ARTIKLER 51

ANALYTIC GRADUATION OF AGE-SPECIFIC FERTILITY RATES

By Eivind Gilje

ANALYTISK GLATTING AV ALDERSSPESIFIKKE FØDSELSRATER

OSLO 1972

STATISTISK SENTRALBYRÅ

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PREFACE

The Central Bureau of Statistics of Norway produces regional population projections which are based in part on regional fertility rates, specific for age of mother. The diagram of the fertility rates for a region, plotted against age of mother, will typically be much less smooth than what is desirable. Because the regional population is usually too small, the diagram may contain considerable irregularities. The Bureau has decided therefore to develop suitable graduation methods.

The papers presented here were written in this context while Mr. Eivind Gilje was employed in the then Study Group for Population Models of the Bureau. Two of them have been published previously. "Fitting curves to age-specific fertility rates" appeared in the Statistical Review of the National Central Bureau of Statistics of Sweden, Third Series, Vol. 7 (1969), pp. 118-134. "The shifted Hadwiger function", which has Dr. L. Yntema, of Amsterdam, as a co-author, appeared in the Skandinavisk Aktuarietidskrift.

"Note on population size and age-specific fertility rates" has not been printed before.

The Central Bureau of Statistics of Norway is grateful to the National Central Bureau of Statistics of Sweden, to Skandinavisk Aktuarietidskrift, and to Dr. Yntema for permission to reprint the two papers.

Central Bureau of Statistics, Oslo, 6 October 1972

Petter Jakob Bjerve

FORORD

Statistisk Sentralbyrås regionale befolkningsprognoser er bl.a. basert på utregning av regionale fødselsrater spesifisert etter morens alder. Diagrammet for fødselsratene for en region, tegnet med morens alder som abscisse, vil som regel være langt mindre glatt enn ønskelig. På grunn av at befolkningen i den enkelte region vanligvis er relativt liten, vil diagrammet vise til dels betydelige uregelmessigheter. Byrået har derfor funnet det nødvendig å utvikle metoder til å glatte dem.

De artiklene som legges fram her, er skrevet i denne sammenheng mens aktuar Eivind Gilje var tilsatt i Byrået. To av dem har vært publisert tidligere. "Fitting curves to age-specific fertility rates" stod i det svenske Statistiska Centralbyråns Statistisk tidskrift, Tredje följden, Årgang 7 (1969), side 118-134. "The shifted Hadwiger fertility function", som aktuar Gilje skrev sammen med dr. L. Yntema, Amsterdam, stod i Skandinavisk Aktuarietidskrift.

"Note on population size and age-specific fertility rates" har ikke vært trykt før.

Statistisk Sentralbyrå vil takke Statistiska Centralbyrån, Skandinavisk Aktuarietidskrift og dr. Yntema for samtykke til opptrykk av de to publiserte arbeidene.

Statistisk Sentralbyrå, Oslo, 6. oktober 1972

Petter Jakob Bjerve

FITTING CURVES TO AGE-SPECIFIC FERTILITY RATES:

SOME EXAMPLES

by Eivind Gilje[x)]

CONTENTS

x) Written in the Study Group for Population Models, The Central Bureau of Statistics of Norway.

Fitting curves to age-specific fertility rates: Some examples

by Eivind Gilje[1]

1. Introduction

1.1. While one would expect fertility to be a fairly smooth function of age, the diagram of a set of observed age-specific fertility rates tends to show a rather rugged curve, particularly in small populations. (See e. g. figure 4 below.) When it is reasonable to ascribe such "irregularities" to accidental circumstances, one may get a better picture of the underlying fertility by fitting some nice mathematical function to the observed values.

There are a large number of functions which could be fitted to observed age-specific fertility rates. In this paper we will give examples of the fitting properties of some of these.

1.2. The problem arose in connection with the making of population projections for Norwegian regions (Gilje, 1968). As the fertility can vary appreciably between the various parts of our widespread country, we wanted separate fertility measures for each of a number of small regions. The observed fertility turned out to fluctuate quite a lot with age, and more in regions with a small population than in regions with a larger.

We have no reason to believe that these "irregularities" are due to particular tendencies in the birth habits of the observed stock of women. We therefore ascribe them to chance fluctuations, and a priori assume that the underlying fertility of the population in such regions is a regular function of some kind. This assumption should justify that we try to fit a mathematical function to the estimates, particularly for small populations. Here we of course must use some discretion. When a population is too small, the chance variations will predominate, and a curve fitted to such observed values has of course little or no interest.

2. Some previous papers related to this problem

2.1. Investigations on curve fitting to age-specific fertility rates have recently been made by Yntema (1969), by Mitra (1967), by Keyfitz (1967), and by Tekse (1967). The latter two give references to many earlier authors.

In this paper, we shall take the studies by Tekse (1967) and Yntema (1953, 1956, and elsewhere) as our starting point.

2.2. Yntema recommends using the following function, originally suggested by Hadwiger (1940, 1941):

[1] Written in the Study Group for Population Models, the Central Bureau of Statistics of Norway.

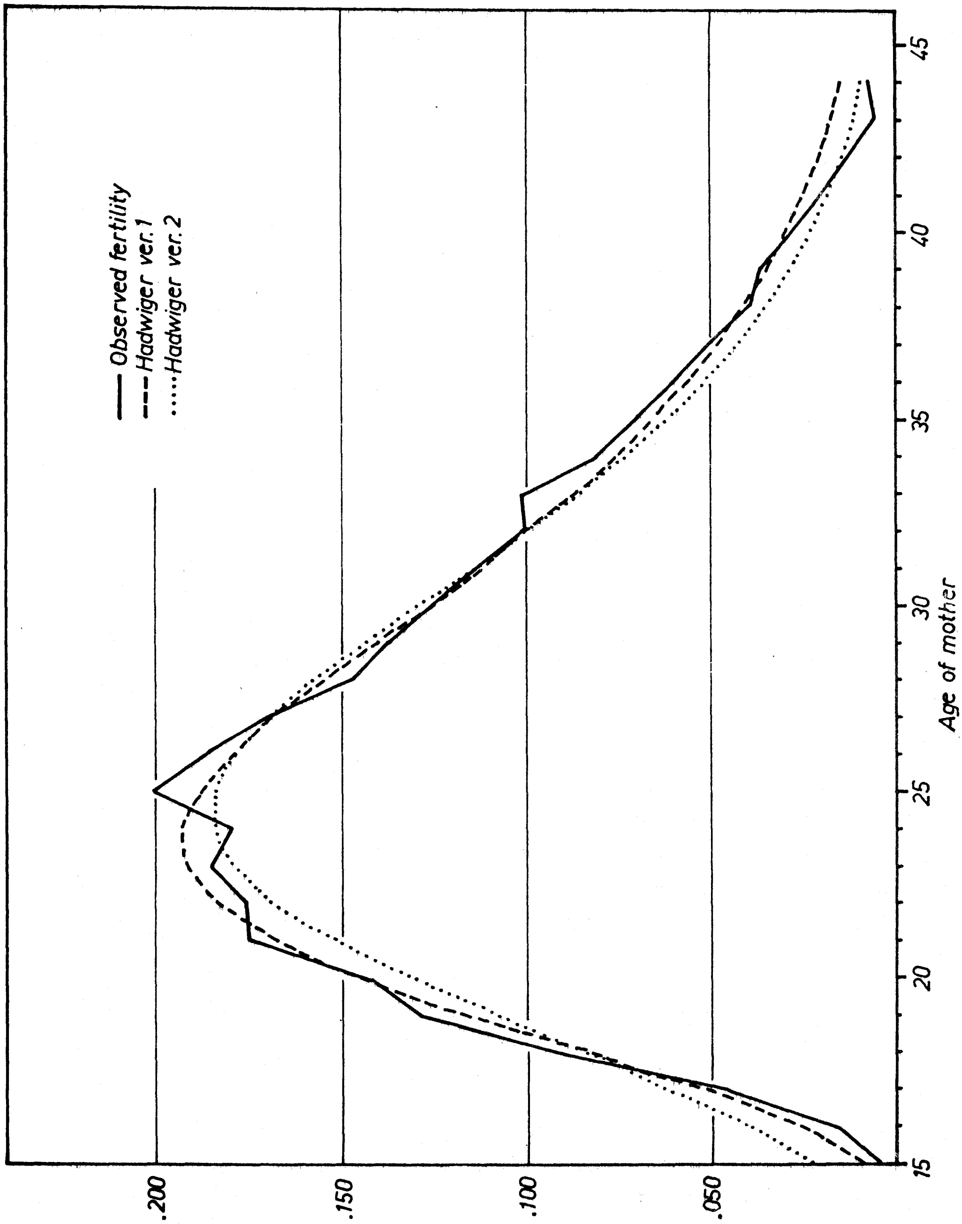

Fig. 1. Observed and smoothed fertility. Norway 1966

$$h_Y(x) = \frac{ab}{c\sqrt{\pi}} \left(\frac{c}{x}\right)^{\frac{3}{2}} exp\left\{-b^2\left(\frac{c}{x}+\frac{x}{c}-2\right)\right\}. \quad (2.1)$$

Here x represents age attained, while a, b and c are parameters to be fitted. If we let $\hat{f}_x$ denote the observed fertility rate at age x, $\hat{R}_0 = \Sigma_x \hat{f}_x$ the corresponding observed gross fertility rate, $\hat{R}_k = \Sigma_x x^k \hat{f}_x$, and $\hat{T} = \hat{R}_1/\hat{R}_0$, Yntema would estimate the parameters a and c by

$$\hat{a} = \hat{R}_0 \text{ and } \hat{c} = \hat{T}, \quad (2.2)$$

respectively. His suggestions for estimators for b amount to

$$\hat{b}_1 = \{\tfrac{1}{2}\hat{R}_1{}^2/(\hat{R}_0\hat{R}_2 - \hat{R}_1{}^2)\}^{\frac{1}{2}},$$
$$\hat{b}_2 = \hat{T}\sqrt{\pi}\hat{f}_{\check{T}}/\hat{R}_0, \quad (2.3)$$

as well as

$$\hat{b} = \tfrac{1}{2}(\hat{b}_1 + \hat{b}_2) \quad (2.4)$$

with a preference for $\hat{b}$. Here $\check{T}$ is the integer value obtained by rounding off $\hat{T}$. The estimators $\hat{a}$, $\hat{b}_1$, and $\hat{c}$ have been found by the method of moments.

Yntema (1953) also investigates a γ-function of the form

$$\gamma_Y(x) = a\left(1 + \frac{x+d}{b}\right)^{bc} exp\{-c(x+d)\} \quad (2.5)$$

as well as that of a normal curve. He considers the corresponding fit to his Dutch data unsatisfactory. We will comment on this later (*4. 7.*).

2.3. Tekse (1967, p. 194) finds the Hadwiger function of (2.1) wholly unsuitable to represent his Hungarian fertility data, and prefers the γ-function in a form like

$$\gamma_T(x) = a(x-14)^b \, exp\{-c(x-14)\}. \quad (2.6)$$

As before x represents age attained, and a, b, and c are parameters to be fitted. (The subtraction of 14 from x in (2.6) corresponds to moving the origin up to age 14.)

Tekse "normalizes" his rates to $f_x^* = \hat{f}_x/\hat{R}_0$, sets $a = c^{b+1}/\Gamma(b+1)$, and estimates b and c by the method of moments and by the maximum likelihood method. (He also lists a "*least squares*" example (Tekse, 1967, table 1). We have been aware (Tekse, 1969), however, that the corresponding column in Tekse's table 1 is not directely comparable to the other columns of that table, nor can it be directly compared to the several least squares estimates which shall be presented below.)

Tekse (1967, p. 197) hypothesises that the Hadwiger function[1] may give a better fit to the rates of a high fertility population, whereas the γ-function of (2.6) may be preferred in the case of lower fertility. He suggests that "at least in Europe, there may exist some connection between the level of fertility on the one hand and the relative fertility intensities of the various age groups, i. e. the form of the fertility function on the other hand." Our results seem to cast some further light on these suggestions.

Tekse also considers some further functions, which we shall leave aside here.

[1] Or alternatively a function due to Mazur.

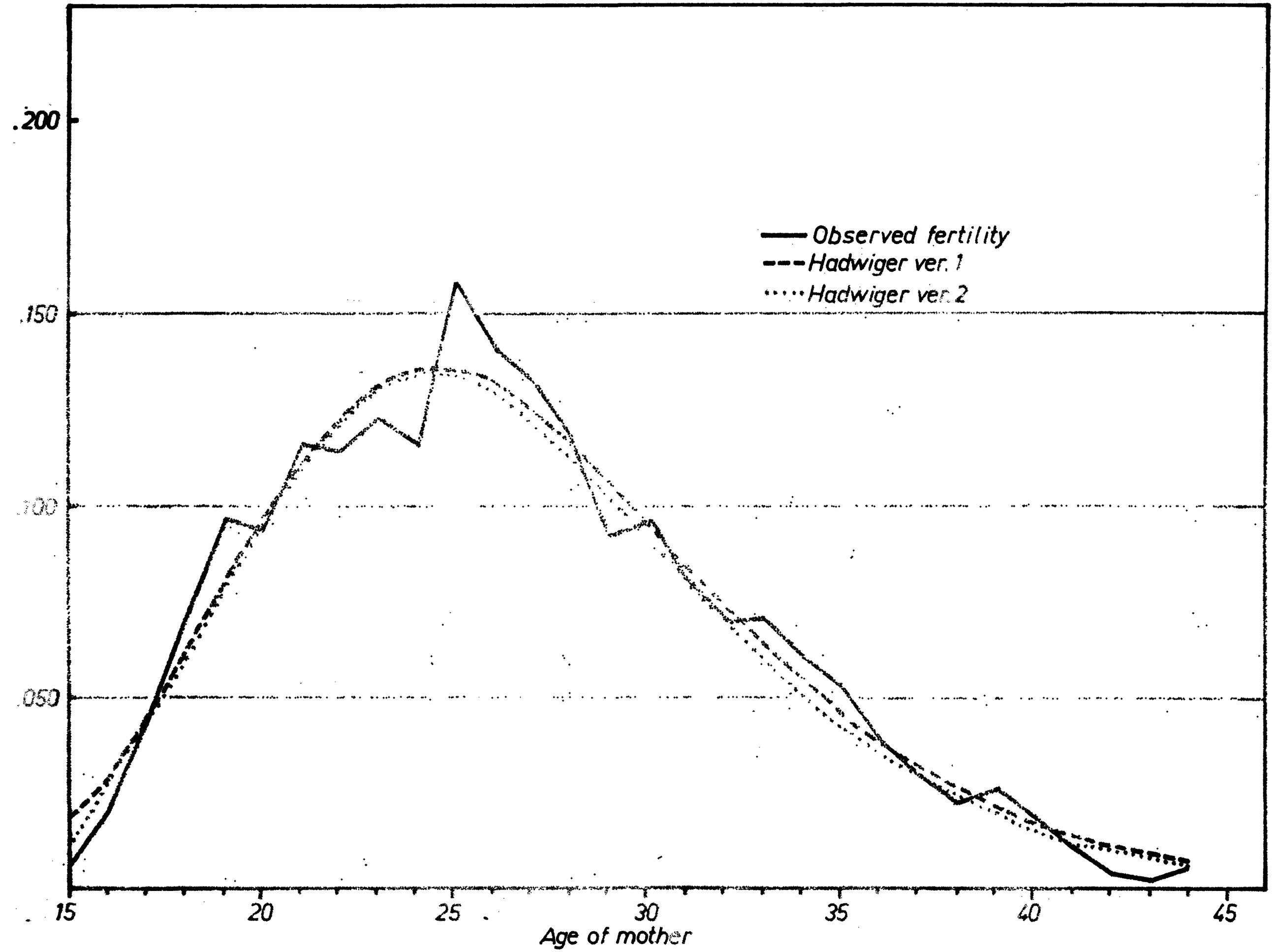

Fig. 2. Observed and smoothed fertility. Oslo 1986

2.4. The estimating methods described in *2.2.* and *2.3.* are by intention convenient for use on a table-calculator as they involve only a fair amount of calculating work. With the electronic computers now available, it is however largely possible to disregard problems caused by laboursome calculations. This gives us a much wider choice of estimating procedures.

As far as we can see, no particular advantage is gained by using the method of moments fully or partially. Instead, we have chosen the method of least squares. Thus we have selected those parameter values which minimize the expression

$$F = \sum_{x=15}^{44} (\hat{f}_x - f_x)^2, \tag{2.7}$$

where $\hat{f}_x$ still represents the observed rate at age x, and f_x represents the value at x of the function to be fitted to the data.

The algorithm used to find the minimum of F is described by Nelder and Mead (1965).

We could alternatively have used the χ^2 minimizing method, but rejected this because of the undesirably great weight laid upon ages with low fertility by this method.

3. Choice of functions to be fitted

3.1. We have used three versions of the gamma function. Two of these build upon the formula

$$\gamma(x) = a(x + d)^b \, exp\{-c(x + d)\} \tag{3.1}$$

and the third version is given by (2.5) (x is still age attained, and a, b, c, and d are parameters as before.)

The three versions are:

Version 1: $\gamma(x)$ with a, b, c, and d estimated by the least squares method (LSM),

Version 2: $\gamma(x)$ with a, b, and c estimated by the LSM, while d was fixed at —14, and

Version 3: $\gamma_Y(x)$ with a, b, c, and d estimated by the LSM.

We observe that version 2 is that of Tekse (1967) given by (2.6). Versions 1 and 3 cannot be transformed into each other by simple parameter transformations. To obtain $\gamma(x) > 0$ for all $x \geqslant 15$, we shall require that $d > -15$ everywhere.

3.2. A simple extension of the function given by (2.1) results from adding a new parameter d to x. This gives

$$h(x) = \frac{ab}{c\sqrt{\pi}}\left(\frac{c}{x+d}\right)^{\frac{3}{2}} \cdot \, exp\left\{-b^2\left(\frac{c}{x+d} + \frac{x+d}{c} - 2\right)\right\}. \tag{3.2}$$

We have used two versions of this generalized Hadwiger function, viz.

Version 1: $h(x)$ with a, b, c, and d estimated by the LSM, and

Version 2: $h(x)$ with b estimated by the LSM, d fixed at 0 and a and c estimated by (2.2).

Again we require that $d > -15$ everywhere.

The idea of introducing an "age correcting" parameter like d has been used independently by Yntema (1969).

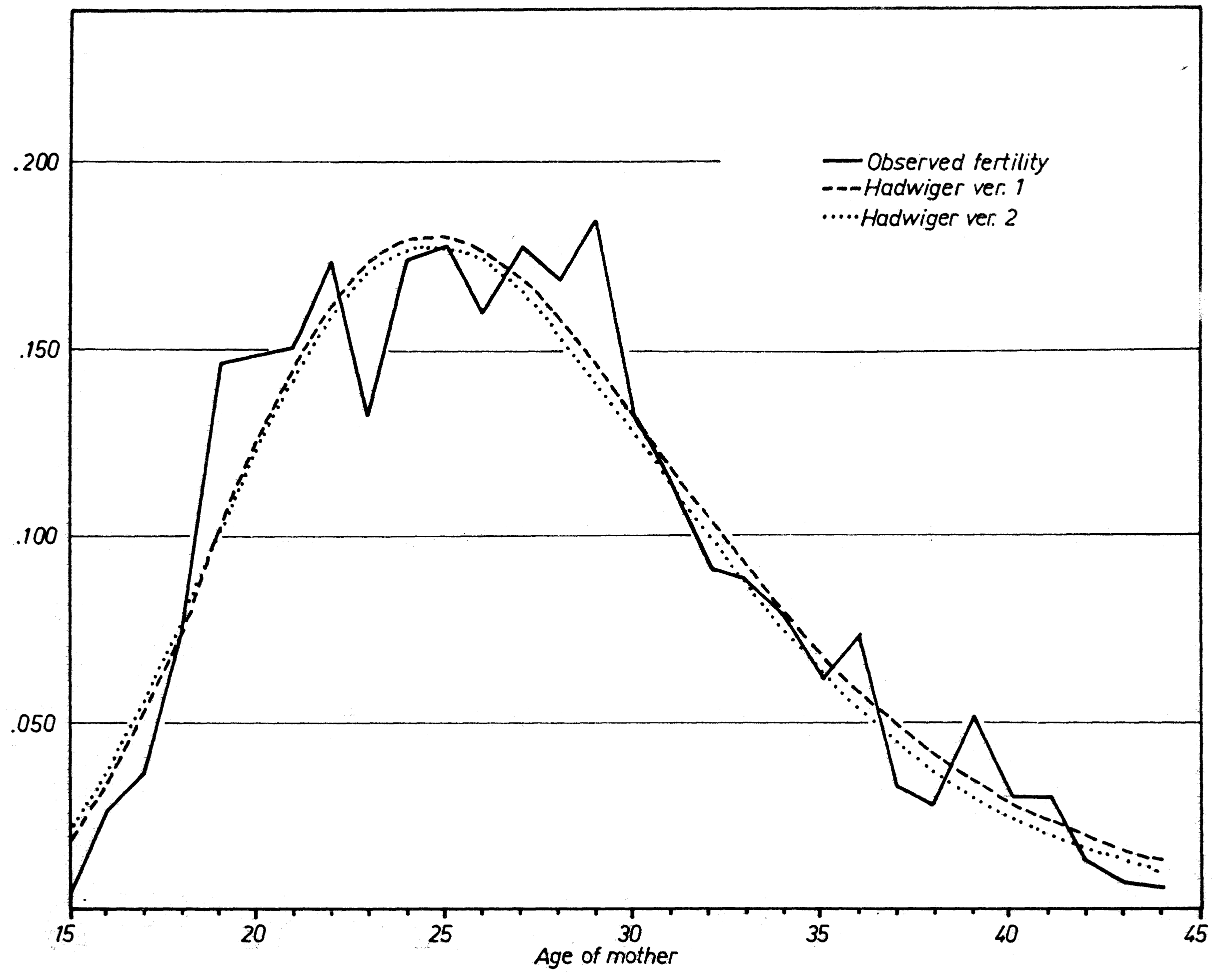

Fig. 3. Observed and smoothed fertility. Stavanger 1966

3.3. The last function we have given examples for, is a fourth degree polynomial of the form:

$$p(x) = a+b\cdot10^{-1}\cdot(x-14)+c\cdot10^{-2}\cdot \cdot(x-14)^2+d\cdot10^{-4}\cdot(x-14)^3+e\cdot10^{-6}\cdot \cdot(x-14)^4. \quad (3.3)$$

In this context $p(x)$ has the disadvantage of giving negative values for certain combinations of x and the parameters a, b, c, d, and e.

Polynomial graduation has previously been recommended by Brass (1960).

3.4. We have also made some trial calculations on the beta function (Pearson I) and the Pearson IV function. Neither of these gave any promising results, so we left them aside.

4. Application to our data

4.1. We have fitted the functions given in 3 above to data from five different sets of fertility rates, viz. 1961 data for Hungary (Tekse, 1967, p. 194) and 1966 data for Norway and for the three Norwegian municipalities of Oslo, Stavanger, and Tromsø. In all cases the rates given are age-specific female fertility rates for offspring of both sexes.

The observed gross fertility rates for the five populations have been listed in column 2 in tables 3 to 7. In columns 3 to 8, we have given the fitted function values. Below these columns, we have listed the parameter values obtained as well as the corresponding least sums of squares of deviations, i. e. the minimum of F in (2.7).

The observed fertility rates and the fitted Hadwiger curves have been drawn in figures 1 to 5. Some of our findings have been summarized in table 1.

In table 2 we have listed the size of the five female populations in question. We see that they vary greatly and that the fit increases strongly with the size of the population. We also see that the fertility varies appreciably both as to fertility level and as to skew of the observed fertility curve.

(We have also made calculations on Swiss data for 1937 given by Hadwiger (1941, p. 33). This gave nothing new, however, and we have not included the results here.)

4.2. The tables show that the two best alternatives are either version 1 of the gamma function or version 1 of the Hadwiger function. For the data from Hungary and Tromsø, the Hadwiger function is best, while the gamma function is best for the data from Stavanger, Oslo and Norway. The difference is small in most cases.

Thus while the second Hadwiger function gives a bad fit to the Hungarian data, as observed by Tekse (1967) already, version 1 actually gives a very good fit.

We cannot find any connection between the fertility level, as measured by $\hat{R}_0$, and the choice between the two functions. (Cf. *2.3.* above.) This holds for a comparison between the two versions 1 of the functions as well as between the two versions 2. Thus version 2 of the gamma function is better than version 2 of the Hadwiger function in all cases except for Oslo. On the other hand the *form* of the fertility curve seems to have some effect. It is notable that the fertility curves for the populations of Hungary and Tromsø are skewer than for the others.

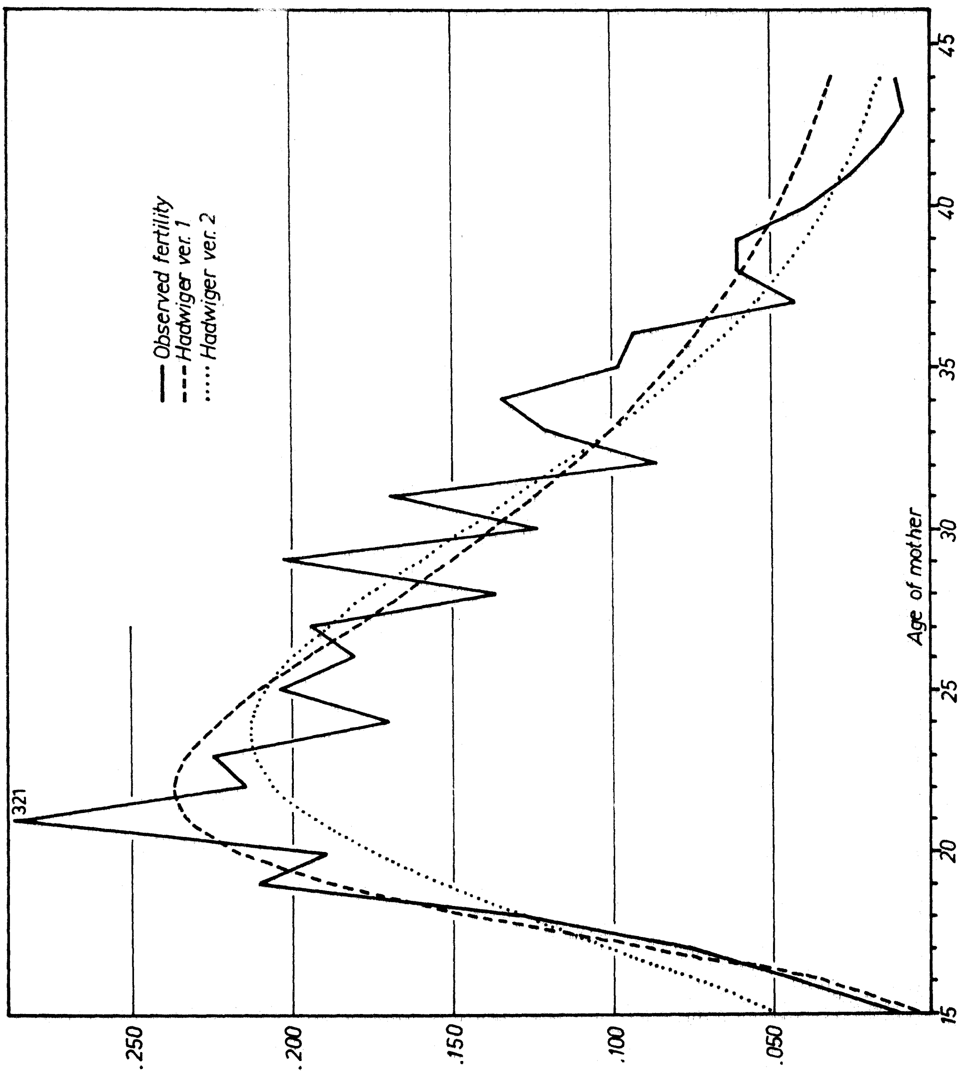

Fig. 4. Observed and smoothed fertility. Tromsø 1966

4.3. The parameter d appears as a measure of the skew of the gamma function. In our data, $\hat{d}$ in version 1 of this function varies in the range between a value slightly exceeding —15 (for Hungary) and a value of —11.13 (for Oslo). It is close to its lower bound of —15 in the populations with a skew fertility curve.

4.4. Similarly d is a measure of the skewness of the Hadwiger function in (3.2). (Cf. table 1 and the figures.)

On comparison of the two versions of the Hadwiger function, the examples make it quite clear that the effect of letting d vary is very strong. We see that $\hat{d}$ in version 1 lies in the interval between —13.048 for Hungary and 0.101 for Oslo. The corresponding differences between the *"F.min."*-values of versions 1 and 2 vary from 3530.4 per cent to 7.8 per cent of *"F.min."* in version 1.

Figures 1 to 5 show the two versions of the Hadwiger function together with the observed values. We see that the more skew the fertility curve is, the better are the relative merits of version 1. On the other hand, the righthand tail of version 1 can be quite "heavy". This is presumably the price we have to pay for the better smoothing at the lower ages.

4.5. In table 1 we have listed the values of $\hat{c}$-$\hat{d}$ and $\hat{T}$ in the two versions of the Hadwiger function. It will be seen that $\hat{c}$-$\hat{d}$ and $\hat{T}$ are of the same size order. We have not been able to explain why this should be so.

From tables 3 to 7 we see that corresponding $\hat{a}$ of the two versions of the Hadwiger function are not much different, while the $\hat{b}$ of version 1 can be much smaller than the corresponding $\hat{b}$ of version 2, particularly for skew fertility curves. In the Norwegian data, formula (2.4) tends to overestimate the corresponding least squares estimate of b by 5 to 10 per cent of the latter. For the Hungarian data, the difference is negligible.

4.6. Comparing our estimates of the parameters of version 2 of the gamma function with Tekse's maximum likelihood estimates (which are his best ones) for the data from Hungary, we find that the corresponding values are quite close to each other (Tekse, 1969):

Our least squares estimates	Tekse's maximum likelihood estimates
$\hat{b} = 2.97$	$p - 1 = 2.82$
$\hat{c} = 0.36$	$\lambda = 0.34$
$\hat{a} = 0.006$	*no third parameter*
F.min. = 0.00116	*F.min.* = 0.00137

Thus very little has been obtained by introducing the new parameter a.

4.7. We see that version 3 of the gamma function is a somewhat unfortunate alternative. We have included it for comparison only.

4.8. The polynomial has the undesirable property of giving negative values for the age 15 and in some cases for the age 44. This can of course be corrected for by replacing all negative values by zero.

The fourth degree polynomial does not give the best fit in any of our examples. Even so, the fit is quite good for some data sets.

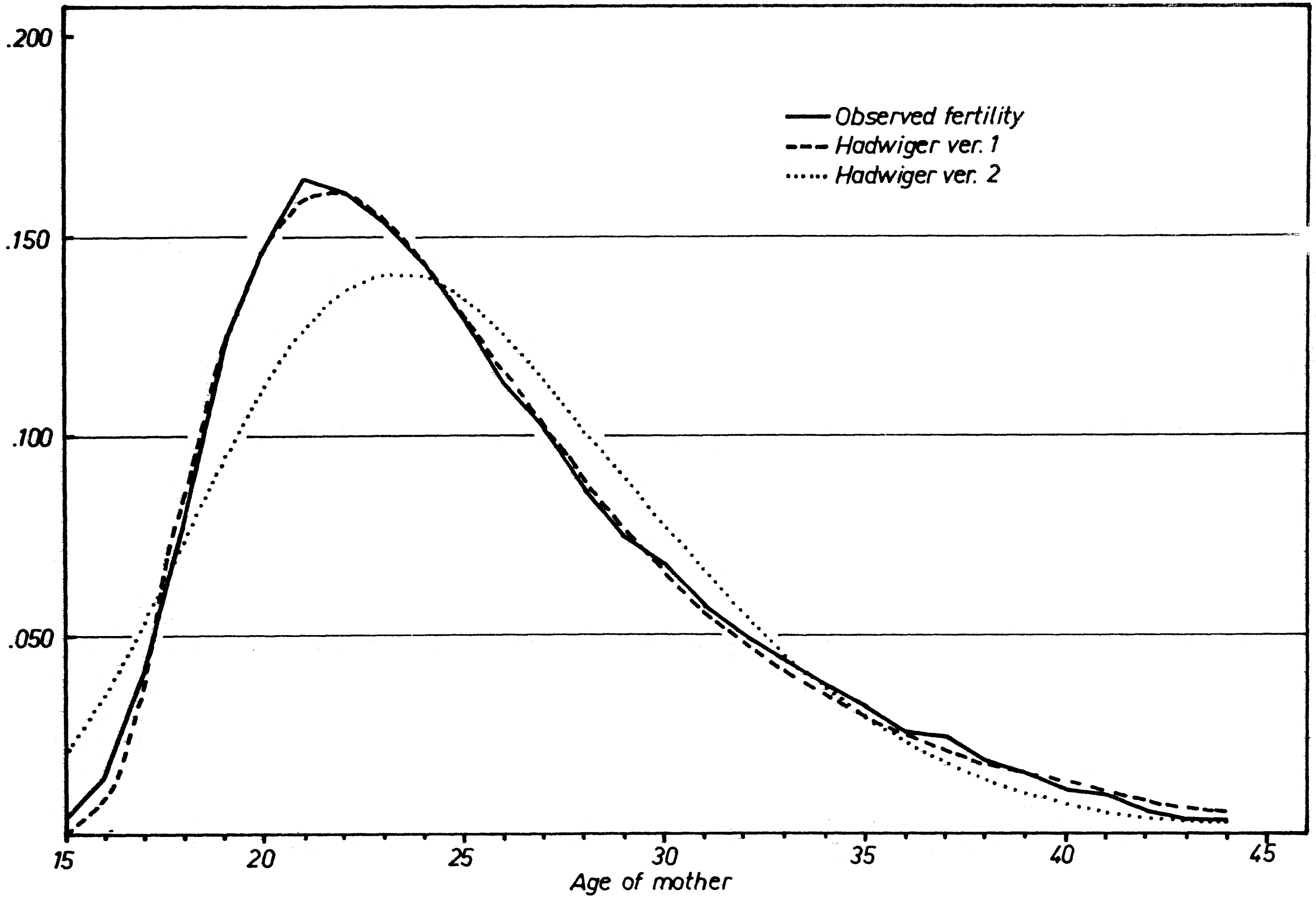

Fig. 5. Observed and smoothed fertility. Hungary 1961

5. Conclusions

On the basis of the investigations reported above, we draw the following conclusions.

(i) The inclusion of the age-correcting parameter *d* in (3.1) and (3.2) may lead to a substantial increase in the fit, as measured by the least sum of squares of deviations.

(ii) On the basis of our results, it is difficult to choose between the best versions of the gamma and the Hadwiger functions. It seems possible that the latter may be the better for skew fertility curves while the former may be preferred for more symmetric curves. In any case the difference in fit seems small.

(iii) If one prefers a simple functional form rather than the very best fit, a polynomial of at least fourth degree is a reasonable choice.

6. References

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Table 1

	Least sum of squares of deviation, mult. by 10^6		Gross fertility rate,	Hadwiger function		
				Version 1		Version 2
	Gamma Version 1	Hadwiger Version 1	$\hat{R}_0$	$\hat{d}$	$\hat{c}-\hat{d}$	$\hat{T}$
Norway	1 183	1 468	2.83	— 9.3	27.5	26.7
Oslo	2 273	2 324	2.00	0.1	26.7	26.5
Stavanger	7 779	7 968	2.73	— 2.8	27.3	26.9
Tromsø	25 856	25 333	3.55	—12.6	28.7	26.5
Hungary	668	168	1.92	—13.0	25.7	25.1

Table 2

Region	Date	Number of women in the age interval 15—44
Norway	31 XII 1965	712 500
Oslo	,,	98 992
Stavanger	,,	15 866
Tromsø	,,	7 021
Hungary	1 VII 1961	2 142 000

Sources: Norwegian data: NOS A 222. [8]
Hungarian data: UN Demographic Yearbook, 1962, pp. 176—7.

Table 3. Norway 1966

Age	Fertility	Gamma functions Version 1	Version 2	Version 3	Hadwiger functions Version 1	Version 2	Polynomial
15	0.00359	0.00276	0.00476	0.04230	0.00875	0.02330	—0.01413
16	0.01619	0.02069	0.02324	0.05634	0.02519	0.03928	0.02979
17	0.04723	0.05086	0.05236	0.07261	0.05128	0.05970	0.06719
18	0.08989	0.08576	0.08597	0.09063	0.08344	0.08323	0.09847
19	0.12861	0.11938	0.11868	0.10965	0.11654	0.10793	0.12402
20	0.14295	0.14790	0.14681	0.12869	0.14605	0.13163	0.14424
21	0.17500	0.16941	0.16837	0.14664	0.16905	0.15234	0.15952
22	0.17556	0.18342	0.18268	0.16233	0.18431	0.16854	0.17027
23	0.18495	0.19033	0.19001	0.17473	0.19191	0.17932	0.17686
24	0.18026	0.19105	0.19114	0.18299	0.19275	0.18441	0.17968
25	0.20124	0.18674	0.18715	0.18658	0.18812	0.18406	0.17913
26	0.18611	0.17857	0.17920	0.18535	0.17938	0.17894	0.17558
27	0.16900	0.16768	0.16841	0.17950	0.16785	0.16995	0.16941
28	0.14690	0.15504	0.15577	0.16954	0.15462	0.15811	0.16100
29	0.13707	0.14145	0.14210	0.15632	0.14059	0.14439	0.15072
30	0.12534	0.12758	0.12810	0.14076	0.12644	0.12969	0.13894
31	0.11300	0.11392	0.11426	0.12384	0.11266	0.11476	0.12603
32	0.10027	0.10081	0.10098	0.10652	0.09960	0.10019	0.11237
33	0.10052	0.08851	0.08850	0.08962	0.08745	0.08640	0.09830
34	0.08073	0.07715	0.07699	0.07379	0.07634	0.07369	0.08421
35	0.06984	0.06683	0.06652	0.05948	0.06631	0.06222	0.07043
36	0.06115	0.05755	0.05713	0.04696	0.05734	0.05205	0.05733
37	0.05085	0.04930	0.04879	0.03634	0.04939	0.04318	0.04527
38	0.04046	0.04203	0.04146	0.02756	0.04240	0.03554	0.03459
39	0.03712	0.03567	0.03507	0.02051	0.03630	0.02904	0.02564
40	0.02851	0.03015	0.02953	0.01497	0.03099	0.02358	0.01877
41	0.02069	0.02539	0.02477	0.01073	0.02639	0.01903	0.01432
42	0.01383	0.02131	0.02070	0.00755	0.02243	0.01527	0.01264
43	0.00725	0.01783	0.01724	0.00522	0.01903	0.01219	0.01405
44	0.00819	0.01487	0.01431	0.00354	0.01612	0.00969	0.01890

Female population in age-interval 15—44: 712 500

Constants:	Gamma functions Version 1	Version 2	Version 3	Hadwiger functions Version 1	Version 2	Polynomial
a	0.009212	0.006	0.187	2.957	2.827	—0.065
b	2.471	2.687	81.901	1.764	2.896 (3.043)[1]	0.545
c	0.266	0.277	2.123	18.239	26.71	—0.367
d	—14.34095	—	—25.243	—9.263	—	0.682
e	—	—	—	—	—	—0.107
F. min.[2]	0.001183	0.001203	0.008489	0.001468	0.003432	0.003341

[1] The figure in parenthesis is $\hat{b}$ calculated by (2.4).
[2] Least sum of squares of deviations.

Table 4. Oslo 1966

Age	Fertility	Gamma functions Version 1	Version 2	Version 3	Hadwiger functions Version 1	Version 2	Polynomial
15	0.00503	0.01169	0.00315	0.02877	0.01664	0.01598	—0.00605
16	0.01912	0.02430	0.01571	0.03863	0.02818	0.02740	0.02409
17	0.04187	0.04112	0.03584	0.05016	0.04301	0.04217	0.04979
18	0.06857	0.06046	0.05937	0.06304	0.06020	0.05936	0.07131
19	0.09560	0.08026	0.08251	0.07674	0.07833	0.07753	0.08891
20	0.09293	0.09859	0.10260	0.09057	0.09582	0.09503	0.10286
21	0.11483	0.11394	0.11818	0.10370	0.11119	0.11032	0.11343
22	0.11374	0.12535	0.12869	0.11531	0.12328	0.12223	0.12087
23	0.12235	0.13243	0.13426	0.12459	0.31141	0.13006	0.12544
24	0.11570	0.13524	0.13542	0.13090	0.13534	0.13360	0.12741
25	0.15724	0.13417	0.13291	0.13383	0.13525	0.13304	0.12704
26	0.14035	0.12985	0.12753	0.13324	0.13161	0.12893	0.12457
27	0.13179	0.12297	0.12007	0.12925	0.12508	0.12197	0.12028
28	0.11666	0.11428	0.11123	0.12224	0.11640	0.11292	0.11440
29	0.09127	0.10443	0.10162	0.11277	0.10632	0.10256	0.10720
30	0.09455	0.09401	0.09172	0.10157	0.09548	0.09156	0.09892
31	0.08049	0.08351	0.08191	0.08934	0.08447	0.08049	0.08982
32	0.06923	0.07329	0.07247	0.07679	0.07370	0.06977	0.08014
33	0.07020	0.06361	0.06357	0.06454	0.06352	0.05971	0.07014
34	0.06056	0.05467	0.05535	0.05305	0.05413	0.05052	0.06006
35	0.05328	0.04656	0.04786	0.04268	0.04566	0.04230	0.05014
36	0.03916	0.03932	0.04113	0.03362	0.03815	0.03508	0.04064
37	0.02879	0.03295	0.03515	0.02594	0.03161	0.02884	0.03179
38	0.02261	0.02741	0.02989	0.01961	0.02598	0.02352	0.02383
39	0.02611	0.02266	0.02529	0.01454	0.02120	0.01903	0.01701
40	0.01760	0.01862	0.02131	0.01057	0.01719	0.01530	0.01157
41	0.01143	0.01521	0.01788	0.00754	0.01385	0.01223	0.00774
42	0.00410	0.01235	0.01495	0.00528	0.01109	0.00971	0.00575
43	0.00220	0.00999	0.01245	0.00363	0.00884	0.00767	0.00585
44	0.00589	0.00804	0.01034	0.00245	0.00701	0.00603	0.00827

Female population in age-interval 15—44: 98 992

Con-stants:	Gamma functions Version 1	Version 2	Version 3	Hadwiger functions Version 1	Version 2	Polynomial
a	0.000076	0.004	0.134	2.069	1.999	—0.041
b	4.766	2.721	81.889	2.921	2.953 (3.226)[1]	0.373
c	0.365	0.278	2.161	26.825	26.54	—0.249
d	—11.13333	—	—25.330	0.101	—	0.451
e	—	—	—	—	—	—0.053
F. min.[1]	0.002273	0.002714	0.003513	0.002324	0.002505	0.002515

[1] See footnotes in table 3.

Table 5. Stavanger 1966

Age	Fertility	Gamma functions			Hadwiger functions		Polynomial
		Version 1	Version 2	Version 3	Version 1	Version 2	
15	0.00328	0.01181	0.00402	0.03689	0.01842	0.02028	—0.00930
16	0.02677	0.02822	0.01993	0.04944	0.03333	0.03473	0.02862
17	0.03697	0.05088	0.04553	0.06414	0.05306	0.05353	0.06149
18	0.07524	0.07713	0.07569	0.08064	0.07628	0.07556	0.08953
19	0.14586	0.10395	0.10572	0.09832	0.10093	0.09908	0.11296
20	0.12265	0.12868	0.13228	0.11636	0.12476	0.12206	0.13201
21	0.15065	0.14937	0.15339	0.13377	0.14573	0.14257	0.14692
22	0.17359	0.16487	0.16825	0.14949	0.16232	0.15905	0.15793
23	0.13158	0.17474	0.17688	0.16252	0.17365	0.17053	0.16531
24	0.17438	0.17914	0.17982	0.17200	0.17950	0.17660	0.16930
25	0.17843	0.17859	0.17792	0.17732	0.18013	0.17742	0.17019
26	0.15886	0.17388	0.17214	0.17817	0.17618	0.17353	0.16825
27	0.17687	0.16588	0.16345	0.17466	0.16851	0.16575	0.16377
28	0.16814	0.15548	0.15274	0.16707	0.15804	0.15501	0.15704
29	0.18537	0.14349	0.14077	0.15606	0.14568	0.14225	0.14837
30	0.13221	0.13062	0.12818	0.14243	0.13226	0.12836	0.13807
31	0.11594	0.11746	0.11550	0.12708	0.11845	0.11408	0.12646
32	0.09068	0.10448	0.10311	0.11090	0.10480	0.10000	0.11386
33	0.08768	0.09202	0.09128	0.09470	0.09172	0.08657	0.10062
34	0.07765	0.08033	0.08020	0.07917	0.07950	0.07411	0.08707
35	0.06550	0.06955	0.06999	0.06484	0.06830	0.06279	0.07357
36	0.07362	0.05978	0.06071	0.05204	0.05821	0.05270	0.06047
37	0.03299	0.05103	0.05237	0.04094	0.04925	0.04385	0.04816
38	0.02796	0.04329	0.04495	0.03159	0.04140	0.03620	0.03699
39	0.05081	0.03651	0.03839	0.02392	0.03460	0.02967	0.02737
40	0.03018	0.03062	0.03265	0.01778	0.02875	0.02416	0.01968
41	0.02994	0.02556	0.02766	0.01297	0.02377	0.01955	0.01432
42	0.01255	0.02124	0.02335	0.00930	0.01956	0.01572	0.01170
43	0.00729	0.01757	0.01964	0.00655	0.01603	0.01258	0.01224
44	0.00469	0.01447	0.01646	0.00454	0.01309	0.01002	0.01636

Female population in age-interval 15—44: 15 866

Con-stants:	Gamma functions			Hadwiger functions		Polynomial
	Version 1	Version 2	Version 3	Version 1	Version 2	
a	0.000570	0.005	0.178	2.843	2.732	—0.052
b	3.844	2.697	81.728	2.533	2.907 (3.076)[1]	0.459
c	0.317	0.268	2.054	24.577	26.94	—0.272
d	—12.23783	—	—25.694	—2.750	—	0.322
e	—	—	—	—	—	0.339
F. min.[1]	0.007779	0.008140	0.010185	0.007968	0.008392	0.007932

[1] See footnotes in table 3.

Table 6. Tromsø 1966

Age	Fertility	Gamma functions Version 1	Gamma functions Version 2	Gamma functions Version 3	Hadwiger functions Version 1	Hadwiger functions Version 2	Polynomial
15	0.01132	0.00000	0.01926	0.07959	0.00495	0.05038	—0.01047
16	0.04264	0.04447	0.05902	0.09753	0.03480	0.07421	0.05731
17	0.07744	0.09977	0.10357	0.11660	0.08817	0.10081	0.11097
18	0.12635	0.14752	0.14467	0.13606	0.14435	0.12808	0.15218
19	0.21122	0.18392	0.17835	0.15510	0.18919	0.15383	0.18250
20	0.18950	0.20883	0.20315	0.17278	0.21849	0.17617	0.20339
21	0.32051	0.22350	0.21911	0.18825	0.23347	0.19375	0.21623
22	0.21515	0.22963	0.22707	0.20068	0.23723	0.20577	0.22229
23	0.22491	0.22898	0.22825	0.20944	0.23299	0.21204	0.22274
24	0.16892	0.22318	0.22398	0.21412	0.22346	0.21283	0.21867
25	0.20385	0.21365	0.21556	0.21450	0.21072	0.20877	0.21105
26	0.17992	0.20157	0.20414	0.21072	0.19626	0.20067	0.20078
27	0.19397	0.18791	0.19074	0.20304	0.18112	0.18946	0.18864
28	0.13596	0.17340	0.17618	0.19202	0.16600	0.17605	0.17534
29	0.20207	0.15865	0.16112	0.17830	0.15135	0.16128	0.16146
30	0.12255	0.14408	0.14609	0.16260	0.13745	0.14588	0.14750
31	0.16854	0.13001	0.13146	0.14572	0.12444	0.13045	0.13388
32	0.08556	0.11665	0.11750	0.12836	0.11241	0.11545	0.12090
33	0.12025	0.10413	0.10440	0.11121	0.10135	0.10122	0.10877
34	0.13366	0.09255	0.09226	0.09478	0.09126	0.08800	0.09760
35	0.09783	0.08192	0.08114	0.07948	0.08209	0.07592	0.08742
36	0.09341	0.07225	0.07105	0.06562	0.07379	0.06504	0.07815
37	0.04167	0.06351	0.06196	0.05335	0.06629	0.05536	0.06960
38	0.06030	0.05567	0.05384	0.04273	0.05953	0.04685	0.06151
39	0.06034	0.04866	0.04662	0.03372	0.05344	0.03943	0.05351
40	0.03766	0.04242	0.04025	0.02623	0.04797	0.03302	0.04514
41	0.02463	0.03690	0.03465	0.02012	0.04306	0.02753	0.03583
42	0.01500	0.03203	0.02975	0.01522	0.03865	0.02285	0.02493
43	0.00901	0.02774	0.02548	0.01136	0.03470	0.01889	0.01169
44	0.01081	0.02399	0.02177	0.00836	0.03115	0.01556	—0.00476

Female population in age-interval 15—44: 7 021

Constants:	Gamma functions Version 1	Gamma functions Version 2	Gamma functions Version 3	Hadwiger functions Version 1	Hadwiger functions Version 2	Polynomial
a	0.054116	0.024	0.215	3.891	3.553	—0.094
b	1.449	1.937	82.516	1.144	2.586 (2.707)[1]	0.922
c	0.196	0.223	1.641	16.092	26.45	—0.886
d	—14.99999	—	—24.593	—12.585	—	3.168
e	—	—	—	—	—	—4.022
F. min.[1]	0.025856	0.027162	0.045543	0.025333	0.036178	0.026822

[1] See footnotes in table 3.

Table 7. Hungary 1961

Age	Fertility	Gamma functions Version 1	Version 2	Version 3	Hadwiger functions Version 1	Version 2	Polynomial
15	0.0041	0.00000	0.00418	0.03313	0.00032	0.02124	—0.02608
16	0.0143	0.01459	0.02275	0.04733	0.00926	0.03583	0.02748
17	0.0398	0.04880	0.05268	0.06433	0.03976	0.05406	0.06914
18	0.0776	0.08672	0.08599	0.08328	0.08258	0.07437	0.10034
19	0.1229	0.11897	0.11589	0.10285	0.12122	0.09466	0.12244
20	0.1466	0.14164	0.13836	0.12129	0.14719	0.11283	0.13671
21	0.1635	0.15415	0.15193	0.13677	0.15955	0.12714	0.14436
22	0.1612	0.15769	0.15692	0.14761	0.16085	0.13650	0.14650
23	0.1540	0.15417	0.15467	0.15265	0.15450	0.14054	0.14415
24	0.1442	0.14560	0.14693	0.15138	0.14349	0.13951	0.13826
25	0.1287	0.13380	0.13548	0.14418	0.13009	0.13411	0.12969
26	0.1132	0.12027	0.12188	0.13198	0.11589	0.12534	0.11923
27	0.1016	0.10616	0.10740	0.11621	0.10190	0.11424	0.10756
28	0.0866	0.09228	0.09298	0.09852	0.08871	0.10184	0.09531
29	0.0753	0.07917	0.07929	0.08050	0.07665	0.08899	0.08301
30	0.0665	0.06717	0.06672	0.06343	0.06583	0.07639	0.07111
31	0.0566	0.05643	0.05550	0.04825	0.05629	0.06452	0.05997
32	0.0489	0.04700	0.04569	0.03545	0.04795	0.05372	0.04987
33	0.0429	0.03885	0.03727	0.02518	0.04073	0.04414	0.04101
34	0.0365	0.03190	0.03016	0.01730	0.03453	0.03584	0.03351
35	0.0317	0.02603	0.02422	0.01151	0.02921	0.02878	0.02742
36	0.0253	0.02112	0.01932	0.00742	0.02468	0.02289	0.02266
37	0.0244	0.01706	0.01532	0.00464	0.02083	0.01804	0.01912
38	0.0184	0.01371	0.01208	0.00281	0.01756	0.01409	0.01658
39	0.0154	0.01097	0.00947	0.00165	0.01480	0.01093	0.01475
40	0.0110	0.00875	0.00739	0.00095	0.01247	0.00842	0.01323
41	0.0099	0.00695	0.00575	0.00052	0.01050	0.00644	0.01157
42	0.0058	0.00551	0.00445	0.00028	0.00884	0.00489	0.00923
43	0.0038	0.00435	0.00343	0.00015	0.00744	0.00370	0.00556
44	0.0026	0.00343	0.00263	0.00008	0.00626	0.00278	—0.00013

Female population in age-interval 15—44: 2 142 000

Constants:	Gamma functions Version 1	Version 2	Version 3	Hadwiger functions Version 1	Version 2	Polynomial
a	0.020013	0.006	0.153	1.963	1.923	—0.093
b	2.198	2.969	82.170	1.373	3.081 (3.068)[1]	0.742
c	0.316	0.364	3.398	12.647	25.19	—0.752
d	—15.00000[2]	—	—23.303	—13.048	—	2.751
e	—	—	—	—	—	—3.454
F. min.[1]	0.000668	0.001161	0.008269	0.000168	0.006099	0.003616

[1] See footnotes in table 3.
[2] This figure is only slightly larger than —15.

MARCUS BOKTR. STHLM 1969

NOTE ON POPULATION SIZE AND AGE-SPECIFIC FERTILITY RATES

By Eivind Gilje[1)]

CONTENTS

1) Written in the Study Group for Population Models, The Central Bureau of Statistics of Norway, February 17, 1969.

1. Introduction

1A. In a recent investigation on the technique of fitting curves to age-specific fertility rates (Gilje, 1969), we encountered the following "problem of small populations" (op. cit., §§ 1A and 1B):

The diagram of a set of observed age-specific fertility rates of a small population tends to show a rather rugged curve, and the irregularities of the curve are apt to increase as the size of the population decreases. Assuming that such ruggedness is due to chance variations only, it is of interest to graduate the observed values, e.g. by fitting a nice mathematical function.

In fact, such a graduation is all the better justified as well as more needed the smaller the population is. (See the diagrams of this paper as well as those of Gilje (1969).)

If the population is too small, however, the chance variations will predominate, and no graduation method will produce much real information on the underlying population fertility. One is then faced with the problem to determine a smallest acceptable population size for the meaningful graduation of a fertility curve.

There does not appear to be any hard and fast answer to this question. Any solution seems to need some room for the good sense of the investigator. Even so, it may be a help to have some rough rules of thumb to guide one's intuition. The purpose of the present paper is to develop a rule or two of this kind.

1B. In the examples given below, the observed fertility rates have been graduated by fitting a Hadwiger function of the form

$$(1.1) \qquad h(x) = \frac{ab}{c\sqrt{\pi}} \left(\frac{c}{x+d}\right)^{\frac{3}{2}} \exp\{-b^2\left(\frac{c}{x+d} + \frac{x+d}{c} - 2\right)\}$$

by the least squares method. Here x represents age attained, while a, b, c, and d are parameters to be fitted.

2. Data and numerical results

2A. Single-year (gross) fertility rates for women specific for each age between 15 and 44 years are available for each municipality in Norway since 1965. (A few of these have been published elsewhere (Gilje, 1969).) We have now selected several additional municipalities, and here present data for five of these with a population of between some 6 000 and some 15 000 persons as of December 31, 1965 (Table 1). The fertility rates have been estimated from data for 1966.

2B. The parameter values given in Table 2 are those which minimize the expression

$$(2.1) \qquad F = \sum_{x=15}^{44} (\hat{f}(x) - h(x))^2$$

where $\hat{f}(x)$ represents the observed fertility rate at age x, and h(x) represents the value at x of the Hadwiger function. The minimum values of F are given in Table 1, column D. In column E we find female total fertility rates for offspring of both sexes.

Observed and fitted fertility rates are given in Table 3 and Figures 1 to 5.

Columns F and G of Table 1 give the average number of females per age and the size of the smallest age group, respectively, of ages 15 to 44 for each of the five municipalities.

3. Tentative conclusions

3A. Any conclusion drawn from an experiment like the present one depends of course on the subjective interpretation of the investigator. We shall end this paper by stating one or two of the things which this investigation has suggested to us.

3B. By studying the fertility curves of the present paper, those of our previous paper on a similar subject (Gilje, 1969), as well as several others at our disposal, we have come to the following tentative conclusion:

The average number of person-years of observation per female single-age group in ages 15 to 44 should probably not be below something like 70 to 75 as a minimum. At the same time, the smallest age groups should not be too small, preferably not below some 50 person-years of observation.

3C. In the Norwegian population as of December 31, 1965, the females in ages 15 to 44 constituted 19.1 per cent of the whole population. (Cf. column C of Table 1.) In a population with an age-sex composition of this kind, the requirements of § 3B correspond to a minimum total population size (counting males and females of all ages) of roughly 12 000 souls.

3D. One can see that the municipality of Nøtterøy falls safely above the minimum limits specified above. Gran represents a border-line case, while Lenvik, Rauma, and Ankenes are too small by our standards.

4. References

Gilje, E. (1969). Fitting Curves to Age-Specific Fertility Rates: Some Examples. Statistical Review, Stockholm, III 7 (2): 118-134.

NOS A 222. Folkemengden etter alder, 31. desember 1965. Central Bureau of Statistics of Norway, 1968.

Table 1. Municipalities selected for investigation

Municipality	Total population[1)] as of Dec. 31, 1965	Number of women in the age interval 15 to 44[1)]	B in per cent of A	Least sum of squares of deviation, mult. by 10^6	Total fertility rate	Average number of females per age in ages 15 to 44	Size of smallest age group in ages 15 to 44
	(A)	(B)	(C)	(D)	(E)	(F)	(G)
Nøtterøy	14 409	2 769	19.22	20 583	2.70	92	54
Gran	11 755	2 131	18.13	32 793	3.18	71	49
Lenvik	10 350	1 904	18.40	52 205	3.85	63	33
Rauma	8 205	1 555	18.95	52 211	2.68	52	25
Ankenes	6 417	1 251	19.50	99 417	3.58	42	29

1) Source: NOS A 222.

Table 2. Least squares estimates of the parameters of the Hadwiger function. Data for 1966

	Nøtterøy	Gran	Lenvik	Rauma	Ankenes
$\hat{a}$	2.818	3.216	3.914	3.033	3.598
$\hat{b}$	1.672	1.780	2.807	1.046	2.198
$\hat{c}$	15.830	15.031	24.554	17.605	18.003
$\hat{d}$	-11.356	-11.164	-2.534	-13.000	-7.896

Table 3. Observed and fitted fertility rates, 1966

Age	Nøtterøy		Gran		Lenvik		Rauma		Ankenes	
	Observed fertility	Hadwiger function	Observed fertility	Hadwiger function	Observed fertility	Hadwiger function	Observed fertility	Hadwiger function	Observed fertility	Hadwiger function
15	-	0.001	-	0.002	-	0.016	-	0.001	-	0.011
16	-	0.009	0.009	0.013	0.029	0.033	-	0.017	0.019	0.032
17	0.037	0.031	0.042	0.042	0.030	0.058	0.030	0.053	0.088	0.067
18	0.041	0.066	0.117	0.088	0.124	0.091	0.114	0.093	0.083	0.112
19	0.081	0.106	0.099	0.142	0.118	0.128	0.123	0.127	0.148	0.162
20	0.214	0.145	0.183	0.191	0.182	0.167	0.118	0.149	0.340	0.208
21	0.172	0.175	0.290	0.228	0.238	0.203	0.232	0.162	0.162	0.244
22	0.189	0.195	0.233	0.249	0.234	0.233	0.140	0.167	0.242	0.268
23	0.185	0.204	0.192	0.256	0.214	0.255	0.146	0.166	0.200	0.278
24	0.207	0.204	0.269	0.252	0.220	0.268	0.231	0.161	0.257	0.276
25	0.149	0.197	0.283	0.238	0.280	0.271	0.114	0.154	0.424	0.264
26	0.240	0.185	0.242	0.219	0.298	0.266	0.122	0.145	0.310	0.246
27	0.141	0.171	0.163	0.197	0.303	0.254	0.172	0.135	0.138	0.223
28	0.148	0.155	0.176	0.174	0.153	0.237	0.115	0.126	0.167	0.198
29	0.109	0.138	0.155	0.151	0.227	0.216	0.040	0.116	0.136	0.173
30	0.146	0.122	0.077	0.130	0.222	0.193	0.114	0.107	0.161	0.149
31	0.155	0.107	0.085	0.111	0.304	0.170	-	0.098	0.156	0.126
32	0.117	0.092	0.113	0.094	0.082	0.148	0.163	0.090	0.109	0.106
33	0.103	0.080	0.180	0.078	0.065	0.126	0.071	0.083	-	0.088
34	0.052	0.068	0.032	0.065	0.067	0.107	0.082	0.075	0.167	0.072
35	0.064	0.058	0.069	0.054	0.067	0.089	0.163	0.069	0.070	0.059
36	0.025	0.050	0.016	0.045	0.083	0.074	0.100	0.063	-	0.048
37	0.035	0.042	0.016	0.037	0.102	0.061	0.095	0.057	0.032	0.039
38	0.011	0.035	0.028	0.030	0.016	0.050	0.034	0.052	-	0.031
39	0.041	0.030	0.041	0.025	0.062	0.040	0.019	0.048	0.050	0.025
40	0.034	0.025	0.011	0.020	0.063	0.032	0.057	0.044	0.089	0.020
41	-	0.021	0.013	0.016	0.040	0.026	0.022	0.040	-	0.016
42	-	0.018	0.023	0.013	0.018	0.020	0.039	0.036	0.026	0.012
43	-	0.015	-	0.011	-	0.016	-	0.033	-	0.010
44	-	0.012	0.026	0.009	0.015	0.013	0.019	0.030	-	0.008

FIGURE 1. OBSERVED AND SMOOTHED FERTILITY.
NØTTERØY, 1966.

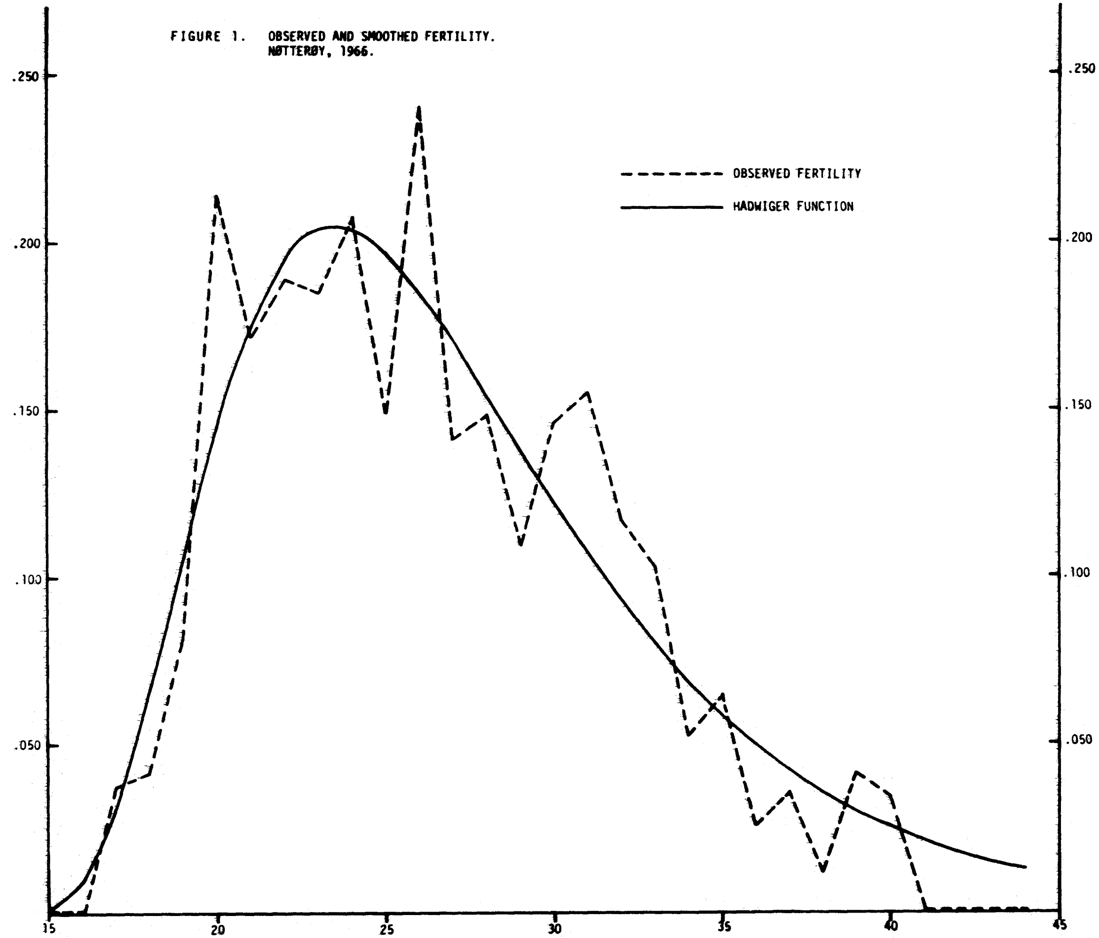


FIGURE 2. OBSERVED AND SMOOTHED FERTILITY.
GRAN, 1966.

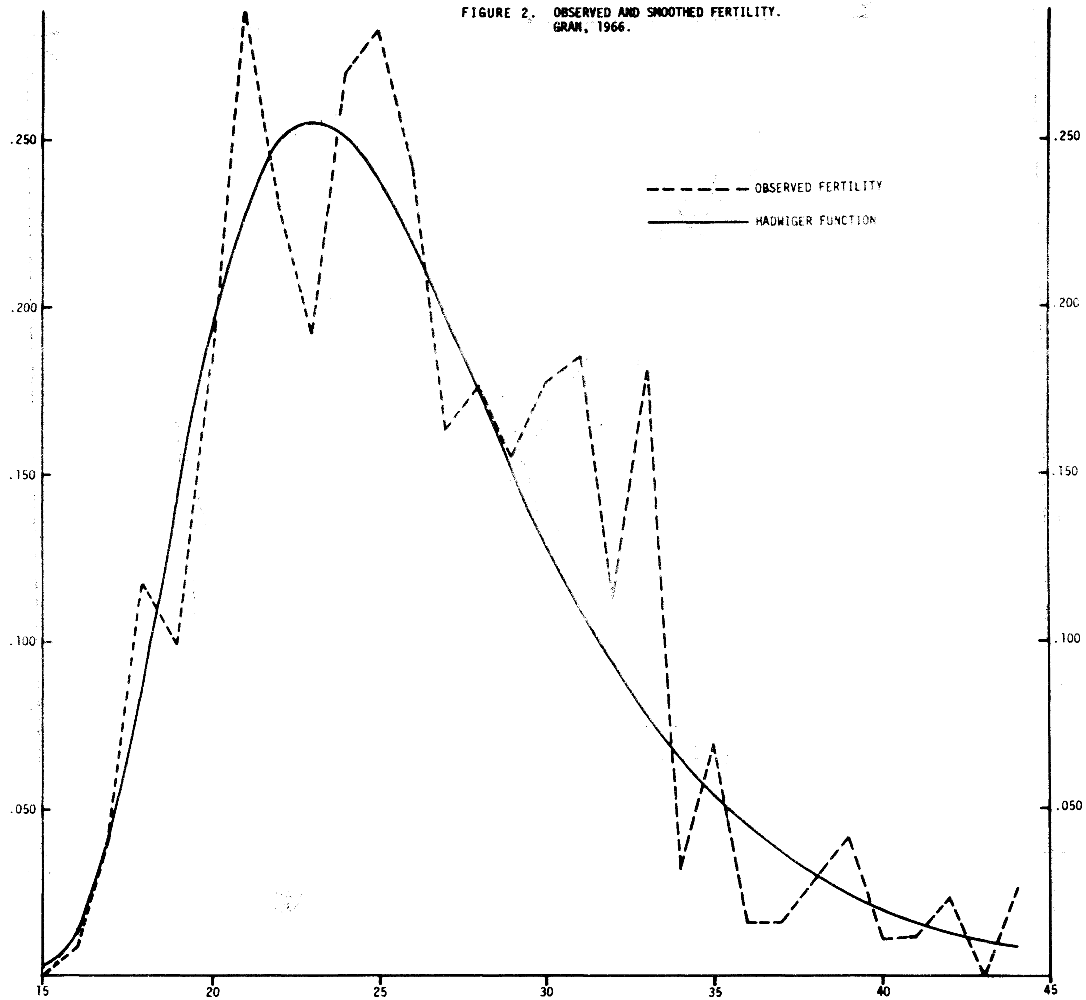


FIGURE 3. OBSERVED AND SMOOTHED FERTILITY.
LENVIK, 1966.

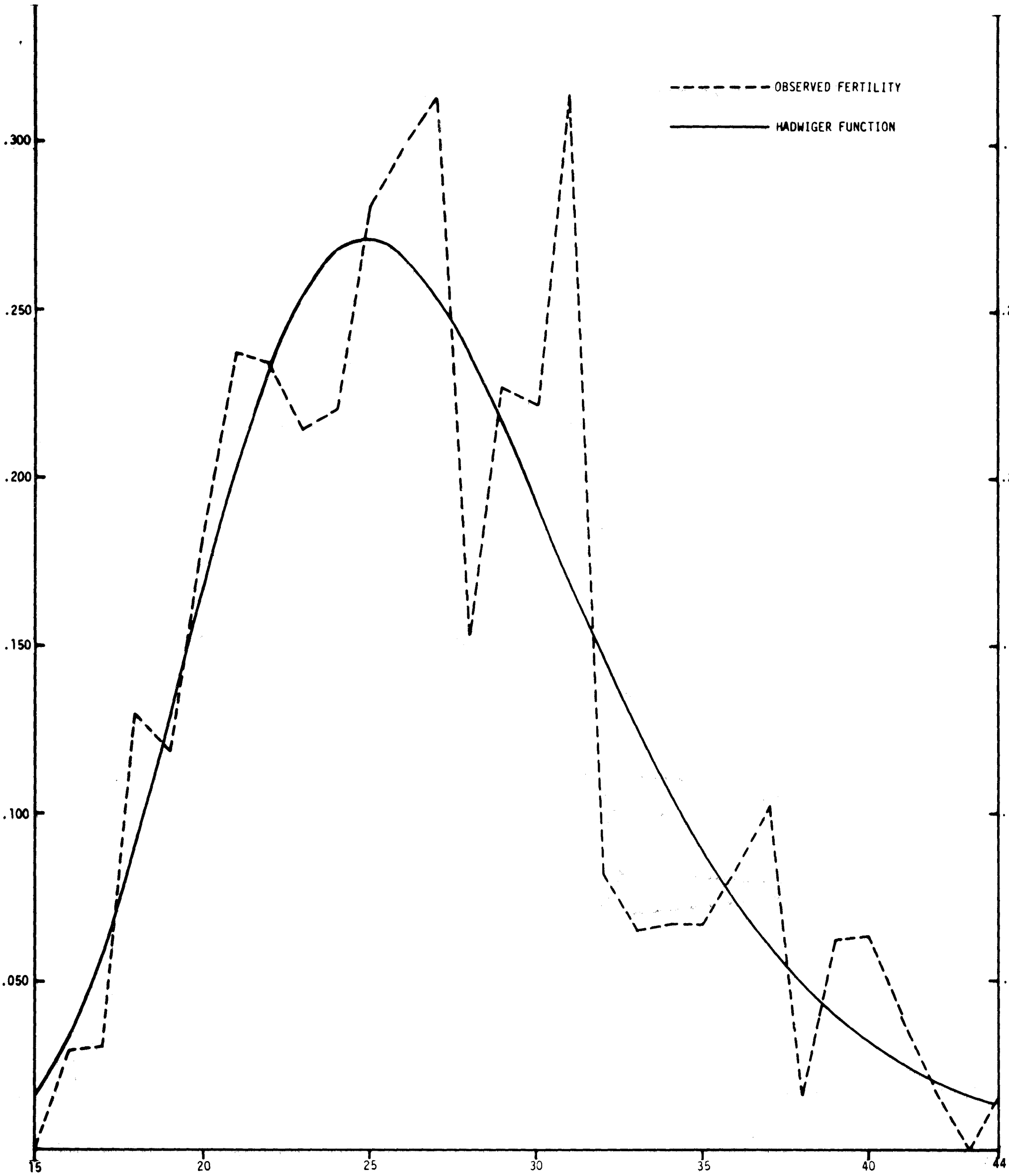


FIGURE 4. OBSERVED AND SMOOTHED FERTILITY.
RAUMA, 1966.

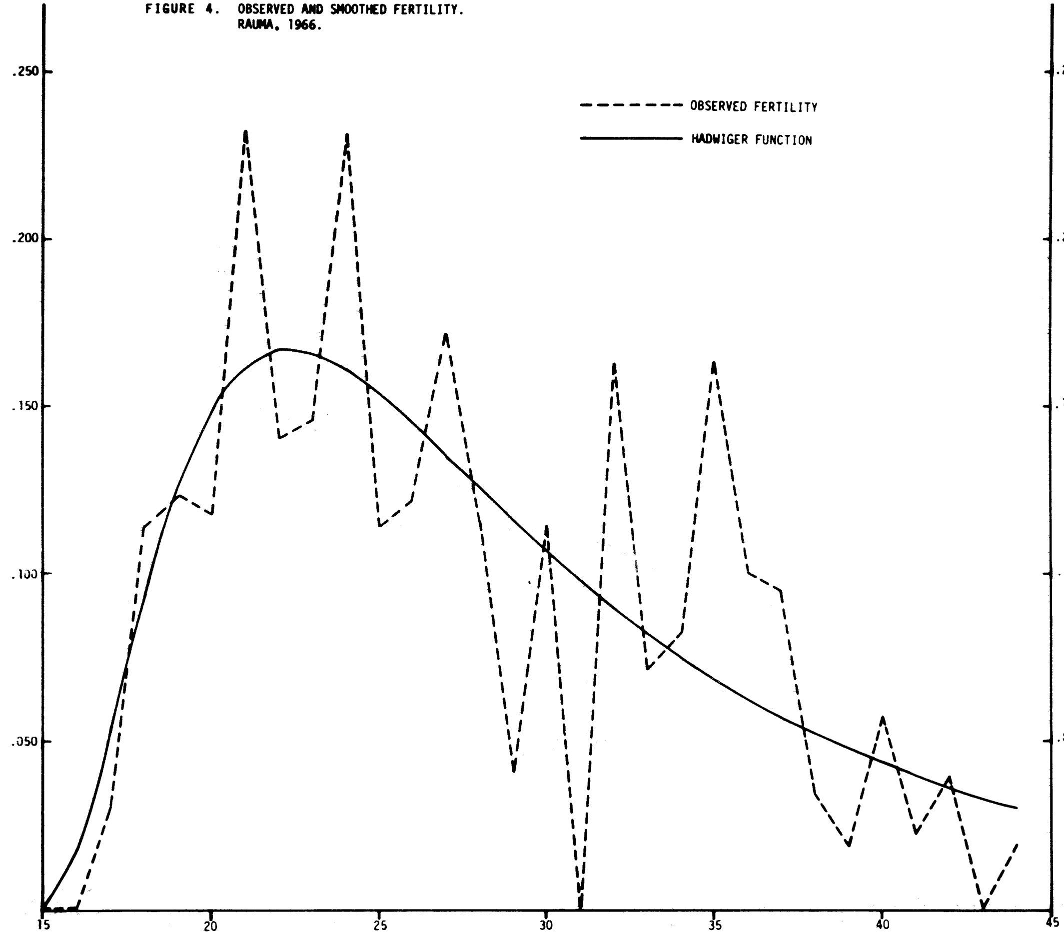
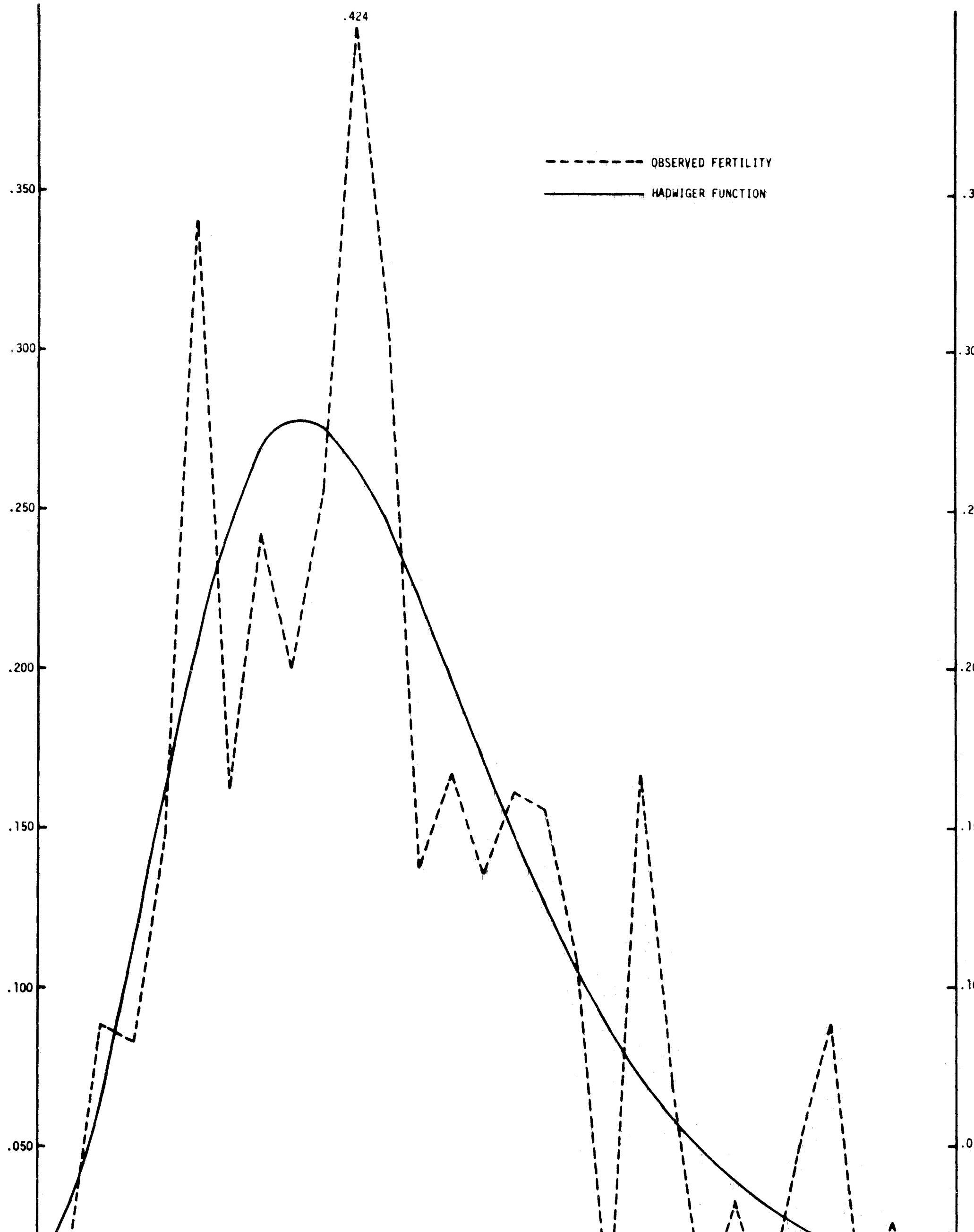


FIGURE 5. OBSERVED AND SMOOTHED FERTILITY. ANKENES, 1966.



THE SHIFTED HADWIGER FERTILITY FUNCTION

By E. Gilje and L. Yntema

CONTENTS

The Shifted Hadwiger Fertility Function

By E. Gilje, Oslo and L. Yntema, Amsterdam

1. Introduction

The age-specific (female) fertility rates $f(y)$ are usually defined by applying the conventional definition of the birth rate to the sub-population consisting of y-aged females. Without further notification only live-born girls are taken into account; in the cases where both live-born girls and boys are considered the rates may be indicated as "total". We distinguish between the above "gross" rates and the "net" rates $p(y)f(y)$, where $p(y)$ refers to the probability that a new-born girl will be alive at age y. Apart from proportionality factors these interdependent rates often show only small differences in a given situation.

In comparing different situations we may, of course, find more important deviations between the observed sets $f_0(y)$. However, in nearly all cases $f_0(y)$ is found to be a unimodal function with slightly varying skewness. Hence it seems justified to look for a mathematical function which for appropriate values of its parameters may provide us with satisfactory graduations of the observed sets in a large number of situations. If such a model can be found, it will serve many purposes. It will be useful for additional specification, completion and correction of scanty population statistics. Extrapolating its (time-dependent) parameters we may use it in population projections; and in a similar way it may be used with actuarial calculations in social insurance (children's allowances, orphan pensions). It may also provide us with general and explicit solutions of the functional equations arising in demometric analysis.

Actuarial applications of the Hadwiger model discussed in this article have been studied by Yntema [6, 7].

2. The Hadwiger fertility model

The choice from the numerous functions which might serve the above purpose, is restricted by demanding something more than general applicability. Thus we may require that the model function is mathematically tractable, containing only a few parameters with immediately clear meanings; to obtain rough estimates of these, preferably only a limited amount of calculations should be involved. Obviously these requirements are not entirely independent. They are, to a large extent, fulfilled by the fertility model

$$f_H(y) = \frac{RH}{T\sqrt{\pi}}\left(\frac{T}{y}\right)^{\frac{3}{2}} \exp\left[-H^2\left(\frac{T}{y}+\frac{y}{T}-2\right)\right] \qquad (y \geqslant 0) \quad (R, T, H > 0) \tag{1}$$

originally suggested by Hadwiger [3]. Intergrating, we find

$$\int_0^\infty f_H(y)\,dy = R \text{ and } \int_0^\infty y f_H(y)\,dy = RT,$$

and thus the parameters R and T represent the gross reproduction rate and the mean age at childbearing, respectively. Using the observed fertility rates, the corresponding moments are

$$\hat{R} = \sum f_0(y) \text{ and } \hat{R}\hat{T} = \sum y f_0(y) \tag{2}$$

which give us estimators for R and T.

The parameter H, however, has no similar "meaning". Aiming at a good fit in the more important central part of the childbearing period we have refrained from making use of seond moments. Instead one can demand that $f_0(\tilde{T}) = f_H(\hat{T})$ where $\tilde{T}$ is the integer value obtained by rounding off $\hat{T}$. Solving this equation by inserting $\hat{R}$ for R and $\hat{T}$ for T in (1), we get as a first estimator for H:

$$\hat{H}_0 = \frac{\hat{T}\sqrt{\pi}}{\hat{R}} f_0(\tilde{T}). \tag{3}$$

Here it should be remarked that a rough graphical graduation of the central values of $f_0(y)$ may be useful before estimating T, if the observed fertility rates show a very rugged curve.

Obviously this method for obtaining rapid estimates of R, T and H, using a desk calculator only, remain useful also if a computer is available. Then, however, better estimates can be obtained by minimizing for instance $\Sigma(f_0 - f_H)^2$. (In the following these two method will be indicated by DSK and LSM respectively.)

3. The shifted Hadwiger function

Until resently the Hadwiger model in this version has given a good fit to the data, even if only DSK-estimates were used. See for instance the estimates for Oslo 1966 in Table 1.

In discussing the fertility rates of Hungary 1956/1962, Tekse [4] demonstrated, however, that the Hadwiger model, in spite of its seemingly general usefulness, sometimes fails to produce an acceptable graduation. This was confirmed by Gilje [2] in a subsequent investigation of local fertility rates in Norway.

This failure is easily understood if we differentiate (1) to find the mode M_H. Then we get the equation $H = [3\, M_H \cdot T/2 \cdot (T^2 - M_H^2)]^{\frac{1}{2}}$. A new estimator for H is obtained from this by inserting $\hat{T}$ for T and $\hat{M}$ for M_H where $\hat{M}$ is the age for which the set $f_0(y)$ reaches its maximum value (see also the commentaries in section 4 about estimating M):

$$\hat{H}_1 = \left[\frac{3\hat{M}\hat{T}}{2(\hat{T}^2 - \hat{M}^2)}\right]^{\frac{1}{2}} \tag{5}$$

Table 1. *Estimated parameters for some different situations*

Parameters	R	T	H	d	M/T	$1-1/a$	Sum of squares of deviation, mult. by 10^6
Rotterdam 1937 (total, gross)							
DSK (non-shifted)	1.81	29.5	3.0	—	0.949	0.900	1 201
LSM	1.87	26.1	2.5	4.04	—	—	1 048
Hungary 1961 (total, gross)							
DSK (non-shifted)	1.92	25.1	3.0	—	0.837	0.916	6 096
DSK (shifted)	1.92	12.9	1.5	12.17	0.837	0.916	824
LSM	1.96	12.6	1.4	13.05	—	—	168
Japan 1963 (total, gross)							
DSK (non-shifted)	1.97	27.2	4.7	—	0.919	0.966	1 846
DSK (shifted using the observed $\hat{M}=25$)	1.97	11.2	1.9	16.00	0.919	0.966	2 590
DSK (shifted using $\hat{M}=26$)	1.97	21.0	3.7	6.18	0.956	0.966	706
LSM	1.95	16.0	2.8	11.23	—	—	218
Oslo 1966 (total, gross)							
DSK (non-shifted)	2.00	26.5	3.2	—	0.943	0.922	3 261
LSM	2.07	26.8	2.9	0.10	—	—	2 324
Norway 1966 (total, gross)							
DSK (non-shifted)	2.83	26.7	2.8	—	0.936	0.906	4 020
LSM	2.96	18.2	1.8	9.26	—	—	1 468

Hence at least

$$\hat{M}/\hat{T} < 1 \tag{6}$$

is required, but apart from that a good graduation is only to be expected if the estimators (3) and (5) give about equal results. Therefore the values $\hat{R}$, $\hat{T}$, $f_0(\tilde{T})$ and $\hat{M}$ must approximately satisfy the relation which is found by setting $\hat{H}_0 = \hat{H}_1$. Writing

$$\hat{a} = \frac{4}{3}\pi \left[\frac{\hat{T}f(\hat{T}_0)}{\hat{R}}\right]^2 \tag{7}$$

we thus obtain instead of (6) the sharper condition

$$\frac{\hat{M}}{\hat{T}} = \frac{\sqrt{1+\hat{a}^2}-1}{\hat{a}} \approx 1 - \frac{1}{\hat{a}} \tag{8}$$

The more this relation is violated, the less satisfactory graduation of the observed fertility rates can be expected using the original Hadwiger model. This is illustrated in Table 1 in the cases of Hungary and Japan.

In order to make the model more flexible, the present writers introduced, independently, a fourth parameter d by shifting the origin of the age-axis [2, 8]. Accordingly the new model is defined by

$$f_{H'.d}(y)=\frac{R'H'}{T'\sqrt{\pi}}\left(\frac{T'}{y-d}\right)^{3/2}\exp\left[-H'^2\left(\frac{T'}{y-d}+\frac{y-d}{T'}-2\right)\right] \tag{9}$$

$(y \geq d)\ (R', T', H' > 0)$

and while using a computer program the LSM-estimation of the four parameters needs no further explanation. In order to obtain DSK-estimators we write $y'=y-d$ and graduate $f_1(y')=f_0(y)$ using (1) with parameters R', T', H'. From (2) and (3) we obtain

$$\left.\begin{aligned}\hat{R}' &= \sum f_1(y') = \sum f_0(y) = \hat{R}\\ \hat{R}'\hat{T}' &= \sum y' f_1(y') = \sum (y-d) f_0(y) = \hat{R}(\hat{T}-d)\\ \hat{H}_0' &= \tilde{T}'\sqrt{\pi} f_1(\tilde{T}')/\hat{R}' = (\tilde{T}-d)\sqrt{\pi} f_0(\tilde{T})/\hat{R}\end{aligned}\right\} \tag{10}$$

where $\hat{R}'$, $\hat{T}'$ and $\hat{H}_0'$ are estimators for R', T' and H_0', respectively. $\tilde{T}'$ is the integer value obtained by rounding off $\hat{T}'$.

In (10) the fourth parameter, d, is assumed known. To find an estimator $\hat{d}$ for this we use (8) which in the new situation takes the form

$$\frac{\hat{M}'}{\hat{T}'}=1-\frac{1}{\hat{a}'} \tag{11}$$

with $\hat{M}'=\hat{M}-\hat{d}$, $\hat{T}'=\hat{T}-\hat{d}$ and $a'=(1-\hat{d}/\hat{T})^2\cdot a$ according to (7). Solving this with regard to d we get the DSK-estimator

$$\hat{d}=\frac{1-1/\hat{a}-\hat{M}/\hat{T}}{1-\hat{M}/\hat{T}}\hat{T}. \tag{12}$$

(Hence, if (8) exactly holds, $\hat{d}=0$ as it should be.)

4. Some examples

In Table 1 we have given estimates for the parameters in these fertility models for some different situations. The estimates indicated by DSK are calculated according to (2) and (3) in the non-shifted cases, and according to (10) and (12) in the shifted cases. The LSM-estimates have been found by minimizing $\Sigma(f_1(y')-f_{H'd}(y))^2$ with regard to the four unknown parameters in (9). M/T and $1-1/a$ refer to (8). We have used the sum of squares of deviation between the observed values and the estimated values as a measure of the goodness of fit.

Observe that the non-shifted DSK-estimates of H are all very close to 3 except in the case of Japan. We have found this same property in several other cases not included in this paper.

The estimates by (12) of the shift-parameter d seem to be good in the case of Hungary but using this estimator in the four remaining cases we get somewhat confusing results. The data for Rotterdam 1937 for instance lead to $\hat{d}=-28.3$ which gives a wholly unsuitable function.

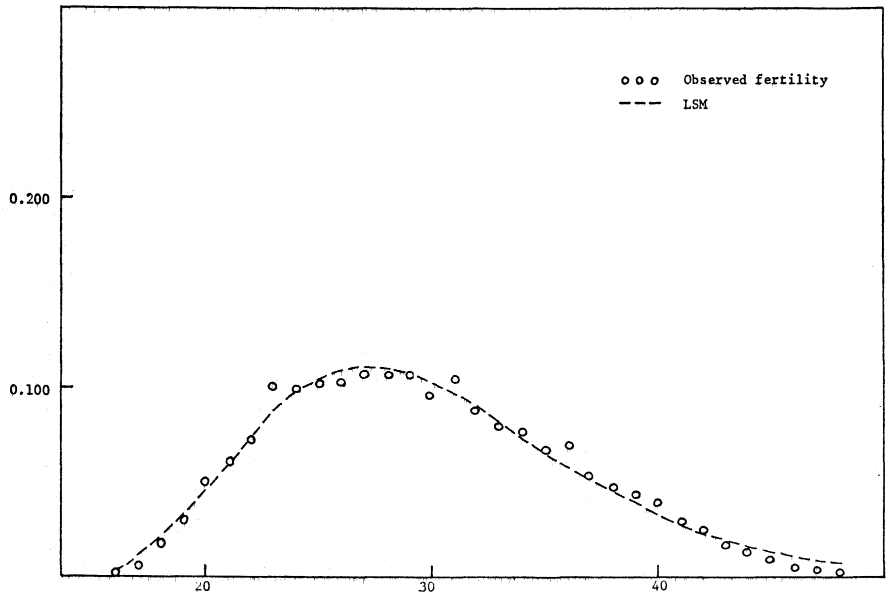


Fig. 1. Observed and smoothed fertility. Rotterdam, 1937.

An examination of the observed fertility rates gives us a clue to where the trouble-maker may be found (Figs. 1 and 2.) The observed mode $\hat{M}$ is not where one should expect it to be, or rather where one wants it to be. As (12) is very sensitive to variations in $\hat{M}$, a maximum observed fertility rate for an "unexpected" age leads to a $\hat{d}$ making no sense. In the Rotterdam-case for

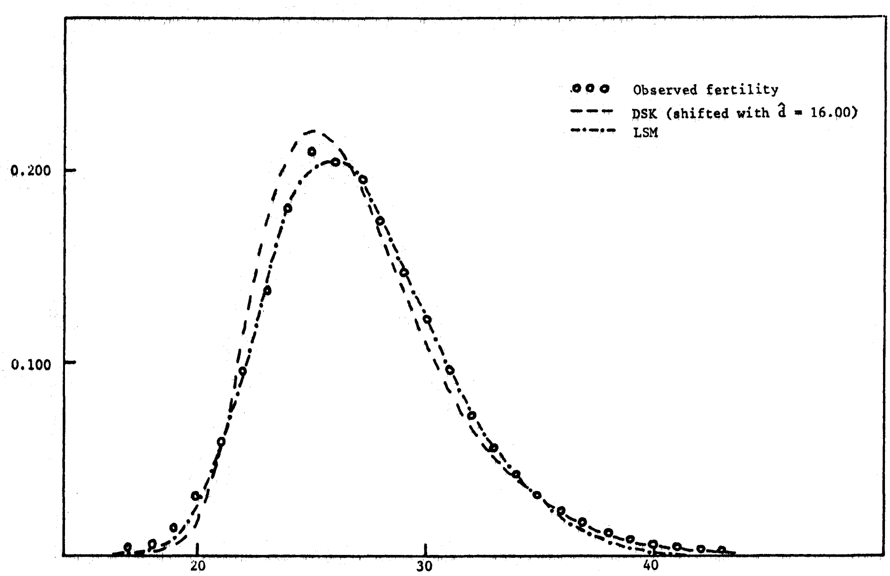


Fig. 2. Observed and smoothed fertility. Japan, 1963.

instance we have found $\hat{M}=28$. If we instead had used $\hat{M}=26$ this would have lead to $\hat{d}=4.7$, which is very close to the LSM-estimate of d. Similar adjustments of $\hat{M}$ can be made for the two Norwegian cases, leading to reasonable shifts in (9).

We have examplified this problem in full in the case of Japan. Here the observed $\hat{M}=25$ leads to $\hat{d}=16.00$ which is too large compared to the shift obtained after using the LSM (Table 1). It is in fact so large that it had been better not using a shift at all according to the respective sums of squares of deviation. In Fig. 2 we have drawn *this* DSK-curve together with the LSM-curve and the observed rates. It is quite obvious that the observed rate for the age 25 is irregular. Just looking at these rates one should expect the mode to come some-

Table 2. *Observed and estimated fertility rates for Japan 1963 (total, gross) multiplied by 10^3*

Age	Observed rates[a]	Shifted DSK-graduation with $\hat{d}=16.00$	Shifted DSK-graduation with $\hat{d}=6.18$	LSM-graduation
17	2	0	1	0
18	5	0	4	1
19	13	1	13	8
20	28	12	32	25
21	58	49	61	57
22	96	108	98	100
23	140	167	138	145
24	183	207	172	182
25	212	223	196	203
26	207	217	204	208
27	196	197	199	197
28	175	170	182	177
29	149	141	158	151
30	125	114	131	124
31	97	90	104	98
32	74	69	80	76
33	56	53	60	57
34	43	40	43	42
35	31	30	31	30
36	23	22	21	21
37	17	16	14	15
38	12	12	10	10
39	9	9	6	7
40	6	6	4	5
41	5	5	3	3
42	3	3	2	2
43	2	2	1	1
44	1	2	1	1
45	1	1	0	1
46	0	1	0	0
47	0	1	0	0
48	0	0	0	0
49	0	0	0	0

[a] Source: Yamaguchi [5].

what later. Thus, setting $\hat{M}=26$, we see from both Tables 1 and 2 that the fit is substantially better.

As well as finding the LSM-estimates minimizing $\Sigma(f_1(y')-f_{H'.d}(y))^2$ we also minimized the sum of absolute differences, $\Sigma|f_1(y')-f_{H'.d}(y)|$, with regard to the unknown parameters. This technique gave approximately the same estimates as the LSM, and we have therefore not included these in the examples.

5. Restrictions

After these apparently satisfactory results we were tempted to test the validity of the shifted model in less recent situations. Therefore we studied the fertility rates of Rotterdam 1870, 1890 and 1909 given by Angenot [1]. In this way certain restrictions were found which, however, seem of little importance in contemporary situations.

In view of their irregular behaviour we may feel suspicious about the reliability of the observed rates for 1870, as shown by Fig. 3, but in spite of this it is clear that the observed $\hat{M}$ is somewhere near age 35. The mean fertile age $\hat{T}$, however, is found to be 32.5, which means that (11) cannot be fulfilled by any shift $\hat{d}$. This is reflected by a "meaningless" LSM-estimate for d, viz. -526 and therefore also for the other parameters. The goodness of fit, though, was remarkably good with a sum of squares of deviation amounting to $14\,002\times10^{-6}$ (see Table 3 for comparison). Who would think of trying estimates like this using the DSK-method, however? Thus, in cases (nowadays very unusual) where $\hat{M}>\hat{T}$, we may, perhaps, obtain "normal" estimates in the Hadwiger-model if the direction of the age-axis is *reversed*.

We have therefore replaced the age-parameter y in (9) by $t=67-y$. The reversed origin of the axis, 67, was chosen out of convenience. Another choice will lead to a different shift, $\hat{d}$, and therefore also different $\hat{T}'$ and $\hat{M}'$, but the resulting estimated fertility rates are of course invariant to where this origin is placed.

Since a good fit was, in view of the observed scatter, not to be expected, the results are not entirely unsatisfactory, as can be seen from Table 3 and Fig. 3.

Table 3. *Estimated parameters for Rotterdam (total, gross)*

	1870[a]		1890		1909	
Parameters	DSK	LSM	DSK	LSM	DSK	LSM
d	—	8.94	—	−81.43	—	−0.70
R	4.533	4.740	4.698	4.880	3.869	4.074
T	34.50	27.09	31.88	113.44	31.09	32.60
H	3.332	2.383	2.913	10.945	2.850	2.900
Sum of squares of deviation, mult. by 10^6	—	21 959	—	12 354	—	8 966

[a] In (9) the age-parameter is replaced by $t=67-y$ in this example.

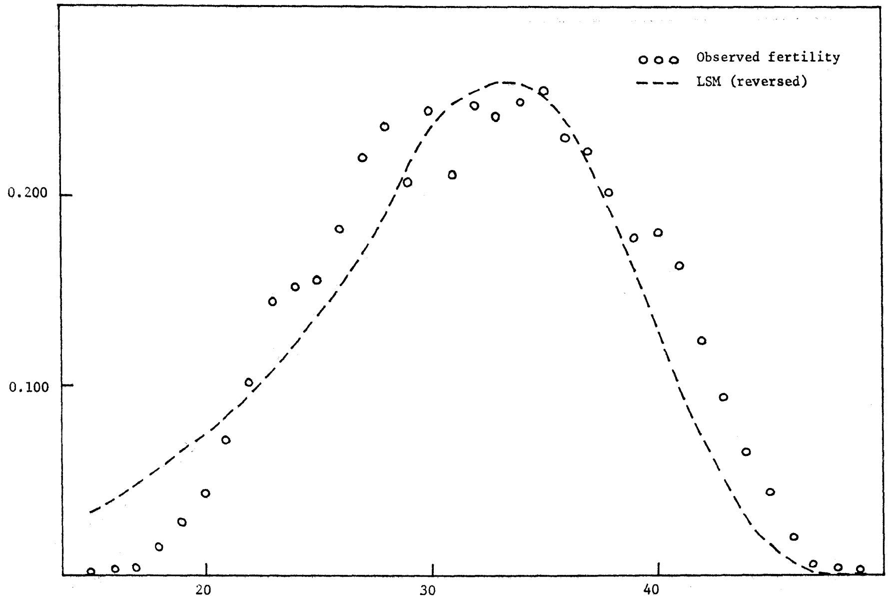


Fig. 3. Observed and smoothed fertility. Rotterdam, 1870.

From Figs. 3, 4, and 5 we see that the same problem as mentioned in section 4 arise when we want to determine the mode. If we try to imagine that the observed rates have already been smoothed, though, it seems that the above relation between mode and mean gradually changes into the opposite direction.

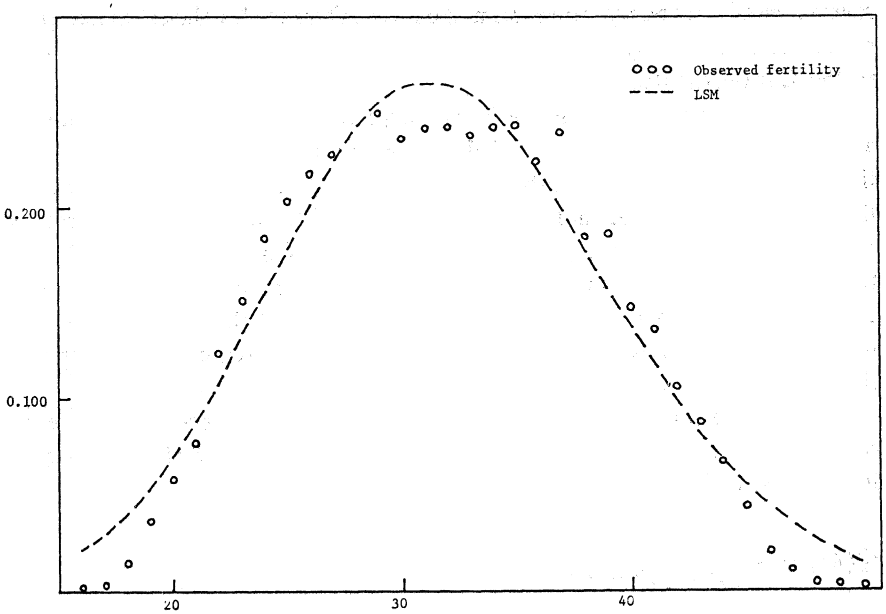


Fig. 4. Observed and smoothed fertility. Rotterdam, 1890.

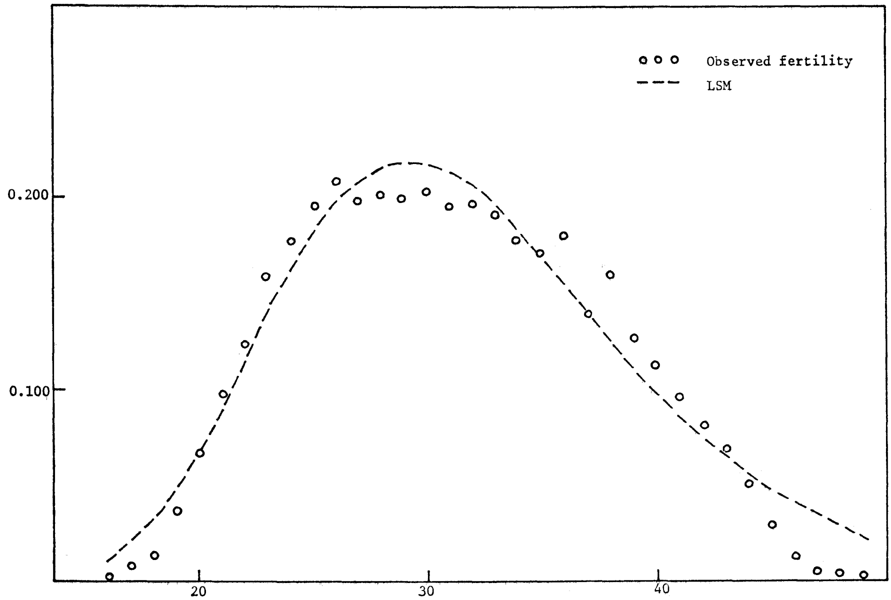


Fig. 5. Observed and smoothed fertility. Rotterdam, 1909.

In 1890 the mode is approximately equal to the mean which should lead to a large shift according to (12). This was also obtained by the LSM. In this situation it is difficult to decide wether the age-axis should be reversed or not. A reversal would probably, however, make little difference for the estimated parameters. In Table 3 only the non-shifted DSK-estimates are given as a comparison to the LSM-estimates.

In the last case, Rotterdam 1909, the inequality, $\hat{M} < \hat{T}$, seems to be fulfilled again. Still, the sum of squares of deviation is remarkably higher in this case than in any of the cases in Table 1. Some of this can be explained by the fact that these sums must be more or less proportional to the reproduction rate R. If we divide the sums of squares of deviation into $\hat{R}$ in the cases of Oslo (the least satisfactory result in Table 1) and of Rotterdam 1909, however, we get $1\,123 \times 10^{-6}$ and $2\,201 \times 10^{-6}$ respectively. The scatter of the observed rates does not seem to justify a difference like this, and we have to search for the explanation elsewhere.

We have the impression that good results could not be expected because the observed rates run rather flatly in the central part of the childbearing age interval. By differentiating (1) twice, we obtain

$$f''_H(M) = -\frac{3}{2}\frac{T^2+M^2}{T^2-M^2}\frac{1}{M^2}f_H(M) \tag{13}$$

which has a large absolute value for $T \simeq M$. Consequently small differences between mode and mean of the observed rates are inconsistent with a very flat curve if the Hadwiger model is to be applied. It is therefore impossible to obtain a good fit using this model if the observed rates are both symmetric and flat around the mean.

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