.$$

Assuming, moreover, that the random variations in these parameters between consumers have zero expectations,

$$(14) \qquad E(\varepsilon_i) \quad = \quad E(e) = 0 \qquad\qquad (i=1,\ldots,N),$$

it follows, by using (2), that

$$(15) \qquad \Sigma\beta_i + \beta \quad = \quad 1,$$

and

$$(16) \qquad \Sigma\varepsilon_i + \varepsilon \quad = \quad 0.$$

By using (2), (3), (4), (5), and (8) to eliminate the starred variables and coefficients in (11) and (12), the expenditure and saving functions can be expressed in terms of the observable variables as

$$(17) \qquad p_i x_i \quad = \quad p_i\gamma_i + \beta_i(y - \Sigma p_j\gamma_j - P\gamma) - \beta_i\{\Sigma p_j U_j + PU\}$$
$$+ p_i U_i + \varepsilon_i(y - \Sigma p_j\gamma_j - P\gamma) - \varepsilon_i\{\Sigma p_j U_j + PU\} \quad (i=1,\ldots,N),$$

$$(18) \qquad Ps \quad = \quad P\gamma + \beta(y - \Sigma p_j\gamma_j - P\gamma) - \beta\{\Sigma p_j U_j + PU\}$$
$$+ PU + \varepsilon(y - \Sigma p_j\gamma_j - P\gamma) - \varepsilon\{\Sigma p_j U_j + PU\},$$

where

$$(19) \qquad U_i \quad = \quad u_i + v_i - w_i \qquad\qquad (i=1,\ldots,N),$$

$$(20) \qquad U \quad = \quad u + v - w.$$

It is readily observed that the composite error terms in (17) and (18), i.e.,

$$(21)\qquad V_i \;=\; p_iU_i + \varepsilon_i(y - \Sigma p_j\gamma_j - P\gamma) - (\beta_i + \varepsilon_i)\{\Sigma p_jU_j + PU\} \qquad (i=1,\ldots,N),$$

$$(22)\qquad V \;=\; PU + \varepsilon(y - \Sigma p_j\gamma_j - P\gamma) - (\beta + \varepsilon)\{\Sigma p_jU_j + PU\},$$

have the property that $\Sigma V_i + V = 0$ *regardless of the assumptions made with respect to the errors and disturbances* u_i, v_i, w_i, ε_i, u, v, w, *and* ε *of the structural form of the model.* Our approach thus automatically ensures that the adding-up restriction is satisfied in the observed variables y, x_i, and s.[4)]

4. The consumption function

The consumption function corresponding to the expenditure and saving functions (17) and (18) can easily be derived. Let c be the total value of observed consumption,

$$(23)\qquad c \;=\; \Sigma p_ix_i \;=\; y - Ps.$$

Eqs. (18) and (22) yield

$$(24)\qquad \begin{aligned} c \;&=\; (1 - \beta)(y - P\gamma) + \beta\Sigma p_j\gamma_j \\ &\quad - PU - \varepsilon(y - \Sigma p_j\gamma_j - P\gamma) + (\beta + \varepsilon)\{\Sigma p_jU_j + PU\} \\ &= (1 - \beta)(y - P\gamma) + \beta\Sigma p_j\gamma_j - V. \end{aligned}$$

4) Confer also Pollak and Wales (1969), whose modification of the Stone Linear Expenditure System (LES) proposed on pp.613-614 emerges as a special case of our model.

We can here interpret $1 - \beta$ as the "average" marginal propensity to consume.[5)]

By elimination of income y from (17) and (24), the expenditure functions can be expressed in terms of total observed consumption expenditure as

$$(25) \quad p_i x_i = p_i \gamma_i + \frac{\beta_i}{1-\beta}(c - \Sigma p_j \gamma_j) + \frac{\beta_i}{1-\beta} V + V_i \qquad (i=1,\ldots,N),$$

whose deterministic part has the same form as in the Stone LES system. We find, not surprisingly, that (25) satisfies the adding-up condition $\Sigma p_i x_i = c$ identically, since $\Sigma\beta_i = 1 - \beta$ and $\Sigma V_i = -V$.[6)]

5. Some remarks on the error distribution

So far, no assumptions have been made with respect to the probability distribution of the errors and disturbances of the model, apart from the assumption of zero expectation of the ε's, (14). Below we shall present one specification and discuss some of its implications.

First, we assume, in the spirit of the Permanent Income Hypothesis of consumption, that the errors of observation in the quantities consumed and in the volume of saving have zero expectations and are uncorrelated with their true values, i.e.,

$$(26) \quad E(u_i | x_1^*,\ldots,x_N^*,s^*) = E(u | x_1^*,\ldots,x_N^*,s^*) = 0 \qquad (i=1,\ldots,N).$$

---

5) It is interesting to note the formal similarity between the deterministic parts of (24) and (17) on the one hand and the consumption function and the expenditure functions derived from the ELES approach, on the basis of a multi-period utility function, on the other. (See Lluch (1973) and Lluch and Williams (1975).) If 'minimum saving' is restricted to zero, i.e., $\gamma = 0$, the deterministic parts of the equations have in fact identically the same form. The stochastic specification of the two models is, however, different, as the standard version of the ELES model includes neither latent structural variables nor random coefficients.

6) Note, however, that y has not been "eliminated" from the error terms of (25), since V_i and V as defined in (21) and (22) are income dependent.

This, in combination with (6) and (8), implies that the errors are also uncorrelated with the true income

$$(27) \qquad E(u_i|y^*) = E(u|y^*) = 0,$$

but correlated with the observed income y. Moreover, it implies

$$E(y) = E(y^*), \; E(x_i) = E(x_i^*), \; E(s) = E(s^*).$$

Second, we assume that the random parts of the coefficients β_i^*, β^*, γ_i^*, and γ^*, as well as the disturbances in the first-order conditions, are uncorrelated with the true income, i.e.

$$(28) \qquad E(\varepsilon_i|y^*) = E(\varepsilon|y^*) = E(v_i|y^*) = E(v|y^*) = E(w_i|y^*) = E(w|y^*) = 0 \quad (i=1,\dots,N).$$

The interpretation of (27) and (28) is that, apart from the prices p_i and P, *y^* is the only truly exogenous structural variable in the demand model.* Even if the u's are uncorrelated with the x_i^*'s and s^*, the same cannot be true for the other random errors in the model: the ε's and the v's are parts of the coefficients on which the individual consumption decisions are based, cf. (2) and (3), and the w's will affect the outcome of the maximization process, given the values of these individual coefficients, cf. (9) and (10).

Third, we assume that

(29) u_i, v_i, w_i, u, v, and w are mutually uncorrelated for i=1,...,N, and uncorrelated with $(\varepsilon_1,\dots,\varepsilon_N,\varepsilon)$.

(The ε's cannot, of course, be mutually uncorrelated, in view of the adding-up restriction (16).) Fourth, all errors and disturbances are assumed to have constant second order moments.

From (19), (20), (27), and (28) we find that

$$(30)\qquad E(U_i|y^*) = E(U|y^*) = 0 \qquad (i=1,\ldots,N),$$

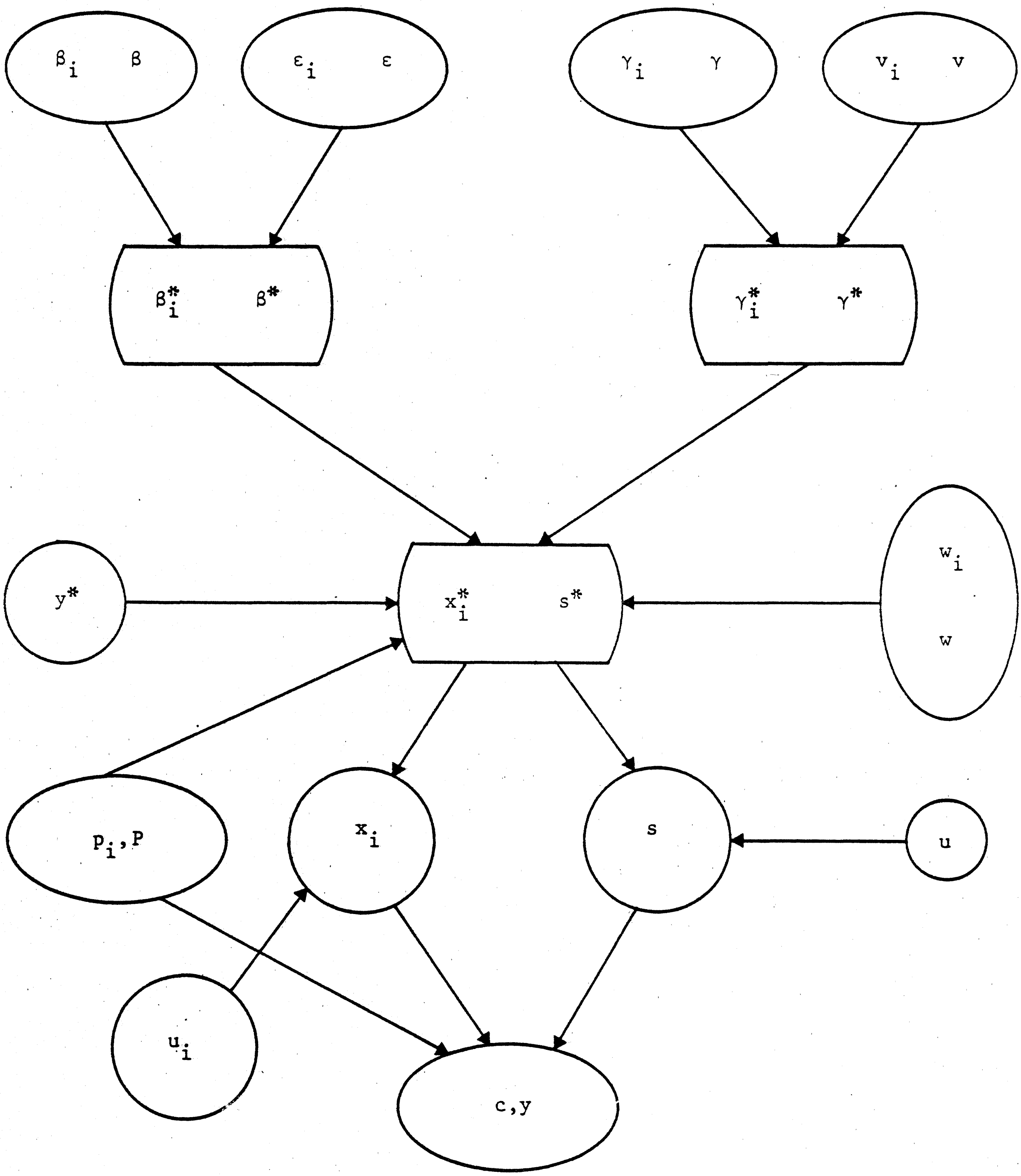
and moreover, using (21), (22), and (29), that

$$(31)\qquad E(V_i|y^*) = E(V|y^*) = 0 \qquad (i=1,\ldots,N).$$

Thus, the composite errors in the expenditure functions (17) (or (25)), the saving function (18), and the consumption function (24) will all be uncorrelated with the true income y^*, *provided that the noncorrelation assumption (29) is satisifed.* But V_i and V will *always* be correlated with the observed values of income, y, saving, Ps, and total consumption expenditure, c. From (21) and (22) we see, moreover, that these errors will show heteroscedasticity, since their second order moments are price dependent. These second order moments will also be functions of the 'minimum quantity' parameters γ_i and γ in the underlying utility function. In figure 1, we have tried to visualize the "causal structure" inherent in this stochastic specification. It clearly illustrates the exogeneity of y^* - no arrow points *to* this variable - and the joint endogeneity of x_i^*, s^*, x_i, s, c, and y.

The noncorrelation assumption (29) may be unduly restrictive for practical applications of the model - in particular if a disaggregate commodity classification is used. Few objections may be raised against assuming that the errors of measurement in consumption and saving, $\underset{\sim}{u} = (u_1 \ldots u_N u)'$ are uncorrelated with the random variations in the demand coefficients, $\underset{\sim}{\varepsilon} = (\varepsilon_1 \ldots \varepsilon_N \varepsilon)'$ and $\underset{\sim}{v} = (v_1 \ldots v_N v)'$, and with the disturbances $\underset{\sim}{w} = (w_1 \ldots w_N w)'$, representing slacks in maximization. But the potential presence of want dependence between commodity groups - recall that the Stone-Geary utility function imposes additive preferences on the structural part of the model - suggests that correlation both within and between the vectors $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{v}$ should be allowed for. Likewise, good reasons may be given for specifying correlation within the errors of measurement vector $\underset{\sim}{u}$, for instance the fact that many modern households make large simultaneous purchases of consumer goods for stock purposes (e.g. foods) in order to save time and transaction costs. The question is, of course, how far in the direction of relaxing (29) to allow oneself to go. Obviously, some restrictions will have to be imposed on the second order moments of the joint error distribution to ensure complete identification of the model. This issue will not be dealt with in the present paper.

Figure 1. Correlation structure of the demand model



REFERENCES

Deaton, A. (1975): *Models and Projections of Demand in Post-War Britain.* (London: Chapman and Hall, 1975.)

LLuch, C. (1973): The Extended Linear Expenditure System. *European Economic Review*, 4. (1973), 21-32.

Lluch, C. and Williams, R. (1975): Consumer Demand Systems and Aggregate Consumption in the USA: An Application of the Extended Linear Expenditure System. *Canadian Journal of Economics*, 8 (1975), 49-66.

Marschak, J. and Andrews, W.H. (1944): Random Simultaneous Equations and the Theory of Production. *Econometrica*, 12 (1944), 143-205.

Nerlove, M. (1965): *Estimation and Identification of Cobb-Douglas Production Functions.* (Amsterdam: North-Holland Publishing Company, 1965.)

Pollak, R.A. and Wales, T.J. (1969): Estimation of the Linear Expenditure System. *Econometrica*, 37 (1969), 611-628.

Theil, H. (1975): *Theory and Measurement of Consumer Demand, Vol. 1.* (Amsterdam: North Holland Publishing Company, 1975.)

ISSUED IN THE SERIES DISCUSSION PAPER

No. 1 I. Aslaksen and O. Bjerkholt: Certainty Equivalence Procedures in the Macroeconomic Planning of an Oil Economy.

No. 2 E. Offerdal: Structural Change, Profitability, and Survival.

No. 3 E. Biørn: On the Prediction of Population Totals from Sample Surveys Based on Rotating Panels.

No. 4 P. Frenger: A Short Run Dynamic Equilibrium Model of the Norwegian Production Sectors.

No. 5 I. Aslaksen and O. Bjerkholt: Certainty Equivalence Procedures in Decision-Making under Uncertainty: an Empirical Application.

No. 6 E. Biørn: Depreciation Profiles and the User Cost of Capital.

No. 7 P. Frenger: A Directional Shadow Elasticity of Substitution.

No. 8 S. Longva, L. Lorentsen, and Ø. Olsen: The Multi-Sectoral Model MSG-4, Formal Structure and Empirical Characteristics.

No. 9 J. Fagerberg and G. Sollie: The Method of Constant Market Shares Revisited.

No. 10 E. Biørn: Specification of Consumer Demand Models with Stochstic Elements in the Utility Function and the First Order Conditions.