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PURGED AND PARTIAL MARKOV CHAINS

By Jan M. Hoem

LUTREDE OG PARTIELLE MARKOVKJEDER

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PREFACE

To create a Norwegian milieu for population studies, the Central Bureau of Statistics of Norway has established a Socio-Demographic Research Unit with the objective of contributing to a better understanding of population behaviour and development through the use of numerical models and other activities. Some of the work done in the Unit is based on the tools of mathematics. The present article, written by the Chief of the Unit, contains elements of this theory. It has been included in the Bureau's series Artikler to make it more easily available to Norwegian readers.

The Central Bureau of Statistics of Norway is grateful to Skandinavisk Aktuarietidskrift for permission to reprint the paper. The original pagination has been retained.

The Bureau simultaneously reprints two other papers of the same character as No. 46 in this series.

Central Bureau of Statistics, Oslo, 16 June 1972

Petter Jakob Bjerve

FORORD

For å skape et norsk miljø for befolkningsstudier har Statistisk Sentralbyrå opprettet en sosiodemografisk forskningsgruppe, hvis målsetting er å bidra til forståelse av befolkningens atferd og utvikling, bl.a. gjennom bruk av numeriske modeller. En del av arbeidet i gruppen er basert på et matematisk modellverktøy. Denne artikkelen, som er skrevet av gruppens leder, inneholder elementer av denne teorien. Den er tatt inn i Byråets serie Artikler for å gjøre den lettere tilgjengelig for norske lesere.

Statistisk Sentralbyrå takker Skandinavisk Aktuarietidskrift for tillatelse til gjenopptrykk av artikkelen. En har beholdt den opprinnelige pagineringen.

Byrået utgir samtidig to andre artikler av samme karakter som nr. 46 i Artikkel-serien.

Statistisk Sentralbyrå, Oslo, 16. juni 1972

Petter Jakob Bjerve

Purged and Partial Markov Chains

By Jan M. Hoem, Oslo

1. Introduction and summary

In multiple decrement theory, disability theory, nuptiality and fertility theory, and in other connections, actuaries and demographers often study what in effect is simple Markov chain models. In these models, the transition probabilities are defined on the basis of the forces of transition (infinitesimal transition probabilities). For various reasons, new models are sometimes constructed by replacing some of the original forces of transition by 0. We shall call the transition probabilities of such a new model *partial probabilities*.

The main purpose of this paper is to generalize and clarify the simple concept of a partial probability. We show how it is defined within the framework of the theory of general time-continuous Markov chains, and consider its relation to the transition probabilities of a second derived model, the *purged* chain, which we also define.

2. Motivating examples

Example 1: Multiple decrement theory

In multiple decrement theory,[1] members of a given stock of individuals are observed to leave the stock due to m causes of decrement. (One common interpretation is in terms of deaths from m different causes.) To each such cause there corresponds a continuous, age-dependent force of decrement, say $\mu_x^{(k)}$ at age x for cause k; $k=1, 2, ..., m$. An individual will not return to the stock once he has left it.

Let ${}_tq_x^{(k)}$ be the probability that an x-year-old member will leave the stock within age $x+t$ due to cause k. Then

$${}_tq_x^{(k)} = \int_0^t \mu_{x+\tau}^{(k)} \exp\left\{-\sum_{\nu=1}^{m}\int_0^\tau \mu_{x+\theta}^{(\nu)}\,d\theta\right\} d\tau.$$

Obviously the value of ${}_tq_x^{(k)}$ will be influenced by the values of the $\mu^{(\nu)}$ for $\nu \neq k$. This is often regarded as a nuisance which one would like to eliminate.[2] A quantity

$${}_tq_{x,k} = \int_0^t \mu_{x+\tau}^{(k)} \exp\left\{-\int_0^\tau \mu_{x+\theta}^{(k)}\,d\theta\right\} d\tau = 1 - \exp\left\{-\int_0^t \mu_{x+\tau}^{(k)}\,d\tau\right\}$$

[1] See e.g. Zwinggi (1945), Simonsen (1966), Sverdrup (1967).
[2] Fix and Neyman (1951, p. 216), Henry (1959, 1963), Hoem (1968 *a*, § 3.2; 1968 *b*, § 3 D).

is then introduced as a measure of the strength of cause k as a cause of decrement. ${}_tq_{x,k}$ would be the probability that an x year old member of the stock will leave it due to cause k within age $x+t$ if only cause k of decrement were at work while the other causes were inactive. The corresponding probability if only causes $k_1, \ldots, k_r$ (with $1 < r < m$) were effective, would be

$$\int_0^t \mu_{x+\tau}^{(k)} \exp\left\{-\sum_{\nu=1}^{r} \int_0^\tau \mu_{x+\theta}^{(k_\nu)} d\theta\right\} d\tau \tag{2.1}$$

provided one of the k_ν equals k.

Following Sverdrup (1967), we shall call ${}_tq_x^{(k)}$ an *influenced* probability. We shall call ${}_tq_{x,k}$ and the quantity in (2.1) *partial* probabilities (DuPasquier, 1913, §4).

Other authors have used terms like "dependent" (Zwinggi, 1945; Jordan, 1952), or "crude" (Chiang, 1968) for the influenced probabilities. ${}_tq_{x,k}$ has been called an "independent" (Zwinggi, 1945) an "absolute" (Jordan, 1952), or a "net" probability (Chiang, 1968). Unfortunately the latter term conflicts with other demographic terminology, as the partial probabilities would correspond to a *gross* decrement table while the influenced probabilities would correspond to a *net* table (Hoem, 1968*b*, chapter 3).

Chiang (1968) and others would call the quantity in (2.1) a partial crude probability.

If *the total force of decrement* is defined as

$$\mu_x = \sum_{\nu=1}^{m} \mu_x^{(\nu)}, \quad \text{then} \quad {}_tp_x = \exp\left\{-\int_0^t \mu_{x+\tau}\, d\tau\right\}$$

is the probability that an x-year old member of the stock will not leave till after age $x+t$. At least in studies of mortality in human populations it is customary to specify some highest living age w such that ${}_tp_x=0$ for $t>w-x$. To make this possible, μ_x must increase without bounds as $x \uparrow w$.

The multiple decrement system is really only a time-continuous Markov chain model with one transient and m absorbing states. The careers of the individuals are regarded as independent sample paths. Such a path is in the transient state as long as the individual belongs to the stock, and upon decrement from the stock, it moves to the absorbing state corresponding to the relevant cause of decrement. The forces of decrement are the infinitesimal transition probabilities of the process.

Example 2: A disability model

In disability theory,[1] one studies, among other things, a Markov chain model with two transient states called "active" and "disabled" and an absorbing state called "dead".[2]

[1] DuPasquier (1913), Simonsen (1966), Sverdrup (1967).
[2] The latter is sometimes split into several absorbing states (Fix and Neyman, 1951; Chiang, 1968).

The infinitesimal transition probabilities are called the force of disablement, say ν_x, the force of recovery ϱ_x, and the forces of mortality μ_x^a for the active population and μ_x^i for the disabled, respectively. The (non-infinitesimal) transition probabilities of this process will be called influenced probabilities in this paper.

A somewhat simpler model results if 0 is substituted for ϱ_x. We shall say that the corresponding disability model (*without* recovery) is partial to the model *with* possible recovery, and its transition probabilities will be called partial probabilities (relative to the original model).

Example 3: A fertility model

Elsewhere (Hoem, 1968*a*) we have considered a very simple fertility model in which an x-year-old parent has a force of mortality $\mu(x)$ and a force of fertility $\phi(x)$, both of which are independent of all other factors than age, such as the previous number of births and their spacing, etc. (This is a stopped Poisson process.) The probability that an x-year-old parent will have k births in the age interval $[x, x+t]$ and survive to age $x+t$ equals

$$P_k(x,t) = \frac{1}{k!}\left\{\int_0^t \phi(x+\tau)\,d\tau\right\}^k \exp\left\{-\int_0^t [\mu(x+\tau)+\phi(x+\tau)]\,d\tau\right\}.$$

The corresponding probability of having k births and dying within age $x+t$ equals

$$Q_k(x,t) = \int_0^t P_k(x,\tau)\,\mu(x+\tau)\,d\tau.$$

The *partial* probability of k births, which occurs if mortality is inactive, is, of course

$$\bar{P}_k(x,t) = \frac{1}{k!}\left\{\int_0^t \phi(x+\tau)\,d\tau\right\}^k \exp\left\{-\int_0^t \phi(x+\tau)\,d\tau\right\}.$$

3. Blanket assumptions

We shall see how the concepts of influenced and partial probabilities appear within a more general theory. Consider a time-continuous Markov chain with a finite or countable state space $\mathbf{I}$ and with time-dependent transition probabilities (Feller, 1957, Chapter XVII. 9) over the fixed time interval $[0,\zeta\rangle$. If a sample path is in state i at time s, let $P_{ij}(s,t)$ be the probability that it will be in state j at time $t>s$.

Let $P_{i\mathbf{A}}(s,t) = \sum_{j\in\mathbf{A}} P_{ij}(s,t)$ for $\mathbf{A}\subseteq\mathbf{I}$.

The following assumptions will be adopted to hold throughout the paper.

Assumption 1. For all i and $j\in\mathbf{I}$,

$$P_{ij}(s,t)\geqslant 0,\ P_{i\mathbf{I}}(s,t) = 1 \quad \text{for } 0\leqslant s<t\leqslant\zeta,$$

$$\lim_{t \downarrow s} P_{ij}(s,t) = P_{ij}(s,s) = \delta_{ij} \quad \text{for}^1 \text{ any } s \in [0, \zeta>,$$

and

$$P_{ij}(s,u) = \sum_{k \in \mathbf{I}} P_{ik}(s,t)\, P_{kj}(t,u) \quad \text{for } 0 \leqslant s < t < u \leqslant \zeta.$$

Assumption 2. To every pair (i,j) of states where $i \neq j$ there corresponds a time-dependent *force of transition*

$$\mu_{ij}(s) = \lim_{t \downarrow s} P_{ij}(s,t)/(t-s) < \infty \text{ for } s \in [0,\zeta>. \tag{3.1}$$

Assumption 3. To every state i there corresponds a time-dependent *total force of decrement*

$$\mu_i(s) = \lim_{t \downarrow s} \{1 - P_{ii}(s,t)\}/(t-s) < \infty \text{ for } s \in [0,\zeta>$$

where $\mu_i(s) = \sum_{j \in \mathbf{I}-i} \mu_{ij}(s)$. (3.2)

We permit the possibility $\mu_{ij}(s) \to \infty$ (or $\mu_i(s) \to \infty$) as $s \uparrow \zeta$ for some state i and some $j \neq i$. (Cf. example 1 above.) The $P_{ij}(s, t)$ will be called the influenced probabilities of the model.

4. The concept of a partial chain

4 A. The main idea behind the partial probabilities, as they appear in the examples of chapter 2, is that they result from the corresponding influenced probabilities when certain forces of transition are replaced by 0. Let, then, $\mathbf{L}$ be a set of pairs (i,j) with $i \in \mathbf{I}$, $j \in \mathbf{I}$, $i \neq j$. We want to construct a set of transition probabilities from the forces of transition $\mu_{ij}(s)$ where $(i,j) \notin \mathbf{L}$, while 0 is substituted for the forces where $(i,j) \in \mathbf{L}$.

Let $\mathbf{K}$ be the set of those $i \in \mathbf{I}$ for which there exists a $j \in \mathbf{I}$ such that $(i,j) \notin \mathbf{L}$ or $(j,i) \notin \mathbf{L}$. For $i \in \mathbf{K}$, $j \in \mathbf{K}$, let

$$\theta_{ij}(s) = \begin{cases} \mu_{ij}(s) & \text{for } (i,j) \notin \mathbf{L}, \\ 0 & \text{for } (i,j) \in \mathbf{L}. \end{cases}$$

Assumption 4. Assume that a set of transition probabilities $P_{ij}(s, t; \mathbf{L})$ can be uniquely constructed over the state space $\mathbf{K}$, with the forces of transition θ_{ij}, and such that the $P_{ij}(s, t; \mathbf{L})$ satisfy assumptions quite similar to assumptions 1 to 3, with

$$\theta_i(s) = \sum_{j \in \mathbf{K}-i} \theta_{ij}(s).$$

We will call the $P_{ij}(s, t; \mathbf{L})$ the partial probabilities corresponding to $\mathbf{L}$, and the Markov chain model with those transition functions will be said to be *partial* relative to the original one.

Assumption 4 will hold automatically if $\mathbf{K}$ is finite and the θ_{ij} are continuous.

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[1] δ_{ij} is a Kronecker delta.

Otherwise, theorems concerning this assumption are given by Reuter and Ledermann (1953), by Chung (1967), and by their references. In examples 1 and 2 above, both **I** and **K** are of course finite. In example 3, assumption 4 holds since a Poisson process results if we let $\mu(x) \equiv 0$.

It is interesting to note that Reuter and Ledermann (1953) actually study a model with $\mathbf{I} = \{1, 2, \ldots\}$ in which they, successively for $n = 1, 2, \ldots$, substitute 0 for all μ_{ij} with i or $j > n$. In our terminology, this corresponds to letting $\mathbf{L} = \{(i, j) : i \neq j, \text{ and } i > n \text{ or } j > n\}$ and $\mathbf{K} = \{(i, j) : i \leq n, j \leq n\}$.

4 B. The partial probability $P_{ij}(s, t; \mathbf{L})$ need not have any interpretation at all as a probability within the original model. In some cases this is different, however. Assume for instance that the sample space has the form $\mathbf{I} = \mathbf{M} \times \mathbf{N}$, where **M** is finite or countable, and where $\mathbf{N} = \{1, 2, \ldots, n\}$ for some positive integer $n \geqslant 2$. Assume further that the forces of transition have the form[1]

$$\mu_{(i,\alpha),(j,\beta)}(s) = \begin{cases} \psi_{ij}(s) & \text{for } i, j \in \mathbf{M}, i \neq j, \alpha = \beta \in \mathbf{N}, \\ \eta_{i\alpha\beta}(s) & \text{for } i = j \in \mathbf{M}, \alpha, \beta \in \mathbf{N}, \alpha \neq \beta, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We let $P_{(i,\alpha),(j,\mathbf{N})}(s, t) = \sum_{\beta \in \mathbf{N}} P_{(i,\alpha),(j,\beta)}(s, t)$, and have

Theorem 1. *Let* $\mathbf{L} = \{((i, \alpha), (i, \beta)) : i \in \mathbf{M}, \beta \in \mathbf{N}, \beta \neq \alpha\}$ *for a given* $\alpha \in \mathbf{N}$. *Then* $P_{(i,\alpha),(j,\alpha)}(s, t; \mathbf{L}) = P_{(i,\alpha),(j,\mathbf{N})}(s, t)$.

Proof. If $i \neq j$, $\lim_{t \downarrow s} P_{(i,\alpha),(j,\mathbf{N})}(s, t)/(t - s) = \psi_{ij}(s)$. Moreover, $\lim_{t \downarrow s}\{1 - P_{(i,\alpha),(i,\mathbf{N})}(s,t)\}/(t - s) = \psi_i(s) = \sum_{j \in \mathbf{M} - i} \psi_{ij}(s)$.

The theorem then follows from assumption 4.

This argument is easily extended to the case where $\mathbf{N} = \{1, 2, \ldots\}$, provided $\lim_{t \downarrow s} P_{(i,\alpha),(j,\beta)}(s, t)/(t - s)$ exists uniformly in β for $(i, \alpha) \neq (j, \beta)$. Our theorem 1 gives a new interpretation of Schweder's theorem 1 (1968).

4 C. A set **A** of states is said to be closed if $P_{i,\mathbf{I}-\mathbf{A}}(s, t) = 0$ for any $i \in \mathbf{A}$ and all $0 \leqslant s < t < \zeta$. Let **R** be the set of states j for which there exists a state $i \in \mathbf{I}$ such that $(i, j) \in \mathbf{L}$. In the following sections, we shall study the case where **R** is closed, as it is in examples 1 and 3 above. If **R** is not closed, as in example 2, the problem can often be re-defined so that a new **R**′ becomes closed. This may be done in the following manner.

Let $\mathbf{I}' = \mathbf{I} \times \{1, 2\}$. For $i \in \mathbf{I}$, $j \in \mathbf{I}$, $i \neq j$, we introduce

$$\nu_{(i,\alpha),(j,\beta)}(s) = \begin{cases} \mu_{ij}(s) & \text{if } \alpha = \beta = 1, (i, j) \notin \mathbf{L}, \\ & \text{if } \alpha = 1, \beta = 2, (i, j) \in \mathbf{L}, \text{ or} \\ & \text{if } \alpha = \beta = 2, \text{ and} \\ 0 & \text{if } \alpha = \beta = 1, (i, j) \in \mathbf{L}, \text{ or} \\ & \text{if } \alpha = 1, \beta = 2, (i, j) \notin \mathbf{L}. \end{cases}$$

[1] An example with $n = 2$ has been given by Hoem (1968 *b*, § 3 C). In the example a transition $(i, 1) \to (i, 2)$ corresponds to an emigration.

Assume that it is possible to uniquely construct a set of transition probabilities $R_{i'j'}(s, t)$ over the state space $\mathbf{I}'$, with the forces of transition $\nu_{i'j'}$, and such that the $R_{i'j'}$ satisfy assumptions quite similar to assumptions 1 to 3.

Now let $\mathbf{I}'_k = \{(i, k) : i \in \mathbf{I}\}$ for $k = 1, 2$; and let $\mathbf{L}' = \{(i', j') : i' \in \mathbf{I}', j' \in \mathbf{I}'_2\}$. Then $\mathbf{R}' = \mathbf{I}'_2$ is closed, and

$$R_{(i,1),(j,1)}(s, t; \mathbf{L}') = P_{ij}(s, t; \mathbf{L}),$$

as follows from assumption 4 by a comparison of forces of transition. If the μ_{ij} similarly uniquely define the $P_{ij}(s, t)$, then

$$P_{ij}(s, t) = R_{(i,1),(j,1)}(s, t) + R_{(i,1),(j,2)}(s, t) = R_{(i,2),(j,2)}(s, t).$$

The idea behind this construction is as follows: Let a sample path of the original chain be $\omega = \{S(t, \omega) : 0 \leqslant t < \zeta\}$. Such a sample path will either at some time $T(\omega) \in [0, \zeta>$ have a first direct transition $i \to j$ for some $(i, j) \in \mathbf{L}$, or have no such transition, in which case we let $T(\omega) = \zeta$. Assume that $T(.)$ is measurable.[1]

A new sample path $\omega' = \{(S(t, \omega), U(t, \omega)) : 0 \leqslant t < \zeta\}$ is then constructed by letting

$$U(t, \omega) = \begin{cases} 1 \text{ for } 0 \leqslant t < T(\omega), \\ 2 \text{ for } T(\omega) \leqslant t < \zeta. \end{cases}$$

Thus ω will be copied in the part $\mathbf{I}'_1$ of the state space $\mathbf{I}'$ until the first direct transition in $\mathbf{L}$ occurs, whence it will be copied in $\mathbf{I}'_2$.

In the rest of this paper, we will take $\mathbf{R}$ to be a closed proper subset of $\mathbf{I}$.

4 D. Let $\mu_{i\mathbf{A}}(.) = \sum_{j \in \mathbf{A}} \mu_{ij}(.)$ for any $\mathbf{A} \subseteq \mathbf{I}$. In some models, such as in example 3, $\mu_{i\mathbf{R}}(.)$ is independent of i for $i \in \mathbf{K}$. It seems intuitively plausible that in such cases the probability $P_{i\mathbf{I}}(s, t)$ may generally be calculated as if the model had two states only, viz. the "transient state" $\mathbf{K}$ and the "absorbing state" $\mathbf{R}$. Thus if $\mu_{i\mathbf{R}}(.) = \gamma(.)$ for all $i \in K$, where $\gamma(.)$ is a continuous function over $[0, \zeta>$, one would expect to have

$$P_{i\mathbf{K}}(s, t) = \exp\left\{-\int_s^t \gamma(\tau)\, d\tau\right\} \tag{4.1}$$

for all $i \in \mathbf{K}$, $0 \leqslant s < t < \zeta$, under rather weak conditions. We prove

Theorem 2. *Assume that the limit in* (3.1) *exists uniformly in* (i, j) *for* $i \in \mathbf{K}, j \in \mathbf{R}$. *Then* (4.1) *holds under the conditions stated for* γ.

Proof. Let $0 \leqslant s < t < t + \Delta t < \zeta$ and $i \in \mathbf{K}$. Then

$$\left|\frac{1}{\Delta t}\left\{P_{i\mathbf{R}}(s, t + \Delta t) - P_{i\mathbf{R}}(s, t)\right\} - P_{i\mathbf{K}}(s, t)\,\gamma(t)\right|$$
$$\leqslant \sum_{k \in \mathbf{K}} P_{ik}(s, t)\left|\frac{1}{\Delta t} P_{k\mathbf{R}}(t, t + \Delta t) - \gamma(t)\right|$$
$$\leqslant \sum_{k \in \mathbf{K}} P_{ik}(s, t) \sum_{j \in \mathbf{R}}\left|\frac{1}{\Delta t} P_{kj}(t, t + \Delta t) - \mu_{kj}(t)\right| \leqslant \varepsilon P_{i\mathbf{K}}(s, t) \text{ for } 0 < \Delta t < \eta(\varepsilon, t),$$

[1] This assumption can certainly be made to hold if $\sup\{\mu_i(s) : i \in \mathbf{I}, 0 \leqslant s < \zeta\} < \infty$, as the original process is then a step-process (Dynkin, 1965, p. 93; Hoem, 1968 *c*, § 2.9).

by the uniform existence of the limit in (3.1). Since $P_{i\mathbf{R}}(s, t+\Delta t) - P_{i\mathbf{R}}(s, t) = -\{P_{i\mathbf{K}}(s, t+\Delta t) - P_{i\mathbf{K}}(s, t)\}$, we get $(\partial/\partial t)\, P_{i\mathbf{K}}(s, t) = -\gamma(t) P_{i\mathbf{K}}(s, t)$, from which the theorem follows.

Formula (4.1) similarly holds if the limit in (3.1) exists uniformly in (i, j) for $i \in \mathbf{K}, j \in \mathbf{K}$ (instead of $j \in \mathbf{R}$ as in the theorem).

5. The purged chain

5A. It sometimes happens that sample paths which end up in the closed subset **R** of the state space **I**, are not presented along with the rest of the data, e.g. because they have been deliberately removed, or because such sample paths simply have not been observed. In studies of the past fertility of women living in some restricted area, for example, data may be collected by asking the women living in the area at a certain date to give an account of their individual fertility histories. In such a case, data will be missing for the women who have died or migrated from the area before the date of observation.

We shall show that the remaining paths may generally be regarded as realizations of a new Markov chain over the state space $\mathbf{K} = \mathbf{I} - \mathbf{R}$, which we shall call the *purged* chain.

5 B. Let **H** be the set of states in **K** from which **R** cannot be reached, and let $\mathbf{J} = \mathbf{K} - \mathbf{H}$. Of course, **H** may be empty. Otherwise **H** is closed. If $\mathbf{H} \neq \emptyset$, we shall assume that also $\mathbf{J} \neq \emptyset$ and that there exists some $i \in \mathbf{J}$, $s \in [0, \zeta>$, where $P_{i\mathbf{H}}(s, \zeta) > 0$, to avoid trivialities.

We wish to avoid that all sample paths end up in **R** by time ζ with probability 1. We also want to secure that some paths starting in **K** may enter **R**. We therefore make

Assumption 5. For any $i \in \mathbf{K}$ and $s \in [0, \zeta>$, $P_{i\mathbf{K}}(s, \zeta) > 0$. There exists an $i \in \mathbf{K}$ and an $s \in [0, \zeta>$ such that $P_{i\mathbf{K}}(s, \zeta) < 1$.

A sample path which does not end in **R** within time ζ will have the (conditional) transition probabilities

$$Q_{ij}(s, t) = P_{ij}(s, t)\, P_{j\mathbf{K}}(t, \zeta) / P_{i\mathbf{K}}(s, \zeta) \text{ for all } i \in \mathbf{K}, j \in \mathbf{K}, 0 \leqslant s < t \leqslant \zeta. \tag{5.1}$$

Specifically

$$Q_{ij}(s, t) = \begin{cases} P_{ij}(s, t) / P_{i\mathbf{K}}(s, \zeta) & \text{for } i \in \mathbf{J}, j \in \mathbf{H}, \\ P_{ij}(s, t) & \text{for } i \in \mathbf{H}, j \in \mathbf{K}. \end{cases} \tag{5.2}$$

It is easily proved that the $Q_{ij}(s, t)$ satisfy conditions similar to those in assumption 1 with **I** replaced by **K** provided the following assumption holds:

Assumption 6. For any given $j \in \mathbf{K}$, $P_{j\mathbf{K}}(., \zeta)$ is continuous from the right in $[0, \zeta>$.

Under this assumption, the forces of transition satisfy

$$\lambda_{ij}(s) = \lim_{t \downarrow s} Q_{ij}(s, t) / (t-s) \tag{5.3}$$

$$= \mu_{ij}(s) P_{j\mathbf{K}}(s, \zeta) / P_{i\mathbf{K}}(s, \zeta) \text{ for } i \neq j, i \in \mathbf{K}, j \in \mathbf{K}, 0 \leqslant s < \zeta.$$

As in (5.2), this formula simplifies somewhat for $i \in \mathbf{J}, j \in \mathbf{H}$, and for $i \in \mathbf{H}, j \in \mathbf{K}$. For each $i \in \mathbf{H}$ a total force of decrement (relative to the $Q_{ij}(s, t)$) exists and equals $\mu_i(s)$. (Cf. the second member of (5.2).) To prove the existence of a finite total force of decrement satisfying a relation similar to (3.2) in the purged process for an $i \in \mathbf{J}$, more restrictive conditions seem necessary. Specifically, if $\mathbf{K}$ is finite,

$$\lambda_i(s) = \lim_{t \downarrow s} \{1 - Q_{ii}(s,t)\} / (t-s) = \sum_{j \in \mathbf{K}-i} \lambda_{ij}(s) \text{ for all } i \in \mathbf{K}.$$

Obviously the sample paths removed from the data can be given a similar treatment.

5 C. (*The time-homogeneous case*)
If the μ_i and the μ_{ij} are independent of s and $P_{ij}(s, s+t) = P_{ij}(t)$, we have

$$\lambda_{ij}(s) = \mu_{ij} P_{j\mathbf{K}}(\zeta - s) / P_{i\mathbf{K}}(\zeta - s),$$

which may genuinely depend on $\zeta - s$. *Even when the transition probabilities of the original Markov chain are stationary, those of the purged chain may thus be time-dependent.*

In this case, $Q_{ij}(s, t)$ will depend on $\zeta - s$ and $\zeta - t$, i.e. generally not on $t - s$ only.

5 D. It has some interest to compare the purged and the partial Markov chain models. Both are derived from the original chain, and both have state space $\mathbf{K}$, but their transition probabilities will, of course, generally be different. By (5.3) and assumption 4, *the transition probabilities of the purged and the partial chain are identical if and only if $P_{i\mathbf{K}}(., \zeta)$ is independent of i for $i \in \mathbf{K}$.* (In that case $\mathbf{H}$ must be empty, since otherwise $\lambda_{ij}(s) = \mu_{ij}(s) / P_{i\mathbf{K}}(s, \zeta) > \mu_{ij}(s)$ for some $i \in \mathbf{J}, j \in \mathbf{H}, s \in [0, \zeta>$, by assumption 4.)

Theorem 2 states conditions sufficient for this to hold. We note that if (4.1) holds, then

$$P_{ij}(s, t) = P_{ij}(s, t; \mathbf{L}) \exp\left\{ - \int_s^t \gamma(\tau) \, d\tau \right\} \tag{5.4}$$

for all i and $j \in \mathbf{K}$, $0 \leqslant s < t < \zeta$. (This follows from (5.1) and the fact that under (4.1), $Q_{ij}(s, t) = P_{ij}(s, t; \mathbf{L})$.) In this case, therefore, a particularly simple relation exists between the influenced and the partial probabilities.

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