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Abstract:
This paper modifies and extends the aggregate equilibrium models for matching markets developed earlier in the literature. Agents in the matching market search for a match among potential partners, including agreements about a flexible contract, such as hours and wage combinations in the labor market. Under general utility representations that are non-transferable and assuming the matching is stable, we derive a probabilistic framework for the probability of realizing a particular match, including the choice of contract. We also show that the popular transferable utility model with transferable utilities can be viewed as a limiting case within our modelling framework. The framework is practical to apply for empirical analysis and is at the same time sufficiently general to accommodate essential features of matching markets with heterogeneous agents.

Keywords: Matching markets, Aggregation, Latent choice sets, Random utility matching models

JEL classification: J22, C51

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Sammendrag

Litteraturen som omhandler modeller for tilpasning i markeder med matching har etter hvert blitt omfattende. Aktører i slike markeder søker etter en passende match blant potensielle partnere. Eksempler på slike markeder er arbeidsmarkedet og ekteskapsmarkedet. I noen tilfeller kan tilpasningen inkludere forhandlinger om kontraktbetingelser. I denne artikkelen generaliserer vi resultater oppnådd tidligere av Dagsvik (2000) og Menzel (2015) for slike modeller. Vi viser blant annet at under generelle forutsetninger medfører vår tilnærming til et praktisk og fleksibelt modellrammeverk som er velegnet til empirisk analyse, og som inkluderer diskrete arbeidstilbudsmodeller (Dagsvik og Jia, 2016)) som spesialtilfelle.

1. Introduction

Many important markets are matching markets, where agents are searching to obtain a match with a suitable partner. Examples include the marriage market, the labor market, and the education market. Most of the literature on two-sided matching markets is theoretical. That is, this literature focusses on analyzing matching under different rules of the game when preferences of the players (suppliers and demanders) in the market are known and fulfill specific requirements. The theoretic analysis of this type of markets begins with the famous article by Gale and Shapley (1962), and since then a large literature has emerged, see Roth and Sotomayor (1990), Hatfield and Milgrom (2005), and Lauermann and Nöldeke (2014) and the references therein.

In contrast to the theoretical literature, the issue addressed in this paper is how to recover the preferences of the agents in the market from data on observed matching outcomes. To this end, it is necessary to develop an equilibrium model for matching behavior that links the distribution of individual preferences to observed matching behavior. A first attempt to deal with this problem in the non-transferable case (NTU) was made by Dagsvik (2000). He obtained equilibrium asymptotic fractions of realizing matches of specific observable types in large markets, with a finite number of observed types of agents on either side of the market, including the choice of contracts from a finite menu. The approach of Dagsvik (2000) consisted in showing that the realizations of a matching game can be viewed as if they were the outcome of a series of discrete choice problems where the suppliers and demanders make choices from their respective choice sets under equilibrium. However, the equilibrium concept of Dagsvik (2000) differs from the one used by Menzel and in this paper. His notion of equilibrium is a probabilistic one in the sense that suitable equilibrium conditions are only supposed to hold "on average." Also, he did not prove that the distribution of the equilibrium choice sets in a finite population actually exists. Apart from a special case, he also did not show that the fractions of realized matches of each type converge (under equilibrium) to the corresponding asymptotic fractions. Recently, Menzel (2015) made an important extension of Dagsvik (2000) by demonstrating that equilibrium choice sets do indeed exist for any matching algorithm. He also extended the framework of Dagsvik (2000) by allowing for continuous individual characteristics of agents. Furthermore, he generalized the matching game to a setting where each pair of agents (supplier and demander) only have knowledge about a random subsample of potential partners, specific to each pair of agents. Under more general distributional assumptions than asserted by Dagsvik (2000), Menzel also derived similar equilibrium choice probabilities for realizing particular matches when the market is large.

Other related approaches that develop aggregate relations for matching markets are based on the transferable utility (TU) assumption (Shapley and Shubik, 1972), and include Gretsky et al. (1999), Choo and Siow (2006), Chiappori et al. (2012), Galichon and Salanie (2010, 2012). See Menzel (2015) for a more complete review of the recent related literature on matching models. In some matching markets, the transferable utility assumption (TU) may seem restrictive. For example, in modern labor – and marriage markets it might be argued that the TU assumption does not seem to correspond to a realistic description of actual matching schemes.

In this paper, we extend the analysis of Dagsvik (2000) and Menzel (2015) in several ways. First, we extend and complement Dagsvik (2000)'s analysis by showing that equilibrium choice sets also exist in the case when flexible contracts are allowed in the matching market. A contract may be a price, a dowry or a non-pecuniary agreement (such as marriage or cohabitation in the marriage market). This type of matching models can be applied to study many different economic phenomena, see, for example, Hatfield and Milgrom (2005). Second, we relax the distributional properties of the utility functions in that we only require some weak regularity conditions to hold and we derive the corresponding aggregate equilibrium model for the general case. This enables us to provide a unified treatment of several types of matching models appearing in the literature, as well as some extensions. It includes several NTU and TU models analyzed recently in the matching literature as special cases. In particular, we show that the TU model of Choo and Siow (2006) emerges as a limiting case of the NTU model (without flexible contracts) as the correlation between utilities across observationally identical potential partners tends towards one. In other words, the TU model of Choo and Siow (2006) can be viewed as a limiting case of a version of our NTU model with interdependent preferences and no flexible contract. This result can thus be interpreted as an NTU foundation of the TU model. Moreover, we show that the model can be extended to accommodate latent random effects that represent match quality or attractiveness between potential pairs of suppliers and demanders. In the presence of such random effects, preferences between suppliers (demanders) and potential partners become correlated. As a special application of this extension, we show that our framework can be accommodated to include the case where each pair of agents only has knowledge about a random sample of potential partners. Menzel (2015) also discusses the latter case and he states particular equilibrium relations which he does not prove. Finally, we discuss welfare analysis and estimation issues when different types of data are available.

We think that our modeling framework (as well as Menzel's formulation) should be of interest in several contexts. First, it might be helpful for addressing more general and realistic stability and equilibrium issues in matching markets, in contrast to many traditional analyses that often are based on stylized frameworks. Second, it offers a convenient structural framework for conducting empirical analyses of matching markets. For example, it allows researchers to identify and estimate the distribution of preferences from observations on the number of realized matches. Third, the approach provides an explicit micro-foundation for macro relations in matching markets without relying on the representative agent postulate or restrictive assumptions about the distribution of preferences and constraints. Fourth, since the model is a structural one, it can be applied to compute distributional and aggregate effects from counterfactual policy reforms, such as the effect of changing the ex ante distribution of the different types of agents in the market, as well as the distributional and welfare effect of economic incentives (e.g., such as taxes/costs and other benefits in the labor- and education markets), depending on the application under study.

In the setting, we consider agents are assumed to have sufficient information to be able to rank over potential partners. However, each agent has, ex-ante, no information about the preferences of other agents. Like Crawford and Knoer (1981), we believe that an analysis of this case can yield useful predictions about markets, where conditions are sufficiently stable over time for agents to have had enough time to learn about their environments. Such an analysis is, at any rate, a necessary prerequisite for the study of the effects of imperfect information.[1]

The matching concept applied in this paper differs from the one applied by Mortensen (1978, 1982), Diamond and Maskin (1979), Jovanovic (1979, 1984), see Mortensen (1988) for a review. In their approach suppliers and demanders are uncertain about who are the

[1] Agents have full information in the sense that they are able to form preference lists of all potential partners in the economy. They have, however, no information ex ante about their equilibrium choice set of available matching candidates.

potential partners and they are therefore unable to make preference lists of potential partners, ex-ante. Recall that in contrast, our framework allows for, in a preliminary stage, that a subset of suppliers and demanders meet (or gather information about each other) randomly. Subsequently, in the second stage, the matching takes place within this random subsample. The match quality random effect mentioned above is somewhat analogous to the match quality parameter introduced by Jovanovic (1979, 1984). Recall that in the setup of Jovanovic suppliers and demanders are also uncertain about the quality of the match at the moment the match is formed, and the quality is revealed gradually over time. This is different from our approach where the quality of the match is revealed instantly upon inspection of the available alternatives in the (current) choice set. In the terminology of Jovanovic (1984), this case corresponds to the match being a pure inspection good. However, one might imagine a similar dynamic extension of our theory where the matching quality random effects are updated similarly according to experience and where separations occur when the match quality falls below a threshold.

The rest of the paper is organized as follows. Section 2 gives an informal summary of the main idea and our analytic approach. In Section 3 we discuss how one can treat a matching market as a series of one-sided discrete choice problems. In Section 4 we derive the probabilistic modeling framework for large markets under equilibrium conditions. To achieve our aggregate equilibrium relations we apply a stochastic representation of the distribution of the preferences of the demanders and the suppliers. The stochastic formulation serves to represent unobserved heterogeneity in the preferences of the agents. This, in turn, allows us to apply the probabilistic theory of discrete choice to develop an equilibrium model, that is, the probability that a given demander (supplier) shall obtain a match with a supplier (demander) at a specific contract. Section 5 discusses various extensions of the model. Section 6 contains results on identification and estimation strategies. Section 7 contains a discussion on measures of welfare and gain from a match.

2. Informal summary of approach and main results

This section provides an informal summary of main results as well as the central ideas of our analytic approach. Similar to Dagsvik (2000) and Menzel (2015) a key idea of our approach is to show how one can “simplify” a two-sided matching game as if it were a series of one-sided choice problems subject to suitably defined agent-specific choice sets. That is, we show how it is possible to define these choice sets such that the solution of the matching game will be the same as if the agents made their choices independently from their respective choice sets of potential partners. Dagsvik (2000) was the first one to pursue this idea but he only discussed the existence of these sets for some special type of stable matchings.[2] In this paper, we prove that such choice sets exist for any stable matching. More importantly, these choice sets are subject to a set of restrictions that result from market competition, and can, in large markets, be characterized by a system of equations (Theorem 1). Menzel (2015) obtained analogous results for the case with no flexible contract.

To deal with possible observed heterogeneity, we group agents into different types according to observed characteristics and model the matching game within a random utility framework so as to allow for unobserved heterogeneity in preferences as well. Our goal is to obtain a structural relationship (model) between the distribution of agents' preferences and the number of realized matches between suppliers and demanders (say) of each combination of observable types. Given such a model one can recover (partially) preferences from observed matching behavior. This setup allows us to establish equilibrium relations that determine the equilibrium choice sets. This is possible because the equilibrium choice set of a supplier (say) is the set of demanders that prefer to be matched with this supplier (conditional on the respective equilibrium choice sets of the demanders), see Section 3.2. However, these equilibrium choice sets are not unique. The key difficulty here is that the agents' choice sets are not only random but also endogenous in the sense that they depend on their own and all the other players' preferences. We show that when the size of the market increases without bounds, the endogeneity is no longer important and the sizes of random equilibrium choice sets become sufficient statistics for these sets. At first glance, this result may not seem intuitive since it implies that, from the analyst's point of view, equilibrium in large markets can be viewed as if individual choice sets were exogenous with deterministic sizes. There are, however, similar results in other fields of economics. For example, within the field of economics of networks, the Aldous-Hoover representation theorem states that any infinitely exchangeable random network can be modelled as if the links were formed independently,

[2] Also, Dagsvik's equilibrium solution concept differs somewhat from the one applied here and by Menzel (2015). Dagsvik's concept is based on the notion of "aggregate" equilibrium. This means that the relations that determine the equilibrium restrictions are only assumed to hold on average across identical repetitions of the matching game.

although the setting obviously appears to involve complicated strategic interaction. A perhaps more familiar example is the competitive market where agents are viewed as price takers, although individual behavior influences market prices. The intuition in our case is that when the market size is large the corresponding equilibrium choice sets also become large. As these choice sets of potential partners increase essentially every level of attractiveness of potential candidates will eventually be represented in the choice sets. Therefore, it does not matter how the members of the choice sets were selected. Menzel (2015) has also obtained a similar result. Our approach to deal with this problem is, however, completely different from his. See Section 4.2 for a more detailed description of our approach.

Once the endogeneity problem is dealt with, one can apply standard results to derive probabilities for an agent's most preferred candidate conditional on his choice set. These probabilities must satisfy particular aggregate equilibrium conditions that follow readily from the corresponding disaggregate equilibrium conditions mentioned above. When the market size increases, we show that the sizes of the choice sets (suitably normalized) converge with probability one to their corresponding unique asymptotic and deterministic values. These values depend on the products of deterministic parts of utilities of the suppliers and the demanders of specific types in a simple way (analogous to Nash products in bargaining theory) (Section 4.1). Since preferences appear in the model solely through "Nash products" this implies that one cannot obtain identification of the distribution of preferences on each separate side of the market without additional data on preference orderings of the suppliers or the demanders.

Next, our analytical approach is extended to more general cases. We show that one can allow utilities of the demanders to be correlated with the utilities of the suppliers. This type of correlation might be due to latent aspects that represent attractiveness, which in our setup is represented by a random effect. This is particularly apparent in the marriage market where the attraction between a particular woman and a man typically cannot be predicted by a third party. We demonstrate (Section 5.1) that in this case the asymptotic number of realized match is determined in a similar way as the baseline case (Section 4.1). With the exception that the "Nash product" is replaced by its expectation which is taken with respect to the latent random effect. This result also allows us to study the case where agents have limited information about potential candidates in the market. Usually, the agents participating in a matching market have limited information about the potential candidates on the opposite side of the market. Similarly to Menzel (2015) we consider a setting in which the matching process takes place in two stages. In the first stage, a subsample of suppliers and demanders meet (or gather information) at random. Here, it is understood that the information process is symmetric in the sense that if supplier s obtains information about demander d, demander d also obtains information about supplier s. In the second stage, the matching takes place as if the subsample obtained in the first stage were the whole market. In this paper, we show that this case can be treated as a special case of the setting with interdependence between the utilities of demanders and suppliers (Section 5.2). Menzel (2015) also states, without proof, an equivalent result.

Another extension we can deal with by using our approach is the case when the preferences of a supplier (demander) of a specific type are correlated with the preferences of a supplier (demander) of the same type (Section 5.3). This case is of interest empirically because it allows the researcher to account for potential latent characteristics that may be correlated across agents of the same observable type. Moreover, this result turns out to be useful for establishing a link between the NTU and TU approach. By using the framework developed in Section 5.3, we show in Section 5.4 that the TU model of Choo and Siow (2006) can be obtained as a limiting case of the NTU as the correlation between utilities of agents of the same type tends towards one (Theorem 7).

We have also attempted to relax the distributional assumptions further (Section 5.5). We find that the main convergence results are still valid (Theorem 8). However, the equations that determine the asymptotic number of realized matches of each combination of types become more complicated.

3. Behavior in matching markets as a discrete choice problem

In this section, we shall first discuss how one can "simplify" a two-sided matching game as if it were a series of one-sided choice problems subject to suitably defined agent-specific choice sets. That is, we show how it is possible to define suitable agent-specific choice sets such that the solution of the matching game will be the same as if the agents made their choices independently from their choice sets of potential partners. Under equilibrium conditions to be discussed below, the choice sets are subject to a set of restrictions that result from the market competition. Similar to Hatfield and Milgrom (2005), we consider an extended setting of the classical matching game (Gale and Shapley, 1962), in which we allow for a finite menu of flexible contracts. Flexible contracts are important in many matching markets. The contracts are not just limited to money. It could be any types of agreements between the two parties, for example, type of cohabitations in the marriage market as discussed by Mourifie and Siow (2014), working hours and job-specific tasks to do in the labor market, and tuition fee in the education market. This setup allows us to take into account restrictions on the set of possible contracts in applications. For example, in the labor market, the set of feasible hours of work is usually constrained due to institutional regulations. Even the wage possibilities are limited in several sectors in the labor market.

As in a standard matching market, there are two kinds of agents which we may term suppliers and demanders, such as women and men in the marriage market and firms and workers in the labor market. Each supplier is looking for a suitable match with a demander, and vice versa. When they form a match, they choose from a set of exogenously given possible contracts. Only one-to-one matchings are considered here. Each agent has preferences over all the combination of potential partners and potential contracts (including the option of remaining unmatched). An agent's characteristics affect his or her own preferences and also enter as attributes in the utility functions of potential partners. However, a priori the agents only know their own preferences and have no information about the preferences of other agents. This implies that the agents have, ex-ante, no information about their "chances" of establishing a match with the respective potential partners.

3.1. Stable matchings with flexible contracts and the extended Deferred Acceptance algorithm.

In the literature on two-sided matchings, the condition that ensures equilibrium is the one of *stable matching*, see Gale and Shapley (1962) or Roth and Sotomayor (1990). It is easy to extend this concept to the case with a finite menu of flexible contracts. This extension has also been made by Crawford and Knoer (1981) for the case where the contracts are wages and later by Hatfield and Milgrom (2005) for more general contracts. However, their approach is different from ours.

Consider a matching where supplier s and demander d are matched together at contract ω, but at least one of the two agents would prefer to be single rather than being matched to the other. Then this matching is said to be *blocked* by the unhappy agent. Second, consider a matching where supplier s and demander d are matched at contract ω, but both prefer to be matched at another contract, or they are not matched to one another but prefer each other at some contract to their assignment in the actual matching. Then the combination (s, d, ω) will be said to *block* the matching. We say that a matching is *stable* if it is not blocked by any individual or combination of agents and contracts.

To show the existence of a stable matching defined above, we modify the Deferred Acceptance algorithm (see Roth and Sotomayor, 1990). Under this algorithm, the matching game takes place in several stages. Only one side of the market (say, demanders) is allowed to make offers. In the first stage, each demander makes an offer to his most preferred combination of contract and supplier. Each supplier rejects the offer from any demander who is unacceptable (i.e. ranked lower than the option of remaining unmatched), and each supplier who receives more than one offer from any demander rejects all but his or her most preferred among these. Any supplier whose offer is not rejected at this point is kept "engaged" (at some contract). At any step, any demander who was rejected at the previous step makes an offer to her next preferred combination of supplier and contract so long as there remains an acceptable offer consisting of a combination of supplier and contract that has not yet been offered. The supplier then rejects any offer from unacceptable demanders and also rejects all but his or her most preferred combination of contract and demander including the combination of contract and demander that was kept engaged from the previous step. The algorithm stops after any step in which no combination of demander and contract is rejected. The matches are now consummated based on the agents' current engagements.

Proposition 1

In a matching market with strict preferences and with a finite set of flexible contract there exists a stable matching.

The proof of Proposition 1 is given in Appendix A. A similar result has also been obtained by Hatfield and Milgrom (2005) in their setting. Note that if it is assumed instead that the suppliers make the offer, the realized matching will be different. Similarly to the original stable matching case, the number of stable matchings for a given market can be large. It is essential that the menu of contracts is finite because in the infinite case with non-monotonic preferences there is no guarantee that the extended Deferred Algorithm described above will converge.

3.2. Equilibrium choice sets in a matching market with flexible contracts

In the following, we shall introduce our definitions of supply, demand and choice sets and use these to introduce an alternative representation of the matching problem.

Let Ω^S denote the set of *N* suppliers and Ω^D the set of *M* demanders, and *W* a finite set (menu) of potential flexible contracts. Let $U_s^S(d,\omega)$ be the utility of supplier *s* of a match with demander *d* at contract ω, $U_d^D(s,\omega)$ the utility of supplier *d* of a match with supplier *s* at contract ω, $U_s^S(0)$ and $U_d^D(0)$ the respective utilities of being self-matched (single). The single option is always available. Let $y_s^S(d,\omega)=1$ if demander *d* belongs to the choice set of supplier *s* at contract ω and zero otherwise. Similarly, let $y_d^D(s,\omega)=1$ if supplier *s* belongs to the choice set of demander *d* at contract ω and zero otherwise. Let $y_s^S=\{y_s^S(d,\omega), d\in\Omega^D, \omega\in W\}$ and $y_d^D=\{y_d^D(s,\omega), s\in\Omega^S, \omega\in W\}$. We realize that y_s^S and y_d^D represent the choice sets of supplier s and demander *d*, respectively.[3] We define the constrained supply function $J_s^S(d,\omega,y_s^S)$, as

$$(3.1)\qquad J_s^S(d,\omega,y_s^S)=\begin{cases}1 & \text{if } U_s^S(d,\omega)\geq \max(\max\limits_{k\in\Omega^D}\max\limits_{v\in W}U_s^S(k,v)y_s^S(k,v),U_s^S(0))\\ 0 & \text{otherwise,}\end{cases}$$

for all $d\in\Omega^D,\omega\in W$. Note that the supply function is defined for *all d*, irrespective of whether or not *d* is available to supplier *s*. Similarly, we define the constrained demand function $J_d^D(s,\omega,y_d^D)$ as

$$(3.2)\qquad J_d^D(s,\omega,y_d^D)=\begin{cases}1 & \text{if } U_d^D(s,\omega)\geq \max(\max\limits_{k\in\Omega^S}\max\limits_{v\in W}U_d^D(k,v)y_d^D(k,v),U_d^D(0))\\ 0 & \text{otherwise,}\end{cases}$$

for all $s\in\Omega^S,\omega\in W$. Recall that here the elements of y_d^D are (given) hypothetical values that represent the choice set of demander *d* and the elements of y_s^S are (given) hypothetical values that represent the choice set of supplier *s*.[4] Our approach based on the constrained supply and demand functions defined above is different from Adachi (2000), but analogous. Menzel (2015)'s approach is similar to Adachi (2000).

[3] To use indicator functions to represent sets is common in Mathematics, see for example, Mirkin (2013).

[4] The notion of choice sets used here is not fully precise, but is chosen for convenience. To be precise, the respective single options must be included.

As in Section 2 let $\{Y_s^S(d,\omega)\}$ and $\{Y_d^D(s,\omega)\}$ denote the corresponding equilibrium choice sets. The equilibrium values of the choice sets satisfy

(3.3) $$Y_d^D(s,\omega) = J_s^S(d,\omega,Y_s^S)$$

and

(3.4) $$Y_s^S(d,\omega) = J_d^D(s,\omega,Y_d^D)$$

for $s \in \Omega^S, d \in \Omega^D, \omega \in W$, where Y_s^S and Y_d^D represent the respective equilibrium choice sets of supplier *s* and demander *d*.

Theorem 1

(i) Suppose that the agents' preferences are strict. Then the system of equations (3.3)-(3.4) has several solutions. Each solution corresponds to a stable matching of the matching market, i.e., there exist choice sets Y_s^S *and* Y_d^D *that satisfy (3.3) and (3.4).*

(ii) If agents choose their most preferred alternative from their respective choice sets $\{Y_s^S\}$ *and* $\{Y_d^D\}$ *(which satisfy (3.3) and (3.4) then the resulting matching will be stable.*

The proof of Theorem 1 is given in Appendix A. The proof actually contains an algorithm which generates the sets satisfying (3.3) and (3.4) for any given stable matching. The results obtained in Theorem 1 imply that matching behavior can be viewed as the outcome of separate one-sided discrete choice problems. Thus, it makes sense to denote the sets Y_s^S and Y_d^D equilibrium choice sets. To be precise, the series of choice problems defined by these agent- specific choice sets will give the same solutions as any matching game that produces a stable matching. The key point in our setup, as will become clearer later, is that Eqs. (3.3) and (3.4) turn out to be a most useful link between the preferences and choice restrictions of agents from both sides of the market that will enable us to address the aggregation problem under equilibrium in a useful way. Note that although we used the modified Deferred Acceptance (DA) algorithm to show the existence of the stable matching, Eqs. (3.3) and (3.4) and Theorem 1 is valid for all possible stable matchings and is not dependent on the actual matching algorithms that generate the matching outcome.

3.3. Observed heterogeneity and aggregated equilibrium conditions

Above we allowed the agents to be heterogeneous with no other restrictions than preferences being fixed and strict. In practical empirical applications, however, only some of the relevant individual characteristics are observed by the analyst. In order to deal with heterogeneity in the analysis, we group agents into different types according to the observed characteristics. Agents within each type are observationally identical but may have different preferences. Similarly to the notation in Section 3.2 let $U_{si}^{S}(d, j, \omega)$ be the utility of supplier *s* of observable type *i* of a match with demander *d* of observable type *j* at contract ω, $U_{dj}^{D}(s, i, \omega)$ the utility of demander *d* of type *j* for a match with supplier *s* of type *i* at contract ω, and the respective utilities of being single.

Let Ω_i^S be the set of suppliers of type *i* and Ω_j^D the set of demanders of type *j*. Define $y_{si}^{S}(d, j, \omega)$ and $y_{dj}^{D}(s, i, \omega)$ as the extension of $y_s^S(d, \omega)$ and $y_d^D(s, \omega)$ defined above. Similarly, let $y_{si}^{S} = \{y_{si}^{S}(d, j, \omega), \omega \in W, d \in \Omega_j^D, j = 1, 2, ...\}$ and $y_{dj}^{D} = \{y_{jd}^{D}(s, i, \omega), \omega \in W, s \in \Omega_i^S, i = 1, 2, ...\}$. We define the supply and demand index functions $J_{si}^{S}(d, j, \omega, y_{si}^{S})$ and $J_{dj}^{D}(d, i, \omega, y_{dj}^{D})$ in an analogous way as in (3.1) and (3.2), namely as

$$J_{si}^{S}(d, j, \omega, y_{si}^{S}) = \begin{cases} 1 & \text{if } U_{si}^{S}(d, j, \omega) \geq \max(\max\limits_{r} \max\limits_{k \in \Omega_r^D} \max\limits_{v \in W} U_{si}^{S}(k, r, v) y_{si}^{S}(k, r, v), U_{si}^{S}(0)) \\ 0 & \text{otherwise,} \end{cases}$$

for all $d \in \Omega_j^D, \omega \in W, j = 1, 2, ...$ Similarly, we define $J_{dj}^{D}(s, j, \omega, y_{dj}^{D})$ as

$$J_{dj}^{D}(s, j, \omega, y_{dj}^{D}) = \begin{cases} 1 & \text{if } U_{dj}^{D}(s, j, \omega) \geq \max(\max\limits_{r} \max\limits_{k \in \Omega_r^S} \max\limits_{v \in W} U_{dj}^{D}(k, r, v) y_{dj}^{D}(k, r, v), U_{dj}^{D}(0)) \\ 0 & \text{otherwise,} \end{cases}$$

for all $s \in \Omega_i^S, \omega \in W, i = 1, 2, ...$

Similarly to (3.3) and (3.4), the equilibrium choice sets are determined by

(3.5) $$Y_{dj}^{D}(s, i, \omega) = J_{si}^{S}(d, j, \omega, Y_{si}^{S})$$

and

(3.6) $$Y_{si}^{S}(d, j, \omega) = J_{dj}^{D}(s, i, \omega, Y_{dj}^{D})$$

where Y_{is}^{S} and Y_{dj}^{D} represent the respective equilibrium choice sets of supplier *s* of type *i* and demander *d* of type *j*. Since the utility functions are random variables so are also the corresponding indicator functions, as well as the equilibrium choice sets.

4. Asymptotic aggregate equilibrium relations

Given a matching market as described above, with flexible contracts and several observable types of agents, we shall in this section discuss asymptotic aggregate results, that is, aggregate equilibrium relations in "large" markets. The reason why we consider the case with large markets is that the more realistic case with smaller markets is too complicated to deal with. That is, we shall establish equilibrium relations for the number of matches between suppliers of a specific type and demanders of another specific type, as a function of the agents' preferences and the fractions of agents of the respective observable types. Let N_i be the number of suppliers of type *i* and M_j the number of demanders of type *j* and let *N* be the total number of suppliers and *M* the total number of demanders.

Let N_i be the number of suppliers of type *i* and M_j the number of demanders of type *j* and let *N* be the total number of suppliers and *M* the total number of demanders. Let $Y_{si}^S(j,\omega)=\{Y_{si}^S(d,j,\omega),d\in\Omega_j^D\}$ and $Y_{dj}^D(i,\omega)=\{Y_{dj}^D(s,i,\omega),s\in\Omega_i^S\}$. Let *C* be a set consisting of non-negative variables and define $\|C\|$ as the sum of the variables in *C* divided by the square root of the number of variables in *C*. For example, we have that

$$\|Y_{si}^S(j,\omega)\|=\frac{1}{\sqrt{N}}\sum_{d\in\Omega_j^D}Y_{si}^S(d,j,\omega).$$

Due to (3.5) and (3.6) we get that under equilibrium the total number of suppliers of type *i* that are in the choice set of demander *d* of type *j* at contract ω, and the total number of demanders of type *j* that are in the choice set of supplier *s* of type *i* are determined by

(4.1) $$\|Y_{dj}^D(i,\omega)\|=\frac{1}{\sqrt{N}}\sum_{r\in\Omega_i^S}J_{ri}^S(d,j,\omega,Y_{ri}^S)$$

and

(4.2) $$\|Y_{dj}^S(i,\omega)\|=\frac{1}{\sqrt{N}}\sum_{r\in\Omega_j^D}J_{ri}^D(s,i,\omega,Y_{ri}^D).$$

Note that it follows from Theorem 1 that there exist solutions to the equations in (4.1, 4.2). These relations represent aggregate equilibrium restrictions because they characterize the number of demanders of type *j* (suppliers of type *i*) in the equilibrium choice set of the supplier of type *i* (demander of type *j*). Note that they only yield necessary conditions for (3.5) and (3.6) to hold, and they may also hold in cases where the matching is unstable. Recall also that there may be many different stable matchings for a given matching problem. Thus, the solutions to the relations in (4.1, 4.2) are not unique.

Assumption 1 (Balanced Market)

The ratios $M/N \to \kappa$, $N_i/N \to \lambda_i^S$, $M_j/M \to \lambda_j^D$ *tend towards positive finite constants, respectively, as* $N \to \infty$.

Assumption 1 is needed to guarantee that when the population gets large, the market composition remains stable.

4.1. Independent utilities

We start our analysis by considering the simplest case where the preferences are independent.

Assumption 2

The utility functions have the structure

$$U_{si}^S(d,j,\omega) = a_i^S(j,\omega)\varepsilon_{si}^S(d,j,\omega)/b(N),\ \ U_{dj}^D(s,i,\omega) = a_j^D(i,\omega)\varepsilon_{dj}^D(s,i,\omega)/b(N),$$

$$U_{si}^S(0) = \varepsilon_{si}^S(0) \ \ \textit{and} \ \ U_{dj}^D(0) = \varepsilon_{dj}^D(0)$$

for $\omega \in W$. *The terms* $a_i^S(j,\omega)$ *and* $a_j^D(i,\omega)$ *are positive and deterministic and the random variables,* $\{\varepsilon_{si}^S(d,j,\omega)\}$, $\{\varepsilon_{dj}^D(s,i,\omega)\}$, $\{\varepsilon_{si}^S(0)\}$ *and* $\{\varepsilon_{dj}^D(0)\}$ *are all positive and independent. The four sets of random variables* $\{\varepsilon_{si}^S(d,j,\omega)\}$, $\{\varepsilon_{dj}^D(s,i,\omega)\}$, $\{\varepsilon_{si}^S(0)\}$ *and* $\{\varepsilon_{dj}^D(0)\}$ *are generated from corresponding four, possibly different, c.d.f. The term* $b(N)$ *is a positive constant that is increasing in N.*

The components $a_i^S(j,\omega)$ and $a_j^D(i,\omega)$ are systematic terms, whereas the corresponding "error" terms represent unobserved heterogeneity in preferences. These error terms may capture the effect of variables that are perfectly known by the individual agent and also the effect of unpredictable fluctuations (to the individual agent) in tastes. The rationale for the latter interpretation is that individuals may have difficulties with evaluating the precise value (to them) of the alternatives, and may, therefore, revise their evaluations depending on their psychological state of mind. However, it is important that the error terms remain unchanged sufficiently long to ensure that the matching algorithm is accomplished each time preferences are altered because otherwise, the matchings will not be stable. We may interpret the term $b(N)$ as representing the mean cost of search. The mean search cost will increase with market size (N) because the matching process in a large market will be costly and time-consuming. This cost factor ensures that when the market becomes large, the probability of remaining unmatched will be bounded away from zero.

Let $F_1^S(u \mid i, j, \omega)$ denote the c.d.f. of $\varepsilon_{si}^S(d, j, \omega)$, $F_1^D(u \mid j, i, \omega)$ the c.d.f. of $\varepsilon_{dj}^D(s, i, \omega)$, $F_0^S(u \mid i)$ the c.d.f. of $\varepsilon_{si}^S(0)$ and $F_0^D(u \mid j)$ the c.d.f. of $\varepsilon_{dj}^D(0)$.

Assumption 3

The c.d.f. $F_1^S(x \mid i, j, \omega)$, $F_1^D(x \mid j, i, \omega)$ *and the sequence* $\{b(N)\}$ *satisfy*

(i) $$\frac{x^\alpha(1-F_1^S(xt \mid i, j, \omega))}{1-F_1^S(t \mid i, j, \omega)} \underset{t\to\infty}{\to} 1, \qquad \frac{x^\alpha(1-F_1^D(xt \mid j, i, \omega))}{1-F_1^D(t \mid j, i, \omega)} \underset{t\to\infty}{\to} 1$$

and

(ii) $$\sqrt{N}(1-F_1^S(b(N) \mid i, j, \omega)) \underset{N\to\infty}{\to} 1 \text{ and } \sqrt{N}(1-F_1^D(b(N) \mid j, i, \omega)) \underset{N\to\infty}{\to} 1.$$

where $\alpha > 0$ *is a constant. Moreover,*

(iii) $$F_0^S(u \mid i) = F_0^S(u), \; F_0^D(u \mid j) = F_0^D(u),$$

(iv) $$E\varepsilon_{si}^S(0)^{-1} < \infty \text{ and } E\varepsilon_{dj}^D(0)^{-1} < \infty.$$

It follows from Extreme value theory that Assumption 3(i, ii) is equivalent to the property that the respective distribution functions of the error terms are in the domain of attraction of the extreme value distribution $\exp(-x^{-\alpha})$, $x > 0$, see for example Proposition 1.11 in Resnick (1987). The condition in Assumption 3(i, ii) is the weakest possible condition that assures that the weak limit of the maximum of independent random variables is a proper random variable defined on the positive real line. In our context we may, with no loss of generality, set $\alpha = 1$. This normalization corresponds to applying a suitable power transform of the utility functions. Recall that we are dealing with ordinal comparisons so that utilities are only unique up to a monotone transformation. Assumption 3 is essentially equivalent to a similar assumption made by Menzel (2015).

Assumption 4

The distributions of the utility of being single have the structure

$$F_0^S(u \mid i) = F_0^D(u \mid j) = \exp(-1/u)$$

for positive u.

Note that Assumption 4 implies Assumption 3(iv), namely that $E\varepsilon_{si}^S(0)^{-1} < \infty$ *and* $E\varepsilon_{dj}^D(0)^{-1} < \infty.$

Menzel (2015) assumes Assumption 4. To motivate this assumption he appeals to extreme value theory by assuming that there are a large number of «outside» elemental options with i.i.d. utilities and thus the most preferred one will have a utility that is the maximum of the utilities of the elemental options and therefore will be (asymptotically) extreme value distributed. However, even if this is the case, the distribution of the outside option may not be regularly varying at infinity with the same index $-\alpha$ as the distributions of the utilities over potential partners and contracts.[5] To ensure that the utility of the single option is regularly varying with the same index as the utilities of the respective matching options one needs to impose the restriction that the error terms of the single and the matching options have "similar" distributions. It is therefore of interest to develop a general framework that does not hinge on Assumption 4. Specifically, the outside option might involve a number of very different underlying alternatives. It is thus desirable to allow for general distributions of the utilities being single.

We are now ready to state the key result of this section. Let $\varphi_i^S(j,\omega)$ be the (equilibrium) asymptotic probability that a given supplier of type *i* shall be matched with some demander of type *j* at contract ω, $\varphi_j^D(i,\omega)$ the asymptotic probability that a given demander of type *j* shall be matched with some supplier of type *i*, $\varphi_i^S(0)$ the asymptotic probability that a supplier of type *i* shall remain single (self-matched), $\varphi_j^D(0)$ the asymptotic probability that a demander of type *j* shall remain single. Let $m_i^S(j,\omega)$ be the asymptotic size, divided by $\sqrt{N}$, of the set of demanders of type *j* that are available to a supplier of type *i* at contract ω in equilibrium and $m_j^D(i,\omega)$ the asymptotic size, divided by $\sqrt{N}$, of the set of

[5] See Resnick (1987), section 0.4 for a definition and discussion of regularly varying functions.

suppliers of type *i* that are available to a demander of type *j* at contract ω in equilibrium. In "finite" populations these sizes will depend on the particular supplier and demander but it follows from the next theorem that the corresponding asymptotic sizes do not. Also, let $NX_{ij}(\omega)$ denote the number of matches with contract ω where the suppliers are of type *i* and the demanders are of type *j*, which is given by

$$X_{ij}(\omega) = \sum_{s\in\Omega_i^S} \sum_{d\in\Omega_j^D} Y_{si}^S(d, j,\omega) Y_{dj}^D(s,i,\omega) / N.$$

Theorem 2

Assume that Assumptions 1 to 4 hold. Then, with $b(N) = \sqrt{N}$,

(4.3) $$\lim_{N\to\infty} \| Y_{si}^S(j,\omega) \| = m_i^S(j,\omega) \qquad \lim_{N\to\infty} \| Y_{dj}^D(i,\omega) \| = m_j^D(i,\omega)$$

and

(4.4) $$\lim_{N\to\infty} X_{ij}(\omega) = \lambda_i^S \varphi_i^S(j,\omega) = m_i^S(j,\omega) m_j^D(i,\omega)$$

with probability 1 where $m_i^S(j,\omega)$ *and* $m_j^D(i,\omega)$ *are positive deterministic terms. Moreover, the asymptotic choice probabilities are uniquely determined by the equations*

(4.5) $$\varphi_i^S(j,\omega) = \frac{a_i^S(j,\omega) m_i^S(j,\omega)}{1+\sum_k \sum_{v\in W} a_i^S(k,v) m_i^S(k,v)}, \quad \varphi_j^D(i,\omega) = \frac{a_j^D(i,\omega) m_j^D(i,\omega)}{1+\sum_k \sum_{v\in W} a_j^D(k,v) m_j^D(k,v)},$$

(4.6) $$\varphi_i^S(0) = \frac{1}{1+\sum_k \sum_{v\in W} a_i^S(k,v) m_i^S(k,v)}, \quad \varphi_j^D(0) = \frac{1}{1+\sum_k \sum_{v\in W} a_j^D(k,v) m_j^D(k,v)},$$

and

(4.7) $$m_i^S(j,\omega) = a_j^D(i,\omega) \varphi_j^D(0) \kappa \lambda_j^D, \quad m_j^D(i,\omega) = a_i^S(j,\omega) \varphi_i^S(0) \lambda_i^S.$$

Theorem 2 is a special case of Theorem 7 (Section 5.5). Theorem 2 implies that the normalized sizes of the equilibrium choice sets, $m_i^S(j,\omega)$ and $m_j^D(i,\omega)$ in a large market are approximately non-stochastic.

In other words, the choice probabilities given in Theorem 2 are the same as in the case where the choice sets were exogenously given and of sizes $m_i^S(j,\omega)\sqrt{N}$ and $m_j^D(i,\omega)\sqrt{N}$, respectively. As we shall see below, this property also holds in the more general case with general distributions of the utility functions and with the correlation between utilities across alternatives (potential partners). The interpretation is that when *N* is large the corresponding equilibrium choice sets also become large. Since these choice sets of potential partners are large essentially every level of attractiveness of potential candidates will be represented in the choice sets.

The structure of the relations (4.3) to (4.7) clearly show the separate "one-sided" discrete choice feature of the model, namely that conditional on the (equilibrium) choice sets determined by (3.5) and (3.6), the choice probabilities in large matching markets have the conventional structure of the Luce model (McFadden, 1973) as if suppliers and demanders make "one-shot" discrete choices with exogenous individual choice sets, provided the market is large. Only aggregate endogeneity (aggregate equilibrium conditions) given in (4.7) needs to be accounted for.

It is also of substantial interest that convergence is with probability one instead of the weaker concept of convergence in probability. Convergence with probability one means that if the population is sufficiently large it will always be true (almost surely) that the sizes of the choice sets divided by $\sqrt{N}$ are approximately equal to their corresponding limiting values $\{m_i^S(j,\omega), m_j^D(i,\omega)\}$, in contrast to the case of convergence in probability. Consider for example a number of matching "experiments" with large markets where the preferences are independent across experiments. If convergence were only in probability it could be the case that the asymptotic model would not necessarily hold in every experiment.

The next result, which is equivalent to Theorem 2, is very useful in the context of identification and estimation. In particular, it shows in a more explicit way how the asymptotic choice probabilities are determined.

Corollary 1

Assumptions 1 to 4 imply that

(4.7) $$\varphi_i^S(j,\omega) = \lambda_j^D \kappa a_i^S(j,\omega) a_j^D(i,\omega) \varphi_i^S(0) \varphi_j^D(0), \quad \varphi_j^D(i,\omega) = \varphi_i^S(j,\omega) \lambda_i^S / \kappa \lambda_j^D,$$

(4.8) $$\varphi_i^S(0) = \frac{1}{1+\sum_k c_{ik} \varphi_k^D(0) \lambda_k^D \kappa} \quad \textit{and} \quad \varphi_j^D(0) = \frac{1}{1+\sum_k c_{kj} \varphi_k^S(0) \lambda_k^S}$$

where

(4.9) $$c_{ij} = \sum_{v \in W} a_i^S(j,v) a_j^D(i,v).$$

The equations in (4.9) represent another way of expressing the aggregate equilibrium conditions. Once the asymptotic choice probabilities have been determined the asymptotic normalized sizes of the choice sets can be recovered from (4.7). Thus, equations (4.8) and (4.9) imply that when the preference parameters $a_i^S(j,\omega)a_j^D(i,\omega)$ have been determined then one can simulate new aggregate equilibrium solutions under alternative population sizes of suppliers and demanders in the respective observable groups. Alternatively, in case the preference parameters depend on policy variables (such as costs and taxes) one can apply the relations above to simulate the effect of changes in these policy variables.

It is also of interest to consider the conditional distribution of the realized contract given the realized matchings of suppliers and demanders in equilibrium. Let $g_{ij}(\omega)$ denote the probability that a particular contract ω is chosen by a supplier of type *i* and a demander of type *j*, given that they have realized a match with each other. From (4.8) we have that

(4.11) $$g_{ij}(\omega)=\frac{\varphi_i^S(j,\omega)}{\sum_{v\in W}\varphi_i^S(j,v)}=\frac{\varphi_j^D(i,\omega)}{\sum_{v\in W}\varphi_j^D(i,v)}=\frac{a_i^S(j,\omega)a_j^D(i,\omega)}{\sum_{v\in W}a_i^S(j,v)a_j^D(i,v)}.$$

The formula in (4.11) is interesting because it suggests that the agents' matching process can be viewed as if it were a two-stage decision: In the first stage the pair of agent decide whether to match with each other. At this stage, it is implicit that the contract is optimally chosen in the second stage. In the second stage, the matched pair chooses the contract by maximizing the Nash product $a_i^S(j,\omega)a_j^D(i,\omega)\xi(\omega),$ where $\xi(\omega),\omega\in W,$ are random error terms that are independent and extreme value (Fréchet) distributed, that is, with c.d.f. $\exp(-1/x), x>0.$ As is well known, the resulting choice probability

$$P(a_i^S(j,\omega)a_j^D(i,\omega)\xi(\omega)=\max_{x\in W} a_i^S(j,x)a_j^D(i,x)\xi(x))$$

is equal to the right-hand side of (4.11).

An interesting question is how fast the equilibrium choice probabilities converge to the corresponding asymptotic values. Dagsvik (2000) and Menzel (2015) report Monte Carlo simulation experiments that seem to suggest that the finite sample choice probabilities are rather close to the corresponding asymptotic ones for small and moderate market sizes.

4.2. Summary of the analytic approach

In this section, we give a brief summary of the analytic approach which should be considerably easier to follow than the rigorous proofs in Appendix A. For simplicity we treat the case with only one observable category on each side of the market. Let h^S and h^D be functions defined by

$$h^S(d,\underset{\sim}{y}^S)=\frac{1}{\sqrt{N}}\sum_{s=1}^{N}J_s^S(d,y_s^S) \quad \text{and} \quad h^D(s,\underset{\sim}{y}^D)=\frac{1}{\sqrt{N}}\sum_{d=1}^{M}J_d^D(d,y_d^D).$$

Recall that $\|Y_d^D\|\sqrt{N}$ is the number of suppliers in the equilibrium choice set Y_d^D of demander *d* and $\|Y_s^S\|\sqrt{N}$ is the number of demanders in the equilibrium choice set Y_s^S of supplier *s*. Recall that by (4.1) and (4.2) it follows that

(4.12) $$\|Y_d^D\|=h^S(d,\underset{\sim}{Y}^S) \quad \text{and} \quad \|Y_s^S\|=h^D(s,\underset{\sim}{Y}^D).$$

Note that since h^D and h^S are stochastic functions it follows that $\|Y_s^S\|$ and $\|Y_d^D\|$ become stochastic. For fixed $\|y_s^S\|$ and $\|y_d^D\|$ define

$$f^S(\|y_s^S\|)=\lim_{N\to\infty}\sqrt{N}EJ_s^S(d,y_s^S) \quad \text{and} \quad f^D(\|y_d^D\|)=\lim_{N\to\infty}\sqrt{N}EJ_d^D(d,y_d^D).$$

It follows from Lemma 6 in Appendix A that these limits exist and are finite. The reason why the expectations of $J_s^S(d,y_s^S)$ and $J_d^D(d,y_d^D)$ only depends on the respective sizes of the choice sets is that the random error terms in the utility functions are i.i.d. For any exogenous $\underset{\sim}{y}^S$ and $\underset{\sim}{y}^D$ we prove in Appendix A that for any positive δ

(4.13) $$|h^S(d,\underset{\sim}{y}^S)-N^{-1}\textstyle\sum_s f^S(\|y_s^S\|)|<\delta \quad \text{and} \quad |h^D(s,\underset{\sim}{y}^D)-N^{-1}\textstyle\sum_d f^D(\|y_d^D\|)|<\delta$$

for all $\underset{\sim}{y}^D$ and $\underset{\sim}{y}^S$ with probability 1 provided *N* is sufficiently large. Let m^S and m^D be determined by the equations

(4.14) $$m^S=f^D(m^D)\kappa \quad \text{and} \quad m^D=f^S(m^S)$$

where $\kappa=M/N$. Suppose for a moment that f^D and f^S were contraction mappings. In that case, one can write

(4.15a) $$|f^S(\|Y_s^S\|)-f^S(m^S)|\le c\,\big|\|Y_s^S\|-m^S\big|$$

and

(4.15b) $$|f^D(\|Y_d^D\|)-f^D(m^D)|\le c\,\big|\|Y_d^D\|-m^D\big|$$

for some c, $0 < c < 1.$ From (4.12), (4.13), (4.14) and (4.15a, b) it follows that for sufficiently large N,

(4.16a) $$\big| \| Y_d^D \| - m^D \big| = \big| h^S(d, \underset{\sim}{Y}^S) - f^S(m^S) \big| \le \big| h^S(d, \underset{\sim}{Y}^S) - N^{-1} \sum_s f^S(\| Y_s^S \|) \big|$$
$$+ N^{-1} \sum_s \big| f^S(\| Y_s^S \|) - f^S(m^S) \big| \le \max_{\underset{\sim}{y}^S} \big| h^S(d, \underset{\sim}{y}^S) - N^{-1} \sum_s f^S(\| y_s^S \|) \big|$$
$$+ N^{-1} \sum_s \big| f^S(\| Y_s^S \|) - f^S(m^S) \big| \le \delta + cN^{-1} \sum_s \big| \| Y_s^S \| - m^S \big|$$
$$\le \delta + c \max_s \big| \| Y_s^S \| - m^S \big|$$

and similarly

(4.16b) $$\big| \| Y_s^S \| - m^S \big| \le \delta + c \max_d \big| \| Y_d^D \| - m^D \big| .$$

When (4.16b) is inserted into (4.16a) it follows that

$$\max_d \big| \| Y_d^D \| - m^D \big| \le \delta + c\delta + c^2 \max_d \big| \| Y_d^D \| - m^D \big| .$$

Since c < 1 the latter equation implies that

(4.17) $$\max_d \big| \| Y_d^D \| - m^D \big| \le \frac{\delta + c\delta}{1 - c^2} = \frac{\delta}{1 - c}.$$

Since (4.17) holds for any $\delta > 0$ with probability 1 if N is sufficiently large it follows that $\| Y^S(d) \|$ converges towards m^D with probability 1 as the population size increases without bounds. Similarly, it follows that $\| Y_s^S \|$ converges towards m^S with probability 1. Accordingly, the equations in (4.6) can be interpreted as the asymptotic equilibrium conditions because in a large population $m^D \sqrt{N}$ $(m^S \sqrt{N})$ is the number of suppliers (demanders) available to a given demander (supplier) and $f^D(m^D)\sqrt{N}$ $(f^S(m^S)\kappa\sqrt{N})$ is the number of suppliers (demanders) that are available to a given demander (supplier), given their (equilibrium) choice sets. From Dagsvik (2000) and in more general cases covered in Appendix A we have that the equations above have a unique positive solution for m^S and m^D.

The proof given in Appendix A is more complicated than the outline above because it is $\log f^S$ and $\log f^D$ that are contraction mappings instead of f^S and f^D but the basic idea and logic of the proof is similar to the outline given in this section.

Similarly, we prove in Appendix A that X converges almost surely towards to the deterministic term $m^S m^D$. Since X can be interpreted as the fraction of suppliers that obtain a match it follows that in the limit this fraction becomes the probability that a supplier shall obtain a match, namely φ^S. Thus, we have that $\varphi^S = m^S m^D$.

Consider for example the special case where the utility functions are i.i.d. with Fréchet distributed error terms. In this case, it follows easily that

(4.18) $$f^S(m^S) = \frac{a^S}{1+a^S m^S} \quad \text{and} \quad f^D(m^D) = \frac{a^D}{1+a^D m^D}.$$

Since choice sets can be treated as if they were exogenous with deterministic sizes in large markets it follows that

(4.19) $$\varphi^S(0) = \frac{1}{1+a^S m^S} \quad \text{and} \quad \varphi^D(0) = \frac{1}{1+a^D m^D}$$

from which it follows, using (4.14) and (4.19) that

(4.20) $$m^S = \kappa a^D \varphi^D(0) \quad \text{and} \quad m^D = a^S \varphi^S(0).$$

If we insert for m^S and m^D given in (4.20) into (4.19) we obtain that

(4.21) $$\varphi^S(0) = \frac{1}{1+c\kappa\varphi^D(0)} \quad \text{and} \quad \varphi^D(0) = \frac{1}{1+c\varphi^S(0)}$$

where $c = a^S a^D$. Using (4.4) we obtain that $X = m^S m^D = \kappa c \varphi^S(0)\varphi^D(0)$, which shows that c can be recovered from the fractions of suppliers and demanders that remain unmatched (given κ). However, without further information one cannot separate a^S from a^D. It follows readily from the relations above that the probability that a supplier shall obtain a match is determined by the following quadratic equation (see Dagsvik, 2000).

$$(1-\varphi^S)(\kappa-\varphi^S)a_S a_M = \varphi^S$$

where $\varphi^S = 1-\varphi^S(0)$. Note that the constrained supply and demand probabilities that follow from (4.21), and the subsequent relations given above, are the same as in the case where the choice sets were exogenously given and of sizes $m^S\sqrt{N}$ and $m^D\sqrt{N}$. This feature also holds in the more general case with general distributions of the utility functions and with correlation between utilities across alternatives (potential partners).

5. Extensions

5.1. Interdependent preferences across suppliers and demanders

We now turn to the case where the utility functions are allowed to be interdependent. There are several and fundamental different types of interdependence. For example, utilities between suppliers and demanders may be correlated. Another type of interdependence arises when utilities across different potential partners are correlated. In this section, we consider the former case. To accommodate this extension we assume that the systematic terms of the utility functions depend on particular random effects in the following way, namely $a_i^S(j,\omega) = a_i^S(j,\omega;\eta_{sd}(i,j))$ and $a_j^D(i,\omega) = a_j^D(i,\omega;\eta_{ds}(j,i))$ where $\{\eta_{sd}(i,j)\}$ is an independent random effect that satisfies the following symmetry property, $\eta_{sd}(i,j) = \eta_{ds}(j,i).$ The interpretation of $\eta_{sd}(i,j)$ is as an affinity attribute that captures the latent attractiveness between supplier *s* and demander *d*.

Assumption 5

The utility functions have the multiplicative separable structure given in Assumption 2 but the systematic terms $a_i^S(j,\omega;\eta_{sd}(i,j))$ *and* $a_j^D(i,\omega;\eta_{sd}(i,j))$ *are allowed to depend on random effects* $\{\eta_{sd}(i,j)\}$ *that are independent of* $\{\varepsilon_{dj}^D(s,i,\omega)\}$, $\{\varepsilon_{si}^S(d,j,\omega)\}$, $\{\varepsilon_{si}^S(0)\}$ *and* $\{\varepsilon_{dj}^D(0)\}$. *Moreover,*

$E\varepsilon_{si}^S(0)^{-1} < \infty,\ E\varepsilon_{dj}^D(0)^{-1} < \infty,\ Ea_i^S(j,\omega;\eta_{sd}(i,j)) < \infty$ *and* $Ea_j^D(i,\omega;\eta_{sd}(i,j)) < \infty.$

We can now prove the following result.

Theorem 3

Suppose that Assumptions 1, 3, 4 and 5 hold, conditional on $\{\eta_{sd}(i,j)\}$. *Then (4.6) and (4.7) hold with* c_{ij} *replaced by* $\tilde{c}_{ij}$ *given by*

$$\tilde{c}_{ij} = \sum_{v\in W} E(a_i^S(j,v,\eta_{sd}(i,j))a_j^D(i,v,\eta_{sd}(i,j)))$$

where the expectation is taken with respect to the random effect. Moreover

$$\varphi_i^S(j,\omega) = \varphi_i^S(0)\varphi_j^D(0)E(a_i^S(j,\omega,\eta_{sd}(i,j))a_j^D(i,\omega,\eta_{sd}(i,j))).$$

The proof of Theorem 3 is given in Appendix A.

We shall now summarize the central idea behind Theorem 3. Note that in this case the utilities have a similar structure as in previous sections apart from the systematic terms which have the structure $a^S = a^S(\eta_{sd})$ and $a^D = a^D(\eta_{sd})$ where $\eta_{sd} = \eta_{ds}$ is a random effect that represents latent attractiveness between supplier *s* and demander *d*. Suppose for simplicity that the random effect is a discrete random variable with support being a finite set and with p.m.f. $g(\eta).$ Then it follows, similarly to the derivations above, that the respective probabilities of being self-matched are given by

(5.1) $$\varphi^S(0) = \frac{1}{1+\sum_k a^S(\eta(k))m^D(k)} \quad \text{and} \quad \varphi^D(0) = \frac{1}{1+\sum_k a^D(\eta(k))m^S(k)}$$

where $m^S(k)\sqrt{N}$ ($m^D(k)\sqrt{N}$) is the asymptotic size of the choice sets of suppliers (demanders) conditional on $\eta_{sd} = \eta(k).$ Similarly to (4.14) it follows that the asymptotic size of the choice set for demanders (suppliers) given the random effect η divided by $\sqrt{N}$ is given by

$$m^D(k) = \frac{a^S(\eta(k))}{1+\sum_r a^S(\eta(r))m^D(r)} = a^S(\eta(k))\varphi^S(0).$$

Hence, the expected normalized size of the choice sets are given by

(5.2) $$m^D(k) = g(\eta(k))a^S(\eta(k))\varphi^S(0) \quad \text{and} \quad m^D(k) = g(k)\kappa a^D(\eta(k))\varphi^D(0).$$

When inserting (5.2) into (5.1) we obtain the result in Theorem 3. Hence, the asymptotic formulas for the equilibrium choice probabilities are as before but with a different interpretation of *c*.

5.2. Matching with limited information about the market

The discussion above relies on the assumption that every agent in the market is able to rank order all the agents on the opposite side of the market. In many applications, this may be unrealistic. For example, in marriage and labor markets the suppliers (demanders) may have limited information about the population of demanders (suppliers) who may be located far away or may be in different social groups. To account for this phenomenon Menzel (2015) has proposed an interesting approach, similar to the two-stage process of Lee and Schwarz (2012). In his modified approach the matching process takes place in two stages. In the first stage, a subsample of suppliers and demanders meet (or gather information) at random and independently of the realized matching outcomes. In the second stage, the matching takes place within the random subsample of those who have met. In order to establish stability (equilibrium) in this case, Menzel (2015) has introduced a modified matching market where the demanders (suppliers) could meet all the potential suppliers (demanders), but where the utilities of potential partners in the original market who have not met are set equal to zero. This modified market continues to satisfy all the assumptions made for a matching game in Section 2.1, with just one exception: it violates the assumption that preferences must be strict. However, this feature does not create any problem for the stability analysis since all these potential partners with whom the agent did not meet are ranked below the "single" option. Thus, Theorem 1 will still be valid. That is, the existence of the stable matching and equilibrium choice sets is still guaranteed.

Note, however, that one cannot apply the same argument as above to generalize the results of Theorems 2 and 3 to the case with this type of two-stage matching. The reason for this is that the utilities of the agents will be perfectly correlated across the two sides of the market in the modified market due to the fact that if demander *d* and supplier *s* do not meet, they both will set their utility of (s, d) equal to zero. Menzel (2015) briefly discussed this problem and correctly pointed out how the aggregate equilibrium relations can be modified so as to hold also in the two-stage case. However, although his conjecture about the equilibrium relations in the two-stage game is correct he does not provide a formal proof of these relations.

In the following, using a slightly modified definition of the constrained demand and supply functions, we are able to extend Theorems 3 and 4 to the case with two-stage matching.

Assumption 6

The matching process is a two-stage matching game, with meeting probability π_{ij}.

It is easy to realize that this two-stage setting can be viewed as a special case of Theorem 4, obtained by letting $\eta_{sd}(i, j)$ be a binary random variable which has outcome 1 (corresponding to meeting) with probability π_{ij} and outcome zero (corresponding to not meeting) with probability $1-\pi_{ij}$ and where $a_i^S(j,\omega;0)=a_j^D(i,\omega;0)=0$ and $a_i^S(j,\omega;1)=a_i^S(j,\omega)$, $a_j^D(i,\omega;1)=a_j^D(i,\omega)$.

Corollary 2

Assume a setting with two-stage matching and that Assumptions 1 to 4 and Assumption 6 hold. Then

$$\varphi_i^S(j,\omega)=\lambda_j^D\kappa\pi_{ij}a_i^S(j,\omega)a_j^D(i,\omega)\varphi_i^S(0)\varphi_j^D(0),\quad \varphi_j^D(i,\omega)=\varphi_i^S(j,\omega)\lambda_i^S/\kappa\lambda_j^D,$$

$$\varphi_i^S(0)=\frac{1}{1+\sum_k c_{ik}\varphi_k^D(0)\lambda_k^D\kappa}\quad \textit{and}\quad \varphi_j^D(0)=\frac{1}{1+\sum_k c_{kj}\varphi_k^S(0)\lambda_k^S}$$

where

$$c_{ij}=\pi_{ij}\sum_{v\in W}a_i^S(j,v)a_j^D(i,v).$$

In some cases, the meeting probabilities may be inversely proportionate to the population size. Consider for example the case where only one side of the market makes the selection of the respective subsamples. Specifically, suppose the demanders make the selection and assume that a demander of type *j* selects a subsample of size n_{ji}. It seems reasonable to assume that n_{ji} is independent of the population size provided the population is not too small. Hence, under random sampling it follows that $\pi_{ij}=n_{ji}/N_i=n_{ji}\lambda_i^S/N$. Thus, when λ_i^S is constant it follows in this case that the meeting probabilities are inversely proportionate to *N*. It is interesting to note that in this case, that is, when π_{ij} is inversely proportionate to *N*, there is no need to introduce the constant $b(N)$ which was needed in Theorems 2, 3 and Corollary 1 to avoid that the asymptotic choice probabilities become degenerate. The intuition for this is that in the case of two-stage matchings the second stage population remains bounded when *N* increases.

In the next section, we consider a more general setting where utilities in the original market are allowed to be correlated across potential partners (but where utilities across the two sides of the market still remain independent). For simplicity, we only consider the original one stage setting. It is, however, straightforward to extend the following analysis to the settings discussed in Sections 5.1 and 5.2.

5.3. Interdependent preferences across observationally identical potential partners

In this section, we shall discuss the case where the utilities across different potential partners are correlated. Thus, we allow the utilities to be dependent on particular random effects that accommodate the preference interdependence between potential alternatives. However, preferences of agents on different sides of the market are independent. This extension may be important both theoretically and empirically because of the restrictive nature of the IIA condition (equivalent to ii type I extreme value distributed error terms). To this end let $a_i^S(j,\omega) = a_i^S(j,\omega; z_{si}^S(j,\omega))$ and $a_j^D(i,\omega) = a_j^D(i,\omega; z_{dj}^D(i,\omega))$ where $\{z_{si}^S(j,\omega), z_{dj}^D(i,\omega)\}$ are independent random effects. For example, $z_{si}^S(j,\omega)$ may be interpreted as representing latent attributes that are common to potential partners of type *j*, contract ω. The variable $z_{si}^S(j,\omega)$ may depend on supplier *s* of type *i* because different suppliers may value a given attribute differently.

Assumption 7

The utility functions have the multiplicative separable structure given in Assumption 2 but the systematic terms $a_i^S(j,\omega; z_{si}^S(j,\omega))$ *and* $a_j^D(i,\omega; z_{dj}^D(i,\omega))$ *are allowed to depend on random effects* $\{z_{si}^S(j,\omega), z_{dj}^D(i,\omega)\}$ *that are independent of* $\{\varepsilon_{dj}^D(s,i,\omega)\}$, $\{\varepsilon_{si}^S(d,j,\omega)\}$, $\{\varepsilon_{si}^S(0)\}$ *and* $\{\varepsilon_{dj}^D(0)\}$.

Moreover, $Ea_i^S(j,\omega; z_{si}^S(j,\omega)) < \infty$, $Ea_j^D(i,\omega; z_{dj}^D(i,\omega)) < \infty$, $E\varepsilon_{si}^S(0)^{-1} < \infty$ *and* $E\varepsilon_{dj}^D(0)^{-1} < \infty$.

Theorem 4

Assume that Assumption 1 and Assumptions 3 to 7 hold. Then

$$\lim_{N\to\infty} \| Y_j^D(s,i,\omega) \| = m_i^S(j,\omega), \quad \lim_{N\to\infty} \| Y_i^S(d,j,\omega) \| = m_j^D(i,\omega)$$

and

$$\lim_{N\to\infty} X_{ij}(\omega) = \varphi_i^S(j,\omega)\lambda_i^S = m_i^S(j,\omega) m_j^D(i,\omega)$$

with probability 1 where the asymptotic choice probabilities are given by

$$\varphi_i^S(j,\omega) = E\left\{\frac{a_i^S(j,\omega;z_{si}^S(j,\omega))m_i^S(j,\omega)}{1+\sum_k\sum_{v\in W} a_i^S(k,v;z_{si}^S(k,v))m_i^S(k,v)}\right\}, \quad \varphi_j^D(i,\omega) = \varphi_i^S(j,\omega)\lambda_i^S / \kappa\lambda_j^D,$$

$$\varphi_i^S(0) = E\left\{\frac{1}{1+\sum_k\sum_v a_i^S(k,v;z_{si}^S(k,v))m_i^S(k,v)}\right\}, \quad \varphi_j^D(0) = E\left\{\frac{1}{1+\sum_k\sum_v a_j^D(k,v;z_{dj}^D(k,v))m_j^D(k,v)}\right\}$$

and where $\{m_i^S(j,\omega), m_j^D(i,\omega)\}$ *are uniquely determined by the equations*

$$m_j^D(i,\omega) = E\left\{\frac{a_i^S(j,\omega;z_{si}^S(j,\omega))\kappa\lambda_j^D}{1+\sum_k\sum_{v\in W} a_i^S(k,v;z_{si}^S(k,v))m_i^S(k,\omega)}\right\}$$

and

$$m_i^S(j,\omega) = E\left\{\frac{a_j^D(i,\omega;z_{dj}^D(i,\omega))\lambda_i^S}{1+\sum_k\sum_v a_j^D(k,v;z_{dj}^D(k,v))m_j^D(k,v)}\right\}.$$

The expectation in the relations above is taken with respect to the random effects in the systematic parts of the utility functions. Theorem 4 follows from more general results which are proved in Appendix A.

We see immediately the similarity between Theorem 5 and Theorem 3, as the result in Theorem 5 follow by taking the expectation of the expressions that are analogous to the ones in (4.2) to (4.4) with respect to the random components of the systematic terms in the utility functions.

Next, we shall discuss the implications of a particular assumption about random effects in the utility functions.

Assumption 8

The utility functions have the structure

$$U_{si}^S(d,j,\omega) = a_i^S(j,\omega)^{1/\theta} z_{si}^S(j,\omega)\varepsilon_{si}^S(d,j,\omega)/N^{\theta/2}, \quad U_{dj}^D(s,i,\omega) = a_j^D(i,\omega)^{1/\theta} z_{dj}^D(i,\omega)\varepsilon_{dj}^D(s,i,\omega)/N^{\theta/2},$$

$$U_{si}^S(0) = z_{si}^S(0)\varepsilon_{si}^S(0) \text{ and } U_{si}^S(0) = z_{si}^S(0)\varepsilon_{si}^S(0)$$

where $\{\varepsilon_{si}^S(d,j,\omega), \varepsilon_{dj}^D(s,i,\omega), \varepsilon_{si}^S(0), \varepsilon_{si}^S(0)\}$ *are independent positive random variable with c.d.f.* $\exp(-1/x), x>0,$ *and* $\{z_{si}^S(j,\omega), z_{dj}^S(i,\omega), z_{si}^S(0), z_{dj}^D(0)\}$ *are independent and identically distributed* *random variables that are generated by a stable distribution*[6] *that is totally skew to the right and with index* θ, $\theta \le 1$.[7]

In Appendix B, we show that Assumption 8 implies that the joint distribution of the random error terms of the utility functions has the structure of a particular multivariate extreme value distribution (type I). Moreover, it follows from well-known results (McFadden, 1984) that the corresponding choice model has a nested multinomial logit structure. Moreover, if demanders *d* and *r* are of the same type, say they are of type *j*, then

$$Corr\left(\log U_{si}^{S}(d, j, \omega), \log U_{si}^{S}(r, j, \omega)\right) = 1 - \theta^{2}.$$

Similarly, if suppliers *s* and *k* are of same type (i) then

$$Corr\left(\log U_{dj}^{D}(s, i, \omega), \log U_{dj}^{D}(k, i, \omega)\right) = 1 - \theta^{2}.$$

Otherwise $U_{si}^{S}(d, j, \omega)$ and $U_{si}^{S}(r, k, \omega')$ are independent if $(d, j, \omega) \ne (r, k, \omega')$ and $U_{dj}^{D}(s, i, \omega)$ is independent of $U_{dj}^{D}(k, r, \omega')$ if $(s, i, \omega) \ne (r, k, \omega')$. Thus, Assumption 8 implies that the preferences of a given agent are correlated across potential candidates of the same type and given contracts, but independent across contracts and across agents of different types.

Theorem 5

Under Assumptions 1, 4, and 8 it follows, as $N \to \infty$, *that* $\| Y_j^{D}(s, i, \omega) \|$, $\| Y_i^{S}(d, j, \omega) \|$ *and* $X_{ij}(\omega)$ *converge to unique equilibrium values* $m_i^{S}(j, \omega)$, $m_j^{D}(i, \omega)$ *and* $m_i^{S}(j, \omega) m_j^{D}(i, \omega)$, *respectively, and the asymptotic choice probabilities are uniquely determined by*

$$\varphi_i^{S}(j, \omega) = (a_i^{S}(j, w) a_j^{D}(i, w) \varphi_i^{S}(0) \varphi_j^{D}(0) \lambda_i^{S} \lambda_j^{D} \kappa)^{1/(2-\theta)} / \lambda_i^{S},$$

$$\varphi_j^{D}(i, \omega) = \varphi_i^{S}(j, \omega) \lambda_j^{D} \kappa / \lambda_i^{S},$$

$$\varphi_i^{S}(0)^{1/(2-\theta)} = \frac{\left(\lambda_i^{S}\right)^{(1-\theta)/(2-\theta)}}{\left(\varphi_i^{S}(0) \lambda_i^{S}\right)^{(1-\theta)/(2-\theta)} + \sum_k \zeta_{ik}(\theta) \left(\varphi_k^{D}(0) \lambda_k^{D} \kappa\right)^{1/(2-\theta)}}$$

and

---

[6] Recall that the stable distribution follows from an extended version of the Central limit theorem. See Samorodnitsky and Taqqu (1994) for a description of stable distribution.

[7] The case where $\theta = 1$ corresponds to a degenerate stable distribution with all mass located at zero.

$$\varphi_j^D(0)^{1/(2-\theta)} = \frac{\left(\lambda_j^D\kappa\right)^{(1-\theta)/(2-\theta)}}{\left(\varphi_j^D(0)\lambda_j^D\kappa\right)^{(1-\theta)/(2-\theta)} + \sum_k \zeta_{kj}(\theta)\left(\varphi_k^S(0)\lambda_k^S\right)^{1/(2-\theta)}}$$

where

$$\zeta_{ij}(\theta) = \sum_{w\in W}(a_i^S(j,w)a_j^D(i,w))^{1/(2-\theta)}.$$

The proof of Theorem 5 is given in Appendix B.

From Theorem 5 it follows that the p.d.f. of the realized contract, given that the matched pair of supplier and demander are of type *i* and *j*, respectively, is given by

$$g_{ij}(\omega) = \frac{\varphi_i^S(j,\omega)}{\sum_{v\in W}\varphi_i^S(j,v)} = \frac{(a_i^S(j,\omega)a_j^D(i,\omega))^{1/(2-\theta)}}{\sum_{v\in W}(a_i^S(j,v)a_j^D(i,v))^{1/(2-\theta)}}.$$

which, similarly to (4.9), is equal to

$$P(a_i^S(j,\omega)a_j^D(i,\omega)\xi_i(j,\omega)^{2-\theta} = \max_{x\in W} a_i^S(j,x)a_j^D(i,x)\xi_i(j,x)^{2-\theta})$$

where $\xi_i(j,\omega),\ \omega\in W$ are independent with standard Fréchet c.d.f. For later reference it is of interest to consider the results of Theorem 6 in the limiting case when $\theta\to 0.$ In this limiting case, we have that, for any given contract; the agents are "almost" indifferent among potential partners of the same type.

Corollary 3

Under the assumptions of Theorem 6 the limiting choice probabilities as $\theta\to 0$ are given by

(5.1) $$\varphi_i^S(j,\omega) = \zeta_{ij}(0)\sqrt{\varphi_i^S(0)\varphi_j^D(0)\lambda_j^D\kappa/\lambda_i^S},$$

(5.2) $$\frac{1}{\sqrt{\varphi_i^S(0)}} - \sqrt{\varphi_i^S(0)} = \sum_k \zeta_{ik}(0)\sqrt{\varphi_k^D(0)\lambda_k^D\kappa/\lambda_i^D}$$

and

(5.3) $$\frac{1}{\sqrt{\varphi_j^D(0)}} - \sqrt{\varphi_j^D(0)} = \sum_k \zeta_{kj}(0)\sqrt{\varphi_k^S(0)\lambda_k^S/\kappa\lambda_j^D}$$

where

(5.4) $$\zeta_{ij}(0) = \sum_{w\in W}\sqrt{a_i^S(j,w)a_j^D(i,w)}.$$

5.4. The transferable case

Shapley and Shubik (1972), and Becker (1973) introduced the transferable utility assumption. In this case, it is assumed, upon a match between a supplier and demander, that a part of the utility of one of the agents in the pair is transferred to the other in order to compensate for participation in the match. Apart from Dagsvik (2000) and Menzel (2015), matching models are usually based on a version of the transferable utility assumption pioneered by Choo and Siow (2006). In our notation, Choo and Siow (2006) assume the following assumption about preferences:

$$U_{si}^S(d,j,\omega)=\alpha_{ij}\exp(\omega)\varepsilon_{si}^S(j),\quad U_{dj}^D(s,i,\omega)=\beta_{ji}\exp(-\omega)\varepsilon_{dj}^D(i),$$

$$U_{si}^S(0)=\varepsilon_{si}^S(0)\quad \text{and}\quad U_{dj}^D(0)=\varepsilon_{dj}^D(0).$$

The assumption above means that potential partners within the same observational groups are perfect substitutes. Provided the stochastic error terms are independent with c.d.f. $\exp(-1/x)$ for positive *x*, we show in Appendix C that the equilibrium relations of the model of Choo and Siow (2006) are determined by the relations[8]

(5.5) $$\varphi_i^S(j)=\sqrt{\alpha_{ij}\beta_{ji}\varphi_i^S(0)\varphi_j^D(0)\lambda_j^D\kappa/\lambda_i^S},$$

(5.6) $$\frac{1}{\sqrt{\varphi_i^S(0)}}-\sqrt{\varphi_i^S(0)}=\sum_k\sqrt{\alpha_{ik}\beta_{ki}\varphi_k^D(0)\lambda_k^D\kappa/\lambda_i^S}$$

and

(5.7) $$\frac{1}{\sqrt{\varphi_j^D(0)}}-\sqrt{\varphi_j^D(0)}=\sum_k\sqrt{\alpha_{kj}\beta_{jk}\varphi_k^S(0)\lambda_k^S/\lambda_j^D\kappa}.$$

It follows from Decker et al. (2013) that a solution of (5.6) and (5.7) exists and is unique. This result also follows from Theorem 4 of Dagsvik (2000). By inspection of the formulas in (5.5) to (5.7) we note that they are exactly the same as the ones in (5.1) to (5.4) given in Corollary 3, with $\alpha_{ij}=a_i^S(j)$ and $\beta_{ji}=a_j^D(i)$. As a result, the next theorem follows:

---

[8] Choo and Siow (2006) express the equilibrium relations of their model in a different way. Note also that here we use the matching probabilities $\varphi_i^S(j)$ while Choo and Siow (2006) used the actually number of matched individuals μ_{ij}. Noting the relations $\mu_{i0}=N_i\varphi_i^S(0)$, $\mu_{j0}=M_j\varphi_j^D(0)$ and $\mu_{ij}=N_i\varphi_i^S(j)$, we see immediately that these two equations are exactly the same.

Theorem 6

The transferable model by Choo and Siow (2006) is equivalent to the structure of a limiting case of the general non-transferable model without flexible contracts given in Corollary 3, obtained when the correlation between the taste-shifters of the potential partners within each observed category is close to one.

Theorem 6 shows that it is possible to interpret the model of Choo and Siow (2006) and similar models in the literature as a special case of our NTU framework.

An interesting question is whether the transferable model fits the data better or worse than the non-transferable model. Consider the case without contracts. In the transferable case, it follows from (5.5) that

$$c_{ij} = \alpha_{ij}\beta_{ji} = \frac{\varphi_i^S(j)^2\lambda_i^S}{\varphi_i^S(0)\varphi_j^D(0)\lambda_j^D\kappa}.$$

Equations (5.5) to (5.7) place no additional restrictions on the parameters $\{c_{ij}\}$. Similarly, in the non-transferable case it follows from Corollary 2 for the special case without contracts that

$$c_{ij} = \pi_{ij}a_i^S(j)a_j^D(i) = \frac{\varphi_i^S(j)}{\varphi_i^S(0)\varphi_j^D(0)\lambda_j^D\kappa}.$$

We state this result in the next corollary.

Corollary 4

In a single cross-section, the transferable and the non-transferable models both fit the data perfectly.

5.5. More general distributional assumptions

The results of Theorem 2 were obtained under Assumption 1 to 4. It is, however, possible to obtain analogous results under weaker assumptions, namely in the case where Assumption 4 is dropped.

Theorem 7

Assume that Assumptions 1, 2, 3 and 6 hold. Let

$$\psi_r(x) = \int_0^\infty \exp(-xv) F_0^r(1/v) dv,$$

for $r = S, D$. *Then the asymptotic choice probabilities are uniquely determined by the equations*

$$\varphi_i^S(j,\omega) = a_i^S(j,\omega) a_j^D(i,\omega) \psi_S(A_i^S) \psi_D(A_j^D) \lambda_j^D \kappa,$$

$$\varphi_i^S(0) = 1 - A_i^S \psi_S(A_i^S) \text{ and } \varphi_j^D(0) = 1 - A_j^D \psi_D(A_j^D),$$

where A_i^S *and* A_j^D *are uniquely determined by the equations*

$$A_i^S = \sum_k c_{ik} \psi_D(A_k^D) \lambda_k^D \kappa, \quad A_j^D = \sum_k c_{kj} \psi_S(A_k^S) \lambda_k^S,$$

and

$$c_{ij} = \pi_{ij} \sum_{v \in W} E(a_i^S(j, v, \eta(i,j)) a_j^D(i, v, \eta(i,j))).$$

for all i *and* j *and* $\omega \in W$.

The proof of Theorem 7 is implied by Lemma 7 which is proved in Appendix A. Thus, we have obtained that even in the case where we only make weak regular assumptions about the distributions of the utilities it is possible to obtain an analytic expression for the equilibrium choice probabilities. After introducing Assumption 4 we mentioned that similarly to Menzel (2015), one could, for example, use extreme value theory to motivate the assumption that utilities of the outside options are extreme value distributed in a more general way than in Assumption 4, namely, as $F_0^S(u \mid i) = \exp(-u^{-\alpha_1})$ and $F_0^S(u \mid j) = \exp(-u^{-\alpha_2})$, for positive u, α_1 and α_2.

The result of Theorem 7 can also be extended to allow for the type of random effects given in Assumption 7.

Similarly to the results obtained in previous sections Theorem 7 implies that one can analyze choice behavior in large two-sided matching markets as if each side of the market were stochastically independent with deterministic sizes of the equilibrium choice sets of available potential partners that only depend on the observed characteristics of the agents. However, the sizes of these choice sets are determined by particular aggregate equilibrium equations that depend on the distribution of preferences of the agents in the market as well as of the number of agents of each type in the market.

6. Estimation

In this section, we shall consider estimation procedures when different types of data are available. We shall base our discussion of the framework given in Section 5.2.

Case I: Information of the whole market is available

In this case, the analyst has data for the behavior of all agents in the market as well as the population sizes of each group of agents. And for each matched pair; we observe their types and realized contracts.

Recall that $X_{ij}(\omega)N$ is the number of matches with contract ω where the suppliers are of type *i* and the demanders are of type *j*, and let NX_{i0}^{S} and NX_{j0}^{D} be the number of single suppliers of type *i* and demanders of type *j*, respectively. When the population is large: $X_{ij}(\omega) \cong \lambda_i^S \varphi_i^S(j,\omega)$, $X_{i0}^S \cong \lambda_i^S \varphi_i^S(0)$ and $X_{j0}^D \cong \kappa \lambda_j^D \varphi_j^D(0)$. Hence, it follows from Corollary 3 that

$$(6.1) \qquad \frac{NX_{ij}(\omega)}{X_{i0}^S X_{j0}^D} \cong \pi_{ij} a_i^S(j,\omega) a_j^D(i,\omega).$$

The relation in (6.1) is convenient in an empirical context because it allows us to recover the structural parameters $\{\pi_{ij} a_i^S(j,\omega) a_j^D(i,\omega)\}$ from data on the number of realized matches of each type and the number of single suppliers and demanders of each type in a very simple way. Since the populations are large (6.1) will provide precise estimates of these parameters. We see immediately that only the product of the meeting probability and the individual mean utilities, $\pi_{ij} a_i^S(j,\omega) a_j^D(i,\omega)$, can be identified. Without further restrictions on the individual preferences or extra information, one cannot separately identify π_{ij} and the mean individual preferences $a_i^S(j,\omega)$ and $a_j^D(i,\omega)$. The empirical analysis of Dagsvik et al. (2001) is based on the special case of (6.1) when no flexible contract is available. Choo and Siow (2006) apply the analogous relation that follows from their transferable model.

Case II: A random sample of agents from one side of the market is available

In this case, we only observe a random sample from one side of the market (for example the supply side) but we know the fractions of each subgroup of agents $\{\lambda_i^S, \kappa\lambda_j^D\}$ and observe a sample of suppliers. For each supplier, we observe whether she or he is matched, and if matched the type of her or his partner and the realized contract.

For supplier *s* of type *i* in the sample define $D_{si}^S(j,\omega)=1,$ if the supplier is matched with a demander of type *j* at contract $\omega,$ and 0 otherwise. Similarly, let $D_{si}^S(0)=1,$ if the supplier is self-matched, and $NZ_{ij}(\omega)=\sum_s D_{si}^S(j,\omega),\ i,j>0,$ the number of suppliers of type *i* in the sample that are matched to a demander of type *j* at contract $\omega,$ NZ_{i0}^S the number of suppliers of type *i* that are self-matched. Also, let $v_{ij}(\omega)=\log a_i^S(j,\omega)+\log a_j^D(i,\omega)+\log \pi_{ij}.$ Hence, the loglikelihood function, in this case, can be written as

$$
\begin{aligned}
logL &= \sum_s \left(\sum_\omega \sum_j D_{si}^S(j,\omega)\log\varphi_i^S(j,\omega) + D_{si}^S(0)\log\varphi_i^S(0) \right) \\
&= \sum_\omega \sum_{i,j} Z_{ij}(\omega)\log\varphi_i^S(j,\omega) + \sum_i Z_{i0}^S \log\varphi_i^S(0).
\end{aligned} \tag{6.2}
$$

Due to Corollary 2 the equation in (6.2) can be written as

$$
logL = \sum_{i,j}\sum_{\omega\in W} Z_{ij}(\omega)v_{ij}(\omega) + \sum_i \lambda_i^S \log\varphi_i^S(0) + \sum_{i,j}\sum_{\omega\in W} Z_{ij}(\omega)\log\varphi_j^D(0)
$$

$$
\begin{aligned}
&= \sum_{i,j}\sum_{\omega\in W} Z_{ij}(\omega)v_{ij}(\omega) - \sum_i \lambda_i^S \log\left(1+\sum_k\sum_{z\in W}\exp(v_{ik}(z))\varphi_k^D(0)\lambda_k^D\kappa\right) \\
&+ \sum_{i,j}\sum_{\omega\in W} Z_{ij}(\omega)\log\varphi_j^D(0).
\end{aligned} \tag{6.3}
$$

Case III: A random sample of suppliers and demanders is available

The difference between case II and case III is that in case II we do not observe the number of demanders of each type that are self-matched, in contrast to case III. As in case II we observe the population fractions, In this case, it follows, similarly to (6.3), that

$$
logL = \sum_\omega \sum_{i,j} Z_{ij}(\omega)\log\varphi_i^S(j,\omega) + \sum_i Z_{i0}^S \log\varphi_i^S(0) + \sum_j Z_{j0}^D \log\varphi_j^D(0) \tag{6.4}
$$

$$
= \sum_\omega \sum_{i,j} Z_{ij}(\omega)v_{ij}(\omega) + \sum_i \lambda_j^S \log\varphi_i^S(0) + \sum_j \kappa\lambda_j^D \log\varphi_j^D(0)
$$

$$= \sum_{i,j} \sum_{\omega \in W} Z_{ij}(\omega) v_{ij}(\omega) - \sum_i \lambda_i^S \log\left(1 + \sum_k \sum_{z \in W} \exp(v_{ik}(z)) \varphi_K^D(0) \kappa \lambda_k^D\right)$$
$$+ \sum_j \kappa \lambda_j^D \log \varphi_{jD}(0).$$

Due to the nonlinearity of the restrictions in Corollary 3 it may be cumbersome to maximize (6.3) or (6.4) by a direct approach. One and alternatively use an iterative approach to compute the likelihood function. Let *n* be the n-th step in the iteration procedure of computing the probabilities of being self-matched and define

$$\varphi_{j,n+1}^D(0) = \frac{1}{1 + \sum_k \sum_{z \in W} \exp(v_{kj}^n(z)) \varphi_{k,n}^S(0) \lambda_k^S} \quad \text{and} \quad \varphi_{i,n}^S(0) = \frac{1}{1 + \sum_k \sum_{z \in W} \exp(v_{ik}^n(z)) \varphi_{k,n}^D(0) \kappa \lambda_k^D},$$

where $v_{ij}^n(\omega)$ denotes the estimate of $v_{ij}(\omega)$ obtained at the n-th step. As starting values of the demander probabilities of being self-matched one can use the fractions of self-matched demanders.

7. Measures of welfare and gain from a match

In this section, we shall discuss measures of welfare. For simplicity, we only consider the two-stage case of Section 5.2 in the special case with no flexible contracts. There are several ways of computing welfare measures in random utility models. One way is to compute Compensating Variation (CV) or Equivalent Variation (EV) measures derived from the individual random utility formulation. Recall, however, that one cannot identify preferences of suppliers and demanders separately without further assumptions or data on preference rankings. Consequently, separate welfare measures for suppliers (demanders) cannot be evaluated without additional data or theory. In the case where money is not involved, it is not possible to compute money metric welfare measures. However, it is possible to compute some measures of total gain for the pair of getting matched.

The first measure we consider for the total gain of a match is

(7.1) $$E(\log U_{si}^S(d,j)) + E(\log U_{dj}^D(s,i)) = \log a_i^S(j) + \log a_j^D(i) + \log \pi_{ij}$$

The formula in (7.1) expresses the total mean utility of the suppliers of type *i* for being matched to a demander of type *j*, and demanders of type *j* of being matched to a supplier of type *i*. It can be computed because $\{a_i^S(j) a_j^D(i) \pi_{ij}\}$ is identified and can thus be recovered from data. This measure is similar to the one proposed by Choo and Siow (2006). Recall that the systematic terms of the utility functions for the single options are normalized to one.

Next, consider an analogous measure which is based on the highest utility an agent can achieve under equilibrium. From the assumptions in Section 4, it follows asymptotically that

(7.2) $$P(\log(\max_{d\in\Omega_j^D}(U_{si}^S(d,j)Y_{dj}^D(s,i)))\le u)=\exp(-a_i^S(j)m_i^S(j)\exp(-u))$$

$$=\exp(-\pi_{ij}a_i^S(j)a_j^D(i)\varphi_j^D(0)\kappa\lambda_j^D\exp(-u)).$$

Consequently, using Corollary 3 it follows that

(7.3) $$P(\log(\max_j\max_{d\in\Omega_j^D}(U_{si}^S(d,j)Y_{dj}^D(s,i)))\le u)=\exp(-\sum\nolimits_j a_i^S(j)m_i^S(j)\exp(-u))$$

$$=\exp(-\sum\nolimits_j \pi_{ij}a_i^S(j)a_j^D(i)\varphi_j^D(0)\kappa\lambda_j^D\exp(-u))=\exp\left(-\frac{1-\varphi_i^S(0)}{\varphi_i^S(0)}\cdot\exp(-u)\right).$$

From (7.3) we get that (apart from an additive constant)

(7.4) $$E(\log(\max_j\max_{d\in\Omega_j^D}(U_{si}^S(d,j)Y_{dj}^D(s,i)))=\log\left(\frac{1-\varphi_i^S(0)}{\varphi_i^S(0)}\right).$$

The interpretation of the expression in (7.4) is as the mean of the highest utility suppliers of type *i* can attain from a match with available demanders of type *j*. It is of interest to consider the interpretation of $\log\max_{d\in\Omega_j^D}(U_{si}^S(d,j)Y_{dj}^D(s,i))$ further. This is the highest utility supplier *s* can obtain. As discussed by Roth and Sotomayor (1990), this value depends on the actual matching rules. For example, given the Deferred-Acceptance algorithm with suppliers making the offers, it is the highest utility supplier *s* that can be attained among all matching algorithms that produce stable matchings. In contrast, it is the lowest value supplier *s* will attain among all algorithms that produce stable matchings in the case where the demanders are making the offers. Similarly, it follows that

(7.5) $$E(\log(\max_i\max_{s\in\Omega_i^S}(U_{dj}^D(s,i)Y_{si}^S(d,j))\mid d\in\Omega_j^D)=\log\left(\frac{1-\varphi_j^D(0)}{\varphi_j^D(0)}\right).$$

The expressions in (7.4) and 7.5) show, however, that the mean indirect utility of being matched, expressed as the logarithm of the odds ratio between the probability of obtaining a match and the probability of remaining unmatched, is independent (asymptotically) of the matching algorithm. Note that it is implicit in (7.4) that supplier *s* is choosing her or his most preferred partner from the (endogenous) set of available potential partners. As we see from the formulas above, these results stem from the fact that the asymptotic equilibrium choice probabilities are independent of the actual matching rules. In finite populations, this might not be true.

The welfare measures given above can be used to make ordinal comparisons of the welfare effect from different reforms, or from changes in the fractions of suppliers or demanders in specific population groups. Due to fact that our model is an equilibrium one it might not be clear, a priori, how different regimes compare in terms of welfare because the two sides of the market may have conflicting interests.

In the case where money is involved, it is also possible to compute money metric welfare measures such as Compensating Variation (CV) and Equivalent Variation (EV).

8. Conclusion

In this paper, we have analyzed a non-transferable structural equilibrium framework suitable for empirical analysis of matching markets. Similarly to Dagsvik (2000), our approach is based on the notion of constrained supply and demand, conditional on the choice set of potential partners. This setup does not mean that the agents are assumed to know their equilibrium choice sets. On the contrary, this setup is to be interpreted as an "as if" rationale and serves only as an analytic device that turns out to be convenient for obtaining aggregate analytic results.

We have demonstrated that one can obtain aggregate equilibrium relations for the resulting number of matched pairs (and singles) under rather weak assumptions about the agents' utility functions. This includes the cases where an agent's utilities are correlated across alternatives (potential partners) and where a supplier's (demander's) utilities of a supplier and a demander are correlated. A special case of the latter representation allows for agent-pair-specific random submarkets (interview stage), consistent with agents meeting potential partners at random in a first stage, and subsequently participate in the matching game in a second stage within the submarkets obtained in the first stage. We believe that this two-stage model extension is very important because it increases the realism of the model dramatically. Another new and particularly interesting finding is that the transferable matching model can be interpreted as a limiting case of a corresponding non-transferable model with correlated preferences, as the correlation between an agent's utilities of different potential partners of the same type approaches one.

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Appendix A

Lemmas and proofs

Proof of Proposition 1

Consider the modified Deferred-Acceptance algorithm. At each step, at least one offer is made by some demander; otherwise, the algorithm will stop at this stage. Since there is only a finite set of possible offers, it follows that this algorithm will stop after a finite number of stages. It is obvious that no single agent will block the matching since the algorithm ensures that only those offers ranked higher than the single option will be accepted by the agent. It remains to show that there is no any combination (s,d,ω) that will block the matching. Assume this is not true, and that at the end of game supplier *s* is matched to demander $\bar{d}$ at contract $\bar{\omega},$ demander *d* is matched to the supplier $\tilde{s}$ at contract $\tilde{\omega},$ but supplier *s* and demander *d* both prefer each other at contract ω to their current matching. Thus, supplier s ranks combination (d,ω) higher than $(\bar{d},\bar{\omega})$ and demander *d* ranks combination (s,ω) higher than $(\tilde{s},\tilde{\omega}).$ Since *d* ranks (s,ω) higher than $(\tilde{s},\tilde{\omega}),$ it must be true that *d* has offered ω to *s* during the matching game. Since *d* is not matched to *s* at contract $\omega,$ it must be the case that *s* has rejected his offer at some previous step, which means that *s* had a better offer than $(d,\omega).$ This means that s prefers $(\bar{d},\bar{\omega})$ to (d,ω), which is a contradiction. Assume next that *s* and *d* are matched to one another at contract ω but both prefer contract $\omega'.$ Then demander *d* must have offered contract ω' to supplier *s* at some previous step and supplier *s* must have rejected this offer at that step. Hence, this case also leads to a contradiction. Thus, we have proved the result stated in Proposition 1.

Q.E.D.

Proof of Theorem 1

It is obvious that if we can prove (ii), (i) follows readily. So, in the following, we will provide only the proof for (ii).

Proof of (i):

Since preferences are strict, it follows from Proposition 1 that stable matchings exist. For any given stable matching, let $\mu(k)$ denote a vector function that matches agent *k* to his chosen combination of supplier and contract, i.e., $\mu(d)=(\mu_1(d),\mu_2(d))$ where $\mu_1(d)$ is the supplier demander *d* is matched to and $\mu_2(d)$ the corresponding contract. If demander d is self-matched (remaining single) we define $\mu(d)=(0,0).$ We claim that the following definition yields equilibrium choice sets: Let $Y_s^S(d,\omega)=0$

if $U_s^S(d,\omega) \geq U_s^S(\mu(s))$ and $Y_s^S(d,\omega)=1$ if $(d,\omega)=\mu(s)$. Consider next the case where $U_s^S(d,\omega)<U_s^S(\mu(s))$, Set $Y_s^S(d,\omega)=0$ if and only if *d* prefers $\mu(d)$ to (s,ω). Otherwise, set $Y_s^S(s,\omega)=1$. The indicator function $Y_d^D(s,\omega)$ is defined in a similar way.

Now it remains to show that the choice sets given by this construction satisfy (3.3) and (3.4).
*Case A***:** Supplier *s* and demander *d* are matched together with contract ω in the stable matching. i.e $\mu(d)=(s,\omega)$ and $\mu(s)=(d,\omega)$. From the analysis above we know that $Y_s^S(d,\omega)=1$. On the other hand, we also know from the analysis above that an alternative (s^*,ω^*) which is ranked above $\mu(d)=(s,\omega)$ by demander *d* is not in *d*'s choice set, i.e. $Y_d^D(s^*,\omega^*)=0$. By definition (3.2), this implies that $J_d^D(s,\omega,Y_d^D))=1$, implying that (3.4) holds. To verify that (3.3) is done in a similar way.

Case B: Supplier *s* and demander *d* are not matched together with contract ω in the stable matching. i.e., $\mu(d)\neq\mu(s)$. If $U_s^S(d,\omega)\geq U_s^S(\mu(s))$ then by definition of stable matching, we must have $U_d^D(s,\omega)<U_d^D(\mu(d))$. Otherwise, both supplier *s* and demander *d* will be better off by switching to the match (s,d,ω). Hence, $Y_s^S(d,\omega)=0$. On the other hand, since we have $U_d^D(s,\omega)<U_d^D(\mu(d))$ this implies that $J_d^D(s,\omega,Y_d^D))=0$. Thus, (3.4) is satisfied in this case.

Consider next the case where $U_s^S(d,\omega)<U_s^S(\mu(s))$ and $U_d^D(s,\omega)>U_d^D(\mu(d))$. In this case it follows that $Y_s^S(d,\omega)=1$. Furthermore, it follows that $J_d^D(s,\omega,Y_d^D)=1$ because $U_d^D(s,\omega)>U_d^D(\mu(d))$ and the fact that an alternative (s,ω) that is ranked above $\mu(d)$ by demander *d* is not in *d*'s choice set. Hence, (3.4) is satisfied also in this case.

Consider next the case where $U_s^S(d,\omega)<U_s^S(\mu(s))$ and $U_d^D(s,\omega)<U_d^D(\mu(d))$. In this case we have that $Y_s^S(d,\omega)=0$. Furthermore, it follows from the inequality $U_d^D(s,\omega)<U_d^D(\mu(d))$ that $J_d^D(s,\omega,Y_d^D))=0$ and the fact that $\mu(d)$ is always in *d*'s choice set. Thus, (3.4) is satisfied. Similarly, we can prove that (3.3) also is satisfied by our above construction of choice sets.

Proof of (ii):
Assume that Y_s^S and Y_d^D satisfy (3.3) and (3.4). Suppose furthermore that the most preferred option of supplier *s* within her choice set is a match with *d* at ω, i.e., $J_s^S(d,\omega,Y_s^S)=1$ and $Y_s^S(d,\omega)=1$. Since $Y_s^S(d,\omega)=1$ (3.4) implies that $J_d^D(s,\omega,Y_d^D)=1$. Thus, we have demonstrated that if *s* prefers a match

with d at contract ω, then (3.3) and (3.4) imply that d prefers a match with s at ω as well. In other words, if the agents' choices satisfy (3.3) and (3.4) a matching is induced.

It remains to show that this matching also is *stable*. First, we claim that under (3.3) and (3.4) there is no option that blocks the matching. Suppose for a moment that this is not true and that (s, d, ω) blocks the matching. Then obviously (d, ω) cannot be in the choice set of supplier s because otherwise, s would have formed a match with d at ω already since we have shown above that rational behavior will always induce a match between the agent and the best alternative within his equilibrium choice set. By (3.4), $Y_s^S(d, \omega) = 0$ implies that $J_d^D(s, \omega, Y_d^D) = 0.$ which means that d will not prefer (s, ω) to his current option which is the best alternative within the choice set of demander d. Therefore (s, d, ω) cannot block the matching. Second, it is easy to see that no agent that is matched will block the matching induced by (3.3) and (3.4) since being matched under (3.3) and (3.4) will guarantee that the self-matched option is not preferred.

Q.E.D.

Lemma 1

Let x be a non-negative real number, n be a non-negative integer and a and b positive real numbers such that b < a < 1. Then

$$| a^x - b^x | \leq \frac{| a - b | a^{-1/\log a}}{-\log a}.$$

Proof of Lemma 1:

Proof of (i): Then

$$a^x - b^x = (a - b)(\sum_{k=1}^{x} a^{x-k} b^k) < (a - b) x a^x.$$

For real x the function xa^x attains its maximum for $x = -1/\log a$ which implies that $xa^x \leq -a^{-1/\log a} / \log a \equiv c.$ Hence, we have proved the lemma.

Q.E.D.

Lemma 2

Let $Z_j, j = 1, 2, ..., n,$ *be independent binary random functions with realizations in {0,1}, and with* $EZ_j = p_j$. *Then*

$$E\left(\sum_{j=1}^{n}(Z_j - p_j)\right)^{2m} < \sum_{k=1}^{m} K_k n^k \overline{p}^k$$

where $K_1, K_2, \ldots,$ *are constants that do not depend on* $\{p_j\}$ *and*

$$\overline{p} = \sum_{j=1}^{n} p_j / n.$$

Proof of Lemma 2:

Let s be a real number and define

$$\psi(s) = E\exp\left(s\sum_{j=1}^{n}(Z_j - p_j)\right).$$

We have that

$$E\exp\left(s\sum_{j=1}^{n} Z_j\right) = \prod_{j=1}^{n}(1 - p_j + p_j e^s)$$

from which it follows that

$$\psi(s) = \prod_{j=1}^{n} g_j(s).$$

where

$$g_j(s) = (1 - p_j)e^{-sp_j} + p_j e^{s(1-p_j)}.$$

Note that

$$\psi^{(r)}(0) = E\left(\sum_{j=1}^{n}(Z_j - p_j)\right)^r.$$

Let $f_j(s) = \log g_j(s).$ Then we get that

(A.1) $$\psi'(s) = \psi(s)\sum_{j=1}^{n} f_j'(s).$$

From (A.1) we have that

(A.2) $$\psi^{(r+1)}(s) = \sum_{k=0}^{r}\left(\psi^{(k)}(s)\binom{r}{k}\sum_{j=1}^{n} f_j^{(r+1-k)}(s)\right),$$

for $r \geq 1.$ It follows readily that

$$g_j^{(r)}(0) = p_j(1 - p_j)((1 - p_j)^{r-1} - (-p_j)^{r-1})$$

which implies that

(A.3) $$g_j^{(r)}(0) \leq p_j(1 - p_j) \leq p_j.$$

Furthermore

$$g_j^{(r+1)}(s) = \sum_{k=0}^{r} \binom{r}{k} g_j^{(k)}(s) f_j^{(r+1-k)}(s) = f_j^{(r+1)}(s) g_j(s) + \sum_{k=1}^{r} \binom{r}{k} g_j^{(k)}(s) f_j^{(r+1-k)}(s)$$

which implies that

(A.4) $$| f_j^{(r+1)}(0) | \le g_j^{(r+1)}(0) + \frac{1}{4} \sum_{k=1}^{r} \binom{r}{k} | f_j^{(r+1-k)}(0) |$$

for $r \ge 1$. We have that

$$f_j'(0) = g_j'(0) = 0, \quad f_j''(0) = g_j''(0) \le p_j.$$

Now suppose there exist constants $\{b_k\}$ such that $| f_j^{(k)}(0) | \le b_k p_j$ for k = 1, 2, …,r. This is true for r = 1 and 2. Then it follows from (A.4) that

$$| f_j^{(r+1)}(0) | \le g_j^{(r+1)}(0) + \frac{1}{4} \sum_{k=1}^{r} \binom{r}{k} | f_j^{(r+1-k)}(0) | \le p_j + \frac{1}{4} \sum_{k=1}^{r} \binom{r}{k} b_k p_j = (1 + \frac{1}{4} \sum_{k=1}^{r} \binom{r}{k} b_k) p_j.$$

Hence, with

$$b_{r+1} = 1 + \frac{1}{4} \sum_{k=1}^{r} \binom{r}{k} b_k$$

it follows that

(A.5) $$| f_j^{(r+1)}(0) | \le b_{r+1} p_j$$

Suppose next that the claim of the Lemma is true for a given m. We shall prove that it holds for $m+1$. We know that it is true for m = 2 because in this case

$$E(\textstyle\sum_{j=1}^{n} (Z_j - p_j))^2 = \displaystyle\sum_{j=1}^{n} VarZ_j = \sum_{j=1}^{n} p_j (1 - p_j) \le n\overline{p}.$$

Since $f_j'(0) = 0$ it follows from (A.2) and (A.5) that

$$\psi^{2(m+1)}(0) = \sum_{k=0}^{2m} \psi^{(k)}(0) \binom{2m}{k} \sum_{j=1}^{n} f_j^{(2m+2-k)}(0) \le \sum_{k=0}^{2m} \psi^{(k)}(0) \binom{2m}{k} \sum_{j=1}^{n} b_{2m+2-k} p_j$$

$$= \sum_{k=0}^{2m} \psi^{(k)}(0) \binom{2m}{k} b_{2m+2-k} n\overline{p} \le \sum_{k=0}^{2m} \sum_{r \le k/2} K_r (n\overline{p})^{r+1} \binom{2m}{k} b_{2m+2-k} = \sum_{r \le m} (n\overline{p})^{r+1} \left(K_r \sum_{k \ge 2r}^{2m} \binom{2m}{k} b_{2m+2-k} \right).$$

The last expression above is a polynomial in $n\overline{p}$ of degree m +1. Hence, the lemma follows by recursion.

Q.E.D.

Before we continue we need to introduce some additional notation. Let [x] denote the ceiling function, that is, the smallest integer that is greater than or equal to x. Let Δ_n be the set of n-dimensional vectors with components that are either zero or one and let C be a set with elements that are non-negative variables. Define $\| C \|$ as the sum of the variables in *C* divided by the square root of the number of variables in *C*.

Lemma 3

Let $J_s(d,y),\ y \in \Delta_n,\ d = 1,2,...,[n\kappa],\ s = 1,...,n,$ *be binary random variables with realizations in* $\{0,1\}$ *with* $EJ_s(d,y) = p_n(\| y \|),$ *where* $p_n(\| y \|)\sqrt{n} < h(\| y \|)$ *for some function h for all n and* $y \in \Delta_n,\ h(\| y \|) \to 0$ *as* $\| y \| \to \infty$ *and* κ *is a positive constant. Moreover, the variables* $J_s(d,y)$ *and* $J_r(d,z)$ *are independent when* $(s,z) \neq (r,y).$ *Let* $\underset{\sim}{y} = (y(1), y(2),..., y(n)),$ *for* $y(k) \in \Delta_n$ *and* $\underset{\sim}{\Delta}_n = \{\underset{\sim}{y} : y(k) \in \Delta_n, k = 1,2,..,n\}.$ *Then*

$$P\left(\max_{\underset{\sim}{y} \in \underset{\sim}{\Delta}_n} \max_{d \le [n\kappa]} \frac{1}{\sqrt{n}} \left| \sum_{s=1}^{n} (J_s(d, y(s)) - p_n(\| y(s) \|) \right| \underset{n\to\infty}{\to} 0 \right) = 1.$$

Proof of Lemma 3:

Let $\varepsilon > 0,\ 0 < \delta < \varepsilon / 2,\ \hat{p}_n(\| y \|) = \max(p_n(\| y \|), \delta)$ for $y \in \Delta_n$ and let β be defined such that $\sqrt{n}\, p_n(\| y \|) \le \delta$ when $\| y \| > \beta.$ This is possible due to the assumptions. Let

$$Z_n(d, \underset{\sim}{y}) = \frac{1}{\sqrt{n}} \sum_{s=1}^{n} (J_s(d, y(s)) - p_n(\| y(s) \|))$$

and

$$\hat{Z}_n(d, \underset{\sim}{y}) = \frac{1}{\sqrt{n}} \sum_{s=1}^{n} (J_s(d, y(s)) - \hat{p}_n(\| y(s) \|)).$$

Since $p_n(\| y(k) \|)\sqrt{n} < h(\| y(k) \|),$ for $y(k) \in \Delta_n$ it follows from Lemma 2 that for some suitable constants $\tilde{h}$ and h^*

(A.6) $$EZ_n(d, \underset{\sim}{y})^{12} < \sum_{r=1}^{6} K_r n^r\, \overline{p}_n(\underset{\sim}{y})^r n^{-6} < \sum_{r=1}^{6} K_r \tilde{h}^r n^{0.5r-6} < \frac{h^*}{n^3}$$

where

$$\overline{p}_n(\underset{\sim}{y}) = n^{-1} \sum_{k=1}^{n} p_n(\| y(k) \|).$$

Hence, for any $\varepsilon > 0,$ it then follows from (A.6) and Chebyshev's inequality that

(A.7) $$P\left(| X_n(d, \underset{\sim}{y}) | > \varepsilon \right) \le \frac{EX_n(d, \underset{\sim}{y})^{12}}{\varepsilon^{12}} \le \frac{h^*}{n^3 \varepsilon^{12}}.$$

Note furthermore that $1-p_n(\|y(k\|)$ and $-p_n(\|y(k)\|),$ can each attain at most $[\beta\sqrt{n}]$ different values when $\|y(k)\|\le\beta.$ Hence, $1-\hat{p}_n(\|y(k\|)$ and $-\hat{p}_n(\|y(k)\|)$ can attain at most $[\beta\sqrt{n}]+1.$ This implies that $|\hat{X}_n(d,\underset{\sim}{y})|$ can attain at most $2n[\beta\sqrt{n}]+2n$ different values. Note that

$$2n[\beta\sqrt{n}]+2n<2n(\beta\sqrt{n}+1)+2n=2n\sqrt{n}(\beta+2/\sqrt{n})\le 2(\beta+2)n\sqrt{n}.$$

Note also that if B_j, j = 1, 2,…, are different events the following inequality holds

$$P\left(\bigcup_j B_j\right)\le\sum_{j\in K}P(B_j)\le|K|\max{}_j P(B_j) \tag{A.8}$$

where K is the index set that corresponds to different B_j and $|K|$ is the number of elements in K.

Note that

$$\frac{1}{\sqrt{n}}|\sum_{s=1}^{n}(\hat{p}_n(\|y(s)\|)-p_n(\|y(s)\|))|\le\delta.$$

Since

$$Z_n(d,\underset{\sim}{y})=\hat{Z}_n(d,\underset{\sim}{y})+\frac{1}{\sqrt{n}}\sum_{s=1}^{n}(\hat{p}_n(\|y(s)\|)-p_n(\|y(s)\|))$$

it follows readily that

$$|\hat{Z}_n(d,\underset{\sim}{y})|-\delta\le|Z_n(d,\underset{\sim}{y})|\le|\hat{Z}_n(d,\underset{\sim}{y})|+\delta. \tag{A.9}$$

Consequently, we obtain from (A.7), (A.8) and (A.9) that

$$P\left(\max_{\underset{\sim}{y}\in\underset{\sim}{\Delta}_n}\max_{d\le[n\kappa]}|Z_n(d,\underset{\sim}{y})|>\varepsilon\right)\le P\left(\max_{\underset{\sim}{y}\in\underset{\sim}{\Delta}_n}\max_{d\le[n\kappa]}|\hat{Z}_n(d,\underset{\sim}{y})|>\varepsilon-\delta\right) \tag{A.10}$$

$$=P\left(\bigcup_{\underset{\sim}{y}\in\underset{\sim}{\Delta}_n,d\le[\kappa n]}(|\hat{Z}_n(d,\underset{\sim}{y})|>\varepsilon-\delta)\right)$$

$$\le 2(\beta+2)n\sqrt{n}\max_{\underset{\sim}{y}\in\underset{\sim}{\Delta}_n}\max_{d\le[\kappa n]}P(|\hat{Z}_n(d,\underset{\sim}{y})|>\varepsilon-\delta)\le 2(\beta+2)n\sqrt{n}\max_{\underset{\sim}{y}\in\underset{\sim}{\Delta}_n}\max_{d\le[\kappa n]}P(|Z_n(d,\underset{\sim}{y})|>\varepsilon-2\delta)$$

$$\le\frac{2(\beta+2)h^*n\sqrt{n}}{(\varepsilon-2\delta)^{12}n^3}=\frac{2(\beta+2)h^*}{(\varepsilon-2\delta)^{12}n\sqrt{n}}.$$

Let A_n be the event

$$A_n=\{\max_{\underset{\sim}{y}\in\underset{\sim}{\Delta}_n}\max_{d\le[\kappa n]}|Z_n(d,\underset{\sim}{y})|>\varepsilon\}$$

and define

$$D=\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}A_n.$$

Recall that the series $\sum_n 1/n^s$ converges when $s>1.$ From (A.10) it follows that

$$\sum_{n=1}^{\infty} P(A_n) \le \frac{2(\beta+2)h^*}{(\varepsilon-2\delta)^{12}} \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} < \infty.$$

From Borel-Cantelli's Theorem it therefore follows that $P(D)=0,$ which means that, with probability one, only finitely many A_n can occur. Hence, the proof is complete.

Q.E.D.

Before we state the next result we need to introduce additional notation. Let $F_0^r(u),$ r = S, D, be two c.d.f. defined on the positive part of the real line, r=S, D. Let $x_i^S(j,\omega)$ and $x_j^D(i,\omega)$ be real numbers and define vectors $x_i^S(j)=(x_i^S(j,\omega_1),x_i^S(j,\omega_2),...),$ $x_i^S=(x_i^S(1),x_i^S(2),...),$ $x^S=(x_1^S,x_2^S,...),$ and define $x_j^D(i),$ x_j^D and x^D similarly. Let Q be the dimension of $(x^S,x^D).$

Lemma 4

Let

$$\psi_r(u) = \int_0^{\infty} e^{-uz} F_0^r(1/z)dz$$

for r = S, D and $u \ge 0.$ *Assume that* $\psi_r(0) < \infty,$ *Let* $\{\tilde{a}_i^S(j,\omega)\}$ *and* $\{\tilde{a}_j^D(i,\omega)\}$ *be positive random functions such that* $E\tilde{a}_i^S(j,\omega) < \infty,$ $E\tilde{a}_j^D(i,\omega) < \infty$ *and* $\{\lambda_i^S\},$ $\{\lambda_j^D\},$ $\{\pi_{ij}\}$ *and* κ *be positive real numbers with* $\lambda_i^S \le 1,$ $\lambda_j^D \le 1$ *and* $\pi_{ij} \le 1.$

numbers, and let

$$K_{ij}^D(x_j^D,\omega) = \log[\pi_{ij} E\{\tilde{a}_j^D(i,\omega)\psi_D(\sum_k \sum_{v\in W} \tilde{a}_j^D(k,v)\exp(x_j^D(k,v)))\}\lambda_j^D \kappa],$$

$$K_{ij}^S(x_i^S,\omega) = \log[\pi_{ij} E\{\tilde{a}_i^S(j,\omega)\psi_S(\sum_k \sum_{v\in W} \tilde{a}_i^S(k,v)\exp(x_i^S(k,v)))\}\lambda_i^S],$$

$K_{ij}^D(x_j^D) = (K_{ij}^D(x_j^D,\omega_1), K_{ij}^D(x_j^D,\omega_2),...),$ $K_i^D(x^D) = (K_{1i}^D(x_1^D), K_{2i}^D(x_2^D),...),$

$K^D(x^D) = (K_1^D(x^D), K_2^D(x^D),...),$ *and define* $K_{ij}^S(x_i^S),$ $K_j^S(x^S)$ *and* $K^S(x^S)$ *similarly. Finally, let* $K(x^S,x^D) = (K^D(x^D), K^S(x^S)).$ *Then* $K(x^S,x^D)$ *is a contraction mapping on* $T(\beta) = \{(x^S,x^D): x_i^S(j,\omega) \le \beta, x_j^D(i,\omega) \le \beta\}$ *into* $T(\beta)$ *where* β *is a real number. Furthermore,* $K(x^S,x^D)$ *has a unique fixed*

point in $R^Q.$

Proof of Lemma 4:

For simplicity, we shall go through the proof only in the case with flexible contracts and only one type of supplier and demander. Let

$$\gamma_r(u)=\int_0^\infty z^{-1}\exp(-u/z)dF_0^r(z)\,,$$

for $r = S, D$,

$$K^S(x^S,\omega)=\log E\{\pi\tilde{a}^S(\omega)\psi_S(\textstyle\sum_{v\in W}\tilde{a}^S(v)\exp(x^S(v))\},$$

$$K^D(x^D,\omega)=\log E\{\pi\kappa\tilde{a}^D(\omega)\psi_D(\textstyle\sum_{v\in W}\tilde{a}^D(v)\exp(x^D(v)))\},$$

$$K^D(x^D)=\{K^D(x^D,\omega)\},\ K^S(x^S)=\{K^S(x^S,\omega)\}\ \text{and}\ K(x^S,x^D)=(K^D(x^D),K^S(x^S)).$$

By applying integration by parts it follows that

$$\text{(A.11)}\qquad \psi_r(u)=\int_0^\infty \exp(-uz)F_0^r(1/z)dz=\int_0^\infty\frac{(1-\exp(-u/z))}{u}\cdot dF_0^r(z)$$

for $r = S, D$. From (A.11) we see that

$$\text{(A.12)}\qquad \gamma_r(u)=\psi_r(u)+u\psi_r'(u).$$

Furthermore, (A.11) implies that

$$\text{(A.13)}\qquad u\psi_r'(u)=\int_0^\infty z^{-1}\exp(-u/z)dF_0^r(z)-\psi_r(u)=-\int_0^\infty\frac{(1-(1+u/z)\exp(-u/z))}{u}\cdot dF_0^r(z).$$

Since $1+u/z\le\exp(u/z)$ when $u/z\ge 0$ it follows that $-u\psi_r'(u)>0$ for positive u. Hence, (A.12) and (A.13) imply that

$$\text{(A.14)}\qquad 0<-E\{\tilde{a}^S(\omega)\textstyle\sum_{v\in W}\tilde{a}^S(v)\exp(x^S(v)))\psi_r'(\sum_{z\in W}\tilde{a}^S(v)\exp(x^S(v)))\})$$

$$=E\{\tilde{a}^S(\omega)\psi_r(\textstyle\sum_{z\in W}\tilde{a}^S(v)\exp(x^S(v)))\}-E\{\tilde{a}^S(\omega)\gamma_r(\sum_k\sum_{z\in W}\tilde{a}_i^S(v)\exp(x^S(v)))\}.$$

From (A.14) we obtain that

$$\text{(A.15a)}\qquad \sum_{\omega'\in W}\left|\frac{\partial K_S(x^S,\omega)}{\partial x^S(\omega')}\right|$$

$$=\frac{-E\{\tilde{a}^S(\omega)\sum_{v\in W}\tilde{a}^S(v)\exp(x^S(v)))\psi_S'(\sum_{v\in W}\tilde{a}^S(v)\exp(x^S(v)))\})}{E\{\tilde{a}^S(\omega)\psi_S(\sum_{v\in W}\tilde{a}^S(v)\exp(x^S(v))\})}=1-G(x^S)\ge 0$$

where

$$G(x^S)=\frac{E\{\tilde{a}^S(\omega)\gamma_r(\sum_{v\in W}\tilde{a}^S(v)\exp(x^S(v)))\}}{E\{\tilde{a}^S(\omega)\psi_S(\sum_{v\in W}\tilde{a}^S(v)\exp(x^S(v)))\}}.$$

Since

$$E\{\tilde{a}^S(\omega)\psi_S(\textstyle\sum_{v\in W}\tilde{a}^S(v)\exp(x^S(v)))\}\le E\{\tilde{a}^S(\omega)\psi_S(0)\}<\infty$$

it follows that $0 < G(x^S)$ and therefore $1 - G(x^S) < 1$ when $x^S(\omega) \le \beta$ for $\omega \in W$. Similarly, it follows that

(A.15b) $$\sum_{\omega' \in W} \left| \frac{\partial K_D(y_D, \omega)}{\partial y_D(\omega')} \right| < 1.$$

Let $\tilde{x}$ and x be two points in $T(\beta)$. By the mean value theorem we have that

(A.16a) $$K^S(\tilde{x}^S, \omega) - K^S(x^S, \omega) = \sum_{v \in W} \frac{\partial K^S(\hat{x}^S, \omega)}{\partial x^S(v)} (\tilde{x}^S(v) - x^S(v))$$

and

(A.16b) $$K^D(\tilde{x}^D, \omega) - K^D(x^D, \omega) = \sum_{v \in W} \frac{\partial K^D(\hat{x}^D, \omega)}{\partial x^D(v)} (\tilde{x}^D(v) - x^D(v))$$

where $\hat{x}^S$ and $\hat{x}^D$ are suitable vectors. Let $w = (w_1, w_2, ...)'$ be a vector and define the metric $||| \cdot |||$ by $||| w ||| = \max_k | w_k |$. By using (A.15a,b) we realize that (A.16a,b) imply that we can find a positive constant $D < 1$ such that

(A.17) $$||| K(\tilde{x}^S, \tilde{x}^D) - K(x^S, x^D) ||| \le D ||| (\tilde{x}^S, \tilde{x}^D) - (x^S, x^D) |||.$$

because $\pi \tilde{a}^S(\omega) < \sum_{v \in W} \tilde{a}^S(v))$ almost surely. Therefore, $K(x^S, x^D)$ is a contraction mapping on R^Q.

We shall next show that $K(x^S, x^D) \in T(\beta)$ whenever $(x^S, x^D) \in T(\beta)$. Using (A.11) we obtain that

(A.18) $$\psi_S(u) < \frac{1}{u}.$$

Hence, (A.18) implies that

$$K^S(0, \omega) = \log E\{\pi \tilde{a}^S(\omega) \psi_S(\textstyle\sum_{v \in W} \tilde{a}^S(v))\} = \log E\{\pi \tilde{a}^S(\omega) \psi_S(\textstyle\sum_{v \in W} \tilde{a}^S(v))\}$$

$$< \log E\left\{ \frac{\pi \tilde{a}^S(\omega)}{\sum_{v \in W} \tilde{a}^S(v))} \right\} < 0$$

because $\pi \tilde{a}^S(\omega) < \sum_{v \in W} \tilde{a}^S(v))$ almost surely. Similarly, it follows that $K^D(0, \omega) < 0$. From (A.17) we thus obtain that

$$||| K(x^S, x^D) - K(0,0) ||| \le D ||| (x^S, x^D) |||$$

which yields

$$K^r(x^r, \omega) \le K^r(0, \omega) + ||| x^r ||| < ||| x^r |||$$

for $r = S, D$. Hence, $K(x^S, x^D) \in T(\beta)$ whenever $(x^S, x^D) \in T(\beta)$ for any positive β. Therefore, $K(x^S, x^D)$ is a contraction mapping on $T(\beta)$. From Blackwell's theorem (Blackwell, 1965) it then follows that the equation $K(x^S, x^D)$ has a unique fixed point in $T(\beta)$. But since β is an arbitrary positive number it implies that $K(x^S, x^D)$ also has only one fixed point of $K(x^S, x^D)$ in R^Q where Q is the dimension of (x^S, x^D). To realize this, suppose that there is another (finite) fixed point $(\overline{x}^S, \overline{x}^D)$ of $K(x^S, x^D)$ in R^Q that does not belong to $T(\beta)$. Then, there exists another $\beta^* > \beta$ such that $(\overline{x}^S, \overline{x}^D) \in T(\beta^*)$. But this leads to a contradiction because we proved above that $K(x^S, x^D)$ has only a unique fixed point in $T(\beta^*)$. Hence, the proof is complete.

Q.E.D.

Lemma 5

Assume that the sequence of functions $h_n(x)$ converges uniformly to $h(x)$ on a set A. If $f(x) > c$ for some positive constant c then $\log h_n(x)$ converges uniformly to $\log h(x)$ on A.

Proof of Lemma 5:

Note first that $\log(1+x) \le x$ for all positive *x* and $\log(1 - x/2) \ge -x$ for $1 > x \ge 0$. To realize this consider the function $x - \log(1+x)$. This function has derivative $1 - 1/(1+x) \ge 0$. is increasing for positive *x*, which implies that $x - \log(1+x) \ge 0$. Similarly, it follows that $\log(1 - x/2) + x$ is decreasing in x so that $\log(1 - x/2) + x > \log 0.5 + 1 \cong 0.31$. Let $\varepsilon > 0$. By assumption

$$-0.5c\varepsilon < h_n(x) - h(x) < c\varepsilon$$

provided *n* is sufficiently large. Accordingly,

$$\log h_n(x) < \log h(x) + \log(1 + \varepsilon c / h(x)) \le \log h(x) + \log(1+\varepsilon) \le \varepsilon$$

and

$$\omega \quad \log h_n(x) > \log h(x) + \log(1 - 0.5\varepsilon c / h(x)) \ge \log h(x) + \log(1 - \varepsilon / 2) \ge -\varepsilon.$$

Hence, it follows that $|\log h_n(x) - \log h(x)| \le \varepsilon$.

Q.E.D.

In the following lemma (Lemma 6) we prove an important result in the general case where the systematic terms are $a_i^S(j, \omega; \eta_{sd}(i,j), z_{si}^S(j,\omega))$ and $a_j^D(i, \omega; \eta_{sd}(i,j), z_{dj}^D(i,\omega))$. That is, in addition to depending on the latent matching variable $\eta_{sd}(i,j)$ (match quality) also depend on a random effects that account for unobservable attributes of potential partners. Here, we assume that $\eta_{sd}(i,j)$ has finite support located at the points $\{\eta(1), \eta(2), \ldots\}$. Let $\tilde{a}_i^S(j,\omega,k) = a_i^S(j,\omega;\eta(k), z_{si}^S(j,\omega))$ and $\tilde{a}_j^D(i,\omega,k) = a_j^D(i,\omega;\eta(k), z_{dj}^D(i,\omega))$. Moreover, let $y_{dj}^D(s,i,\omega,q) = 1$ if demander *d* of type *j* belongs to the choice set of supplier *s* of type *i* at contract ω and match quality $\eta(q)$, and zero otherwise. Let $y_{dj}^D(i,\omega,q) = \{y_{dj}^D(s,j,\omega,q), s \in \Omega_i^S\}$ and define $y_{si}^S(d,j,\omega,q)$ and $y_{si}^S(j,\omega,q)$ similarly.

Lemma 6

Assume that Assumptions 1,3,5 and 7 hold and that the matching random effect $\eta_{sd}(i,j)$ *has finite support. Then*

$$\sqrt{N} E((J_{si}^S(d,j,\omega,y_{si}^S) \mid \eta_{sd}(i,j) = \eta(q))$$

converges uniformly to

$$E\{\tilde{a}_i^S(j,\omega,q) \int_0^\infty \exp(-z \sum_{(k,v,r)\neq(j,\omega,q)} \tilde{a}_i^S(k,v;r) \| y_{si}^S(k,v,r) \| - z\tilde{a}_i^S(j,\omega,q) \| y_{si}^S(j,\omega,q) \|) F_0^S(1/z) dz\}$$

and

$$\sqrt{N} E((J_{dj}^D(s,i,\omega,y_{dj}^D) \mid \eta_{sd}(i,j) = \eta(q))$$

converges uniformly to

$$E\{\tilde{a}_j^D(i,\omega,q) \int_0^\infty \exp(-z \sum_{(k,v,r)\neq(i,\omega,q)} \tilde{a}_j^D(k,v,r) \| y_{dj}^D(k,v,r) \| - z\tilde{a}_j^D(i,\omega,q) \| y_{dj}^D(i,\omega,q)) \|) F_0^D(1/z) dz,$$

as $N \to \infty$.

Proof of Lemma 6:
For simplicity, we only present the proof for the case with one observable type of suppliers and demanders and no flexible contract. The proof in the general case is entirely analogous. Assume first that some of the components of $y^D(s)$ are positive. For simplicity and with no real loss of generality suppose the support of the distribution of η_{sd} is given by the set $\{\eta(1),\eta(2),\eta(3)\}$. For simplicity, let $H_{qN}(v) = F_1^S(vb(N)/\tilde{a}^S(q))$ and $G_q(v) = \exp(-\tilde{a}^S(q)/v), q = 1,2,3.$ We have that

(A.19) $\quad \sqrt{N} E(J_s^S(d,y_s^S) \mid \eta_{sd} = \eta(q))$

$$= \sqrt{N} E\left(\int_0^\infty dH_{qN}(v) H_{qN}(v)^{\|y_s^S(q)\|\sqrt{N} - \delta(s,q)} \prod_{r\neq q} H_{rN}(v)^{\|y_s^S(r)\|\sqrt{N}} F_0^S(v) dv \right)$$

$$= \frac{\sqrt{N}}{[\| y^D(s,q) \| \sqrt{N} + 1 - \delta(s,q)]} E\left(\int_0^\infty d\left(H_{rN}(v)^{[\|y^D(s,q)\|\sqrt{N}+1-\delta(s,q)} \right) \prod_{r\neq j} H_{rN}(v)^{\|y^D(s,r)\|\sqrt{N}} F_0^S(v) dv \right)$$

where $\delta(s,q) = 1$ if demander *d* is in the choice set of supplier *s* given that $\eta_{sd} = \eta(q)$. We have that

(A.20) $\quad d\left(H_{1N}(v)^{\|y_S^S 1)\|\sqrt{N}} \right) H_{2N}(v)^{\|y_S^S(2)\|\sqrt{N}} H_{3N}(v)^{\|y_S^S(3)\|\sqrt{N}} - d\left(G_1(v)^{[\|y_s^S(1)\|]} | \right) G_2(v)^{[\|y_s^S(2)\|]} G_3(v)^{[\|y_s^S(3)\|]}$

$$= (d\left(H_{1N}(v)^{\|y_s^S(1)\|\sqrt{N}} \right) - d\left(G_1(v)^{[\|y_s^S(1)\|]} \right)) H_{2N}(v)^{\|y_s^S(2)\|\sqrt{N}} H_{3N}(v)^{[\|y_s^S(3)\|\sqrt{N}]}$$

$$+(H_{2N}(v)^{\|y_s^S(2)\|\sqrt{N}} - G_2(v)^{[\|y_s^S(2)\|]}) d\left(G_1(v)^{[\|y_s^S(1)\|]} \right) H_{3N}(v)^{\|y_s^S(3)\|\sqrt{N}}$$

$$+(H_{3N}(v)^{\|y_s^S(3)\|\sqrt{N}} - G_3(v)^{[\|y_s^S(3)\|]}) d\left(G_1(v)^{[\|y_s^S(1)\|]} \right) G_2(v)^{[\|y_s^S(2)\|]}.$$

As a consequence we obtain that

(A.21) $$E\left(\int_0^\infty (d\left(H_{rN}(v)^{\|y_s^S(j)\|\sqrt{N}} \right) \prod_{r\neq j} H_{rN}(v)^{\|y_s^S(r)\|\sqrt{N}} - dG_j(v)^{[\|y_s^S(j)\|]} \prod_{r\neq j} G_r(v)^{[\|y_s^S(r)\|]}) F_0^S(v) \right)$$

$$= E\left(\int_0^\infty (d\left(H_{1N}(v)^{\|y_s^S(1)\|\sqrt{N}} \right) - d\left(G_1(v)^{[\|y_s^S(1)\|]} \right)) H_{2N}(v)^{\|y_s^S(2)\|\sqrt{N}} H_{3N}(v)^{\|y_s^S(3)\|\sqrt{N}} F_0^S(v) \right)$$

$$+E\left(\int_0^\infty (H_{2N}(v)^{\|y_s^S(2)\|} - G_2(v)^{[\|y_s^S(2)\|]}) d\left(G_1(v)^{[\|y_s^S(1)\|]} \right) H_{3N}(v)^{\|y_s^S(3)\|} F_0^S(v) \right)$$

$$+E\left(\int_0^\infty (H_{3N}(v)^{\|y_s^S(3)\|\sqrt{N}} - G_3(v)^{[\|y_s^S(3)\|]}) d\left(G_1(v)^{[\|y_s^S(1)\|]} \right) G_2(v)^{[\|y_s^S(2)\|]} F_0^S(v) \right).$$

Consider the integral

$$E\left(\int_0^\infty (d\left(H_{1N}(v)^{\|y_s^S(1)\|\sqrt{N}} \right) - d\left(G_1(v)^{[\|y_s^S(1)\|]} \right)) H_{2N}(v)^{\|y_s^S(2)\|\sqrt{N}} H_{3N}(v)^{\|y_s^S(3)\|\sqrt{N}} F_0^S(v) \right).$$

From Lemma 1we obtain that

(A.22) $$| H_{qN}(v)^{\|y_s^S(q)\|\sqrt{N}} - G_q(v)^{[\|y_s^S(j)\|]} | \leq c \,| H_{qN}(v)^{[\sqrt{N}]} - G_q(v) |$$

where c is a suitable constant. From (A.22), the mean value theorem for integrals, and subsequent integration by parts we get, for some $\tilde{x}$, that

(A.23) $$| E\left(\int_0^\infty (d\left(H_{1N}(v)^{\|y_s^S(1)\|\sqrt{N}} \right) - d\left(G_1(v)^{[\|y_s^S(1)\|]} \right)) H_{2N}(v)^{\|y_s^S(2)\|\sqrt{N}} H_{3N}(v)^{\|y_s^S(3)\|\sqrt{N}} F_0^S(v) \right) |$$

$$= | E(H_{2N}(\tilde{x})^{\|y_s^S(2)\|\sqrt{N}} H_{3N}(\tilde{x})^{\|y_s^S(3)\|\sqrt{N}} \int_0^\infty (d\left(H_{1N}(v)^{\|y_s^S(1)\|\sqrt{N}} \right) - d\left(G_1(v)^{[\|y_s^S(1)\|]} \right)) F_0^S(v)) |$$

$$= | E(H_{2N}(\tilde{x})^{\|y_s^S(2)\|\sqrt{N}} H_{3N}(\tilde{x})^{\|y_s^S(3)\|\sqrt{N}} \int_0^\infty G_1(v)^{[\|y_s^S(1)\|]} - H_{1N}(v)^{\|y_s^S(1)\|\sqrt{N}} dF_0^S(v)) |$$

$$\leq E(\int_0^\infty | G_1(v)^{[\|y_s^S(1)\|]} - H_{1N}(v)^{\|y_s^S(1)\|\sqrt{N}} | dF_0^S(v)) \leq cE(\int_0^\infty | G_1(v) - H_{1N}(v)^{[\sqrt{N}]} | dF_0^S(v)).$$

By Proposition 1.11 in Resnick (1987) it follows that $H_{1N}(v)^{[\sqrt{N}]} - G_1(v) \to 0$ as $N \to \infty$. Hence, Lebesgue's dominated convergence theorem implies that

$$E(\int_0^\infty |G_1(v)-H_{1N}(v)^{[\sqrt{N}]}|dF_0^S(v))\to 0$$

as $N\to\infty$ which shows that the first integral on the right hand side of (A.21) converges to zero as $N\to\infty.$ Similarly

$$|E\int_0^\infty (H_{2N}(v)^{\|y_s^S(2)\|\sqrt{N}}-G_2(v)^{[\|y_s^S(2)\|]})d(G_1(v)^{[\|y_s^S(1)\|]})H_{3N}(v)^{\|y_s^S(3)\|\sqrt{N}}F_0^S(v)|$$

$$\le[\|y_s^S(1)\|]E(\int_0^\infty |H_{2N}(v)^{\|y_s^S(2)\|\sqrt{N}}-G_2(v)^{[\|y_s^S(2)\|]}|G_1(v)^{[\|y_s^S(1)\|]-1}dG_1(v))$$

$$\le c[\|y_s^S(1)\|]E(\int_0^\infty |H_{2N}(v)^{[\sqrt{N}]}-G_2(v)|G_1(v)^{[\|y_s^S(1)\|]-1}dG_1(v))$$

$$\le c\beta E(\int_0^\infty |H_{2N}(v)^{[\sqrt{N}]}-G_2(v)|G_1(v)^{[\|y_s^S(1)\|]-1}dG_1(v))\le c\beta E(\int_0^\infty |H_{2N}(v)^{[\sqrt{N}]}-G_2(v)|dG_1(v))$$

which shows that also the second integral on the right hand side of (A.21) converges to zero as $N\to\infty$ due to Lebesgue’s dominated convergence theorem. Similarly, it follows that the third integral on the right hand side of (A.21) converges towards zero when $N\to\infty.$ Accordingly, it follows from (A.19) that, as $N\to\infty,$

$$\sqrt{N}E(J_s^S(d,y_s^S)|\eta_{sd}=\eta(j))\to E\left(\int_0^\infty d(G_j(v)^{[\|y_s^S(j)\|]})\prod_{r\ne j}G_r(v)^{[\|y_s^S(r)\|]}F_0^S(v)dv\right)$$

which proves the first part of the lemma. The proof of the second part of the lemma is analogous.

Q.E.D.

Before we turn to the next result we need some additional notation. Let $\tilde{a}_i^S(j,\omega)=a_i^S(j,\omega;z_{si}^S(j,\omega),\eta_{sd}(i,j,\omega))$ and $\tilde{a}_j^D(i,\omega)=a_j^D(i,\omega;z_{dj}^D(i,\omega),\eta_{sd}(i,j,\omega))$ where $\{z_{si}^S(j,\omega),z_{dj}^D(i,\omega),\eta_{sd}(i,j,\omega)\}$ are independent random effects and $\eta_{sd}(i,j,\omega)=\eta_{ds}(j,i,\omega)$.

Lemma 7

Assume that Assumptions 1, 2, 5, 6 and 7 hold. Let

$$\psi_r(x)=\int_0^\infty \exp(-xv)F_0^r(1/v)dv,$$

for $r=S,D.$ *Then*

$$\lim_{N\to\infty}\|Y_j^D(s,i,\omega)\|=m_i^S(j,\omega),\quad \lim_{N\to\infty}\|Y_i^S(d,j,\omega)\|=m_j^D(i,\omega)$$

where $\{m_i^S(j,\omega)\}$ *and* $\{m_j^D(i,\omega)\}$ *are uniquely determined by the equations*

$$m_i^S(j,\omega)=E\{\tilde{a}_j^D(i,\omega)\psi_D(\sum_k\sum_{z\in W}\tilde{a}_j^D(k,z)m_j^D(k,z))\}\lambda_j^D\kappa$$

and

$$m_j^D(i,\omega)=E\{\tilde{a}_i^S(j,\omega)\psi_S(\sum_k\sum_{z\in W}\tilde{a}_i^S(k,z)m_i^S(k,z))\}\lambda_i^S$$

for all i and j and $\omega\in W$.

Proof of Lemma 7:

We shall only give the proof for the special case with no flexible contracts and one type of suppliers and one type of demanders. Let Δ be the set of vectors with components that are either zero or one. As previously, let $y_s^S(d)=1$ if demander *d* is in the choice set of supplier *s* and zero otherwise. Similarly, $y_d^D(s)=1$ if supplier s is in the choice set of demander *d* and zero otherwise. Let $y_s^S=\{y_s^S(d), d\in\Omega^D\}$ and $y_d^D=\{y_d^D(s), s\in\Omega^S\}$. Thus, exogenous choice sets of supplier *s* can be represented by y_s^S and similarly the exogenous choice set of demander *d* can be represented by y_d^D. Let Y_s^S and Y_d^D denote the corresponding equilibrium values.

Let $\underset{\sim}{y}^S=\{y_s^S, s\in\Omega^S\}$ $\underset{\sim}{y}^D=\{y_d^D, d\in\Omega^D\}$ and similar definition for the corresponding equilibrium variables (capital *Y*'s), h^S and h^D be functions defined by

$$h^S(d,\underset{\sim}{y}^D)=\frac{1}{\sqrt{N}}\sum_{s=1}^{N}J_s^S(d,y_s^S)\quad\text{and}\quad h^D(s,\underset{\sim}{y}^D)=\frac{1}{\sqrt{N}}\sum_{d=1}^{M}J_d^D(s,y_d^D).$$

Recall that $\|Y_d^D\|\sqrt{N}$ is the number of suppliers in the equilibrium choice set of demander *d* and $\|Y_s^S\|\sqrt{N}$ is the number of demanders in the equilibrium choice set of supplier *s*. By (4.1) and (4.2) it follows that

(A.24) $$\|Y_d^D\|=h^S(d,\underset{\sim}{Y}^S)\quad\text{and}\quad\|Y_s^S\|=h^D(s,\underset{\sim}{Y}^D).$$

Then Lemma 3 implies that for any $\delta>0$ the following inequalities

(A.25) $$|h^S(d,\underset{\sim}{y}^S)-Eh^S(d,\underset{\sim}{y}^S)|<\delta\quad\text{and}\quad|h^D(s,\underset{\sim}{y}^D)-Eh^D(s,\underset{\sim}{y}^D)|<\delta$$

hold for any *d* and $\underset{\sim}{y}^S\in\Delta,\underset{\sim}{y}^D\in\Delta$ with probability 1 provided *N* is sufficiently large. Let

(A.26) $$f^S(x)=\pi E(\tilde{a}^S\int_0^\infty\exp(-\tilde{a}^S zx)F_0^S(1/z)dz)\quad\text{and}\quad f^D(x)=\pi E(\tilde{a}^D\int_0^\infty\exp(-\tilde{a}^D zx)F_0^D(1/z)dz)$$

for $x\geq 0$. From Lemma 6 it follows that for any *d* and $\underset{\sim}{y}^S\in\Delta,\underset{\sim}{y}^D\in\Delta$ that

(A.27) $$|\sqrt{N}EJ_s^S(d,y_s^S)-f^S(\|y_s^S\|)|<\delta\quad\text{and}\quad|\sqrt{N}EJ_d^S(s,y_d^D)-f^D(\|y_d^D\|)|<\delta$$

with probability 1 provided *N* is sufficiently large. Let m^S and m^D be determined by the equations

(A.28) $$m^S = \kappa f^D(m^D) \quad \text{and} \quad m^D = f^S(m^S).$$

From Lemma 4 we have that the equations above have a unique positive solution for m^S and m^D. It now follows from (A.26), for $\underset{\sim}{y}^S \in \Delta, \underset{\sim}{y}^D \in \Delta$, that

(A.29a) $$|Eh^S(d,\underset{\sim}{y}^S) - N^{-1}\sum_s f^S(\| y_s^S \|)| \le \frac{1}{N}|\sum_s[\sqrt{N}EJ_s^S(d,y_s^S) - f^S(\| y_s^S \|)]|$$

$$\le \frac{1}{N}\sum_s|\sqrt{N}EJ_s^S(d,y_s^S) - f^S(\| y_s^S \|)| < \delta$$

and similarly that

(A.29b) $$|Eh^D(s,\underset{\sim}{y}^D) - \kappa^{-1}N^{-1}\sum_d f^D(\| y_d^D \|)| < \delta$$

with probability 1, provided *N* is sufficiently large. Thus, from (A.24), (A.25) and (A.29a, b) we obtain that

(A.30a) $$|h^S(d,\underset{\sim}{y}^S) - N^{-1}\sum_s f^S(\| y_s^S \|)| \le |h^S(d,\underset{\sim}{y}^S) - Eh^S(d,\underset{\sim}{y}^S)|$$

$$+|Eh^S(d,\underset{\sim}{y}^S) - N^{-1}\sum_s f^S(\| y_s^S \|)| < 2\delta$$

with probability 1 and similarly that

(A.30b) $$|h^D(s,\underset{\sim}{y}^D) - N^{-1}\sum_d f^D(\| y_d^D \|)| < 2\delta$$

with probability 1 if *N* is sufficiently large. Let β be such that $\beta > \max(m^S, m^D)$, $f^S(\beta) \le \delta$ and $f^D(\beta) \le \delta$. This is possible since $f^r(x)$, $r = S, D$, are decreasing towards zero. Hence, we obtain that

$$N^{-1}|\sum_s(f^S(\min(\| y_s^S \|,\beta)) - f^S(\| y_s^S \|))| \le N^{-1}\sum_s|f^S(\min(\| y_s^S \|,\beta)) - f^S(\| y_s^S \|)| < \delta.$$

Consequently, it follows that

(A.31a) $$|h^S(d,\underset{\sim}{y}^S) - N^{-1}\sum_s f^S(\min(\| y_s^S \|,\beta))| \le |h^S(d,\underset{\sim}{y}^S) - N^{-1}\sum_s f^S(\| y_s^S \|)|$$

$$+N^{-1}|\sum_s(f^S(\min(\| y_s^S \|,\beta) - f^S(\| y_s^S \|))| < 3\delta.$$

Similarly, it follows that

(A.31b) $$|h^D(s,\underset{\sim}{y}^D) - \kappa N^{-1}\sum_d f^D(\min(\| y_d^D \|,\beta))| \le 3\delta.$$

Note that

$$f^S(\min(\| y_s^S \|,\beta)) \ge f^S(\beta) > 0 \quad \text{and} \quad f^D(\min(\| y_d^D \|,\beta)) \ge f^D(\beta) > 0$$

Hence, Lemma 5 applies and (A.31a,b) imply that

(A.32a) $$|\log h^S(d,\underset{\sim}{y}^S) - \log\left(N^{-1}\sum_s f^S(\min(\| y_s^S \|,\beta))\right)| < 2\delta$$

and

(A.32b) $$|\log h^D(s,\underset{\sim}{y}^D) - \log\left(N^{-1}\sum_d f^D(\min(\| y_d^D \|,\beta))\right)| < 2\delta$$

with probability 1 if N is sufficiently large. By Lemma 4 $\log f^S(e^z)$ and $\log f^D(e^z)$ are contraction mappings on the set $T(\beta)=\{z : z \le \beta\}$, that is

(A.33) $|\log f^S(x)-\log f^S(z)| \le c\,|\log x - \log z|$ and $|\log f^D(x)-\log f^D(z)| \le c\,|\log x - \log z|$

for some positive $c<1$ provided x and z belongs to $T(\beta)$.

We are now ready to complete the proof. From (A.33) it follows that

(A.34a) $$|\log\left(N^{-1}\sum_s f^S(\min(\|Y_s^S\|,\beta))\right)-\log f^S(m^S)|$$
$$\le \max_s |\log\left(f^S(\min(\|Y_s^S\|,\beta))\right)-\log f^S(m^S)|$$
$$\le c\max_s |\log\min(\|Y_s^S\|,\beta)-\log m^S|$$

and similarly

(A.34b) $$|\log\left(N^{-1}\sum_d f^D(\min(\|Y_d^D\|,\beta))\right)-\log(\kappa f^D(m^D))|$$
$$\le c\max_d |\log\min(\|Y_d^D\|,\beta)-\log m^D|.$$

Note furthermore that since $\beta>\max(m^S,m^D)$ we have that

(A.35a) $$|\log\|Y_d^D\|-\log m^D| \ge |\log\min(\|Y_d^D\|,\beta)-\log m^D|$$

and

(A.35b) $$|\log\|Y_s^S\|-\log m^S| \ge |\log(\min\|Y_s^S\|,\beta)-\log m^S|.$$

Finally, from (A.28), (A.30a,b), (A.34a,b) and (35a,b) we obtain that with probability 1

(A.36a) $$\max_d |\log\|Y_d^D\|-\log m^D| = \max_d |\log h^S(d,\underset{\sim}{Y}^S)-\log f^S(m^S)|$$
$$\le \max_d |\log h^S(d,\underset{\sim}{Y}^S)-\log\left(N^{-1}\sum_s f^S(\min(\|Y_s^S\|,\beta))\right)|$$
$$+|\log\left(N^{-1}\sum_s f^S(\min(\|Y_s^S\|),\beta))\right)-\log f^S(m^S)|$$
$$\le 3\delta + c\max_s |\log\min(\|Y_s^S\|,\beta)-\log m^S| \le 3\delta + c\max_s |\log\|Y_s^S\|-\log m^S|$$

and

(A.36b) $$\max_s |\log\|Y_s^S\|-\log m^S| \le 3\delta + c\max_d |\log\|Y_d^D\|-\log m^D|$$

if N is sufficiently large. When combining (A.36a) and (A.36b) we obtain that

$$\max_d |\log\|Y_d^D\|-\log m^D| \le 3\delta+3\delta c+c^2\max_d |\log\|Y_d^D\|-\log m^D|$$

which implies that

(A.37a) $$\max_d |\log\|Y_d^D\|-\log m^D| \le \frac{3\delta(1+c)}{1-c^2}=\frac{3\delta}{1-c}$$

with probability 1 if N is large enough. In the same way it follows that also

(A.37b) $$\max_s |\log\|Y_s^S\|-\log m^S| \le \frac{3\delta}{1-c}$$

with probability 1 if N is large enough. Hence, with probability 1 the sizes of the choice sets normalized by b tend towards the deterministic terms m^D and m^S which are determined by (A.28).

Similarly, in the general case with several observable groups of suppliers and demanders and flexible contracts we get that $\| Y_{si}^S(j,\omega)\|$ converges towards $m_i^S(j,\omega)$ with probability one and $\| Y_{dj}^D(j,\omega)\|$ converges towards $m_j^D(i,\omega)$ with probability one. Hence, Lemma 7 follows where $\{m_j^D(i,\omega)\}$ and $\{m_i^S(j,\omega)\}$ are determined by the equations

$$m_j^D(i,\omega)=\pi_{ij}E\{a_i^S(j,\omega)\lambda_i^S\int_0^\infty \exp(-z\sum_k\sum_{v\in W}a_i^S(k,v)m_i^S(k,v))F_0^S(1/z)dz)\}$$

$$\kappa m_i^S(j,\omega)=\pi_{ij}E\{a_j^D(i,\omega)\lambda_j^D\int_0^\infty \exp(-z\sum_k\sum_{v\in W}a_j^D(k,v)m_j^D(k,v))F_0^S(1/z)dz\}.$$

By Lemma 5 the system of equations above have a unique solution. Thus, the proof of Lemma 7 is complete.

Q.E.D.

Lemma 8

Under the assumptions of Lemma 7 it follows that

$$\lim_{N\to\infty} X_{ij}(\omega)=\lambda_i^S\varphi_i^S(j,\omega)=m_i^S(j,\omega)m_j^D(i,\omega)$$

with probability 1, where the asymptotic choice probabilities of being self-matched are given by

$$\varphi_i^S(0)=1-\sum_k\sum_{z\in W}m_i^S(k,z)m_k^D(i,z)/\lambda_i^S,$$

$$\varphi_j^D(0)=1-\sum_k\sum_{z\in W}m_j^D(k,z)m_k^S(j,z)/\lambda_j^S\kappa$$

for all i and j and $\omega\in W$.

Proof of Lemma 8:

Consider next the asymptotic behavior of X. Let $p_N(\| y_s^S\|)=EJ_s^S(d,y_s^S)$. Due to the result we just proved above it follows that there exists a positive β such that $\| Y_s^S\|<\beta$ and $\| Y_d^D\|<\beta$ with probability 1. We have that

(A.38) $$|X-m^Dm^S|=|\frac{1}{N}\sum_d\sum_s Y_s^S(d)Y_d^D(s)-m^Dm^S|$$

$$\le|\frac{1}{N}\sum_d\sum_s(J_s^S(d,Y_s^S)-p_N(\| Y_s^S\|))Y_d^D(s)|$$

$$+\frac{1}{N\sqrt{N}}\left|\sum_s(\sqrt{N}\,p_N(\|Y_s^S\|)-f^S(\|Y_s^S\|))Y_d^D(s)\right|$$

$$+\frac{1}{N\sqrt{N}}\left|\sum_s(f^S(\|Y_s^S\|)-m^D)Y_d^D(s)\right|+\frac{m^D}{N}\sum_s\left|\|Y_s^S\|-m^S)\right|.$$

Let $d^*(s)$ denote the chosen demander by supplier s in equilibrium. Then

$$\frac{1}{N}\sum_d\sum_s(J_s^S(d,Y_s^S)-p_N(\|Y_s^S\|))Y_d^D(s)=\sum_s(J_s^S(d^*(s),Y_s^S)-p_N(\|Y_s^S\|))$$

because when the matching is stable $Y_d^D(s)=1$ when $d=d^*(s)$, and $Y_d^D(s)=0$ otherwise. Hence, when N is large it follows from Lemma 3 that for any $\delta>0$

$$\left|\frac{1}{N}\sum_d\sum_s(J_s^S(d,Y_s^S)-p_N(\|Y_s^S\|))Y_d^D(s)\right|=\left|\sum_s(J_s^S(d^*(s),Y_s^S)-p_N(\|Y_s^S\|))\right|$$

$$\le\frac{1}{N}\max_{\underset{\sim}{y}^S\in\Delta_N}\left|\sum_s(J_s^S(d^*(s),y_s^S)-p_N(\|y_s^S\|))\right|\le\frac{1}{N}\max_{\underset{\sim}{y}^S\in\Delta_N}\max_{d\le[N\kappa]}\left|\sum_s(J_s^S(d,y_s^S)-p_N(\|y_s^S\|))\right|\le\delta.$$

Moreover, Lemma 6 implies that with probability 1

$$\left|\sqrt{N}\,p_N(\|Y_s^S\|)-f^S(\|Y_s^S\|)\right|<\delta$$

if N is sufficiently large. Above we proved that $\left|\|Y_s^S\|-m^S\right|<\delta$ with probability 1 if N is sufficiently large. Due to the fact that $f^S(x)$ is continuous it follows from Slutsky's Theorem that

(A.39) $$\left|f^S(\|Y_s^S\|)-m^D\right|=\left|f^S(\|Y_s^S\|)-f^S(m^S)\right|<\delta$$

with probability 1 if N is sufficiently large. Therefore, it follows from (A.38) to A.39) that

$$\left|X-m^Dm^S\right|<\delta+2\delta\sum_s\|Y_s^S\|/N+\delta m^D\le\delta(1+2\beta+m^D)$$

with probability 1 if N is sufficiently large. Hence, X converges with probability 1 towards m^Dm^S as $N\to\infty$.

Q.E.D.

Appendix B

The case with interdependent preferences

Lemma 9

Assumption 8 implies that the joint distributions of the error terms of the suppliers and the demanders have the structure

(B.1) $$P\left(\bigcap_{d\in\Omega_j^D}(z_{Sij}(\omega)\varepsilon_{si}^S(d,j,\omega)\le x_{si}(d,j,\omega))\right)=\exp\left(-\sum_j\left(\sum_{d\in\Omega_j^D}x_{si}(d,j,\omega)^{-1/\theta}\right)^{\theta}\right)$$

and

(B.2) $$P\left(\bigcap_{s\in\Omega_i^S}(z_{Sji}(\omega)\varepsilon_{dj}^D(s,i,\omega)\le x_{dj}(s,i,\omega))\right)=\exp\left(-\sum_i\left(\sum_{s\in\Omega_i^S}x_{dj}(s,i,\omega)^{-1/\theta}\right)^{\theta}\right),$$

where $\theta\in(0,1]$. *The parameter* θ *has the interpretation*

(B.3) $$Corr\left(\log U_{si}^S(d,j,\omega),\log U_{si}^S(r,k,\omega)\right)=Corr\left(\log U_{dj}^D(s,i,\omega),\log U_{dj}^D(k,r,\omega)\right)=1-\theta^2$$

for $d,r\in\Omega_j^D$ *and* $s,k\in\Omega_i^S$. *Otherwise* $\varepsilon_{si}^S(d,j,\omega)$ *and* $\varepsilon_{si}^S(r,k,\omega')$ *are independent if* $(d,j,\omega)\ne(r,k,\omega')$ *and* $\varepsilon_{dj}^D(s,i,\omega)$ *is independent of* $\varepsilon_{dj}^D(k,r,\omega')$ *if* $(s,i,\omega)\ne(r,k,\omega')$.

Proof of Lemma 9:

By Assumption 8 it follows by using the properties of the Stable distributions $S_\theta(1,1,0)$ and $S_\theta(1,1,0)$ (see Samorodnitsky and Taqqu, 1994) that

$$P\left(\bigcap_{d\in\Omega_j^D}(z_{Sij}(\omega)\varepsilon_{si}^S(d,j,\omega)\le x_s(d,j,\omega))\right)=E\exp\left(-\sum_j z_{Sij}\sum_{d\in\Omega_j^D}x_{si}(d,j,\omega)^{-1}\right)$$

$$=\exp\left(-\sum_j\left(\sum_{d\in\Omega_j^D}x_{si}(d,j,\omega)^{-1/\theta}\right)^{\theta}\right)$$

and

$$P\left(\bigcap_{s\in\Omega_i^S}(z_{Sji}(\omega)\varepsilon_{dj}^D(s,i,\omega)\le x_{dj}(s,i,\omega))\right)=E\exp\left(-\sum_i z_{Dji}\sum_{s\in\Omega_i^S}x_{dj}(s,i,\omega)^{-1}\right)$$

$$=\exp\left(-\sum_i\left(\sum_{s\in\Omega_i^S}x_{dj}(s,i,\omega)^{-1/\theta}\right)^{\theta}\right).$$

From Tiago de Oliveira (1973) follows. Hence, the proof is complete.

Q.E.D.

Lemma 10

Assume that

$$\tilde{a}_i^S(j,\omega) = a_i^S(j,\omega) z_i^S(j,\omega), \ \tilde{a}_j^D(i,\omega) = a_j^D(i,\omega) z_j^D(i,\omega),$$

$$F_0^S(z) = P(z^S(0)\varepsilon^S(0) \le z) \ \textit{and} \ F_0^D(z) = P(z^D(0)\varepsilon^D(0) \le z)$$

where $a_i^S(j,\omega)$, $a_j^D(i,\omega)$ *are positive constants,* $\{\varepsilon^S(0), \varepsilon^D(0)\}$ *are independent positive random variable with c.d.f.* $\exp(-1/x), x > 0,$ *and* $\{z_i^S(j,\omega), z_j^D(i,\omega), z^S(0), z^D(0)\}$ *are independent and identically distributed random variables that are generated by a stable distribution*[9] *that is totally skew to the right and with index* θ, $\theta \le 1$.[10] *Then the conclusion of Lemma 4 holds.*

Proof of Lemma 10:

For simplicity we shall only go through the proof for the case with only one type of demanders and suppliers. From Samorodnitsky and Taqqu (1994) it follows that

$$E\exp(-u(\textstyle\sum_{v\in W} a^S(v) z^S(v) y(v) + z^S(0)))\} = \exp(-cu^\theta(\textstyle\sum_{v\in W} a^S(v)^\theta y(v)^\theta + 1)\}$$

for real $y(v) \ge 0$, $v \in W$. By differentiating the above equation with respect to $uy(\omega)$ we obtain that

$$E\{a^S(\omega) z^S(\omega) \exp(-u(\textstyle\sum_{v\in W} a^S(v) z^S(v) y(v))) + z^S(0)))\}$$

$$= c\theta a^S(\omega)^\theta u^{\theta-1} y(v)^{\theta-1} \exp(-cu^\theta(\textstyle\sum_{v\in W} a^S(v)^\theta y(v)^\theta + 1)\}.$$

By using the same notation as in Lemma 4 the result above implies that

$$E\{\tilde{a}^S(\omega)\psi_S(\textstyle\sum_{v\in W} \tilde{a}^S(v) \exp(x^S(v))\}$$

$$= E\{a^S(\omega) z^S(\omega) \int_0^\infty \exp(-u(\textstyle\sum_{v\in W} a^S(v) z^S(v) \exp(x(v)) + z^S(0)))\} du$$

$$= c\theta a^S(\omega)^\theta \exp((\theta-1)x(\omega)) \int_0^\infty u^{\theta-1} \exp(-cu^\theta(\textstyle\sum_{v\in W} a^S(v)^\theta \exp(\theta x(v)) + 1)\} du$$

$$= \frac{a^S(\omega)^\theta \exp((\theta-1)x(\omega))}{\sum_{v\in W} a^S(v)^\theta \exp(\theta x(v)) + 1}.$$

Hence, it follows that

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[9] Recall that the stable distribution follows from an extended version of the Central limit theorem. See Samorodnitsky and Taqqu (1994) for a description of stable distribution.

[10] The case where $\theta = 1$ corresponds to a degenerate stable distribution with all mass located at zero.

$$K^S(x,\omega) = \log(\pi a^S(\omega)^\theta) + (\theta-1)x(\omega) - \log\left(\sum_{v\in W} a^S(v)^\theta \exp(\theta x(v)) + 1\right)$$

which implies that

$$|\sum_{v\in W} \frac{\partial K(x,\omega)}{\partial x(v)}| = 1 - \frac{\theta}{\sum_{v\in W} a^S(v)^\theta \exp(\theta x(v)) + 1} \le 1 - \frac{\theta}{\sum_{v\in W} a^S(v)^\theta \exp(\theta\beta) + 1} < 1.$$

The rest of the proof is similar to the proof of Lemma 4.

Q.E.D.

Proof of Theorem 5:

Due to Lemma 10 it follows, as in Lemma 7, that the sizes of the equilibrium choice sets, divided by $\sqrt{N}$, converge with probability one toward the corresponding unique asymptotic values as $N \to \infty$. .

Furthermore, it follows that $X_{ij}(\omega)$ converges with probability one towards $\lambda_i^S \varphi_i^S(j,\omega)$.

Similarly to Dagsvik (2000, pp. 40-41) one can apply McFadden's formula for Generalized Extreme Value random utility models (McFadden, 1984), which by (B.1) and (B.2) imply, asymptotically, that

(B.4) $$m_i^S(j,\omega) = \pi_{ij}\sqrt{M_j}\, P(U_{dj}^D(s,i,\omega) \ge \max(\max_{v\in W}\max_k \max_r U_{dj}^D(r,k,v)Y_{dj}^D(r,k,v), U_{dj}^S(0)))$$

$$= \frac{\sqrt{M_j}\,\pi_{ij} a_j^D(i,\omega)(\sqrt{N_i}\, m_j^D(i,\omega))^{\theta-1}}{R_j^D}.$$

where

(B.5) $$R_j^D = b_j + \sum_k \sum_{z\in W} a_j^D(k,z)(\sqrt{N_i}\, m_j^D(k,z))^\theta ..$$

Similarly, it follows that

(B.6) $$m_j^D(i,\omega)\sqrt{N} = \frac{N_i \pi_{ij} a_i^S(j,\omega)(\sqrt{M_j}\, m_i^S(j,\omega))^{\theta-1}}{R_i^S},$$

where

(B.7) $$R_i^S = b_i + \sum_k \sum_{z\in W} a_i^S(k,z)(\sqrt{M_j}\, m_i^S(k,z))^\theta.$$

It is, however, not evident how the constant *b* should depend on *N*. From (B.4) and (B.6) we obtain that

(B.8) $$m_j^D(i,\omega)\sqrt{N_i} = \left(\frac{\pi_{ij}^\theta a_i^S(j,\omega) N_i}{R_i^S}\right)^{1/\theta(2-\theta)} \left(\frac{a_j^D(i,\omega) M_j}{R_j^D}\right)^{(\theta-1)/\theta(2-\theta)}$$

and

(B.9) $$m_i^S(j,\omega)\sqrt{N} = \left(\frac{\pi_{ij}^{\theta} a_j^D(i,\omega) M_j}{R_j^D}\right)^{1/\theta(2-\theta)} \left(\frac{a_i^S(j,\omega) N_i}{R_i^S}\right)^{(\theta-1)/\theta(2-\theta)}.$$

Moreover, after some straight forward calculus, one obtains that

(B.10) $$N_i \varphi_i^S(j,\omega) = M_j \varphi_j^D(i,\omega) = \left(\frac{\pi_{ij}^{\theta} a_i^S(j,\omega) N_i}{R_i^S} \cdot \frac{a_j^D(i,\omega) M_j}{R_j^D}\right)^{1/2-\theta},$$

(B.11) $$\varphi_i^S(0) = \frac{b}{R_i^S} \quad \text{and} \quad \varphi_j^D(0) = \frac{b}{R_j^D}.$$

By inserting the equations in (B.11) into (B.10) we obtain that

(B.12) $$N_i \varphi_i^S(j,\omega) = M_j \varphi_j^D(i,\omega) = \left(\frac{\pi_{ij}^{\theta} a_i^S(j,\omega) \varphi_i^S(0) N_i}{b} \cdot \frac{a_j^D(i,\omega) \varphi_j^D(0) M_j}{b}\right)^{1/(2-\theta)},$$

By summing over contracts and demander groups it follows from (B.12) that

(B.13) $$1-\varphi_i^S(0) = \frac{1}{N_i} \sum_k \sum_{\omega\in W} \left(\frac{\pi_{ij}^{\theta} a_i^S(k,\omega) \varphi_i^S(0) N_i}{b} \cdot \frac{a_k^D(i,\omega) \varphi_k^D(0) M_k}{b}\right)^{1/(2-\theta)}$$

and

(B.14) $$1-\varphi_j^D(0) = \frac{1}{M_j} \sum_k \sum_{\omega\in W} \left(\frac{\theta_{ij}^{\theta} a_k^S(j,\omega) \varphi_k^S(0) N_k}{b} \cdot \frac{a_j^D(k,\omega) \varphi_j^D(0) M_j}{b}\right)^{1/(2-\theta)}$$

If (B.13) and (B.14) are to be independent of the population size *b* must depend on *N*. A suitable choice is to let

$$\frac{1}{N}\left(\frac{N}{b}\right)^{2/(2-\theta)} = 1$$

which yields $b = N^{\theta/2}$. Hence, in this case we obtain that

(B.15) $$\varphi_i^S(j,\omega) = (\pi_{ij}^{\theta} a_{iS}(j,w) a_{jD}(i,w) \varphi_i^S(0) \varphi_j^D(0) N_i M_j / N^2)^{1/(2-\theta)} N / N_i.$$

Furthermore, when inserting (B.11) into (B.8) and (B.9) we obtain that

(B.16) $$m_j^D(i,\omega)\sqrt{N} = \left(\frac{\pi_{ij}^{\theta} a_i^S(j,\omega) \varphi_i^S(0) N_i}{b}\right)^{1/\theta(2-\theta)} \left(\frac{a_j^D(i,\omega) \varphi_j^D(0) M_j}{b}\right)^{(\theta-1)/\theta(2-\theta)}$$

and

(B.17) $$m_i^S(j,\omega)\sqrt{N} = \left(\frac{\pi_{ij}^{\theta} a_j^D(i,\omega) \varphi_j^D(0) M_j}{b}\right)^{1/\theta(2-\theta)} \left(\frac{a_i^S(j,\omega) \varphi_i^S(0) N_i}{b}\right)^{(\theta-1)/\theta(2-\theta)}.$$

From (B.16) and (B.17) it follows readily that

$$\varphi_i^S(0)^{1/(2-\theta)} = \frac{\left(N_i / N\right)^{(1-\theta)/(2-\theta)}}{\left(\varphi_i^S(0)N_i / N\right)^{(1-\theta)/(2-\theta)} + \sum_k \zeta_{ik}(\theta)\left(\varphi_k^D(0)M_k / N\right)^{1/(2-\theta)}}$$

and

$$\varphi_j^D(0)^{1/(2-\theta)} = \frac{\left(M_j / N\right)^{(1-\theta)/(2-\theta)}}{\left(\varphi_j^D(0)M_j / N\right)^{(1-\theta)/(2-\theta)} + \sum_k \zeta_{kj}(\theta)\left(\varphi_k^S(0)N_k / N\right)^{1/(2-\theta)}}$$

where

$$\zeta_{ij}(\theta) = \sum_{w \in W} (\pi_{ij}^{\theta} a_i^S(j,w) a_j^D(i,w))^{1/(2-\theta)}.$$

Note finally that since the nested logit modelling framework use here can be interpreted as a random effect model, the result of Lemma 6 guarantees the existence and uniqueness of the solution of the equation system above. This completes the proof.

Q.E.D.

Appendix C

Here, we derive the equilibrium relations of the model of Choo and Siow (2006). In this model utilities are transferable and potential partners within each observational category are perfect substitutes. The utility functions are specified as follows:

(C.1) $$U_{si}^{S}(d,\omega_{ij}) = \alpha_{ij}\exp(-\omega_{ij})\varepsilon_{si}^{S}(j), \qquad U_{si}^{S}(0) = \varepsilon_{si}^{S}(0),$$

(C.2) $$U_{d}^{D}(s,\omega_{ij}) = \beta_{ji}\exp(\omega_{ij})\varepsilon_{dj}^{D}(i) \text{ and } U_{jd}^{S}(0) = \varepsilon_{jd}^{D}(0)$$

where the random terms $\varepsilon_{si}^{S}(0)$, $\varepsilon_{jd}^{D}(0)$, $\varepsilon_{s}^{S}(j)$, $\varepsilon_{d}^{D}(i)$, $s \in \Omega_{i}^{S}$, $d \in \Omega_{j}^{D}$, $i = 1, 2, \ldots, j = 1, 2, \ldots$, are independent and distributed according to $\exp(-1/u)$, for positive u.

From (C.1) and (C.2), and standard results in the theory of discrete choice it follows that the respective probabilities of realizing a match and of being single are given by

(C.3) $$\varphi_{i}^{S}(j,\omega_{ij}) = \frac{\alpha_{ij}\exp(-\omega_{ik})}{1+\sum_{k}\alpha_{ik}\exp(-\omega_{ik})}, \qquad \varphi_{j}^{D}(i,\omega_{ij}) = \frac{\beta_{ji}\exp(\omega_{ij})}{1+\sum_{k}\beta_{jk}\exp(\omega_{kj})},$$

(C.4) $$\varphi_{i}^{S}(0) = \frac{1}{1+\sum_{k}\alpha_{ik}\exp(-\omega_{ik})} \quad \text{and} \quad \varphi_{j}^{D}(0) = \frac{1}{1+\sum_{k}\beta_{jk}\exp(\omega_{kj})},$$

and where the equilibrium relations are determined by the conditions

(C.5) $$N_{i}\varphi_{i}^{S}(j,\omega_{ij}) = M_{j}\varphi_{j}^{D}(i,\omega_{ij}), \text{ for all } (i,j).$$

From (C.3) to (C.5) it follows that

(C.6) $$\varphi_{si}^{S}(j,\omega_{ij}) = \alpha_{ij}e^{-\omega_{ij}}\varphi_{i}^{S}(0) \quad \text{and} \quad \varphi_{j}^{D}(i,\omega_{ij}) = \beta_{ji}e^{\omega_{ij}}\varphi_{j}^{D}(0).$$

From (C.5) and (C.6) we realize that the equilibrium condition in (C.5) can be expressed as

(C.7) $$N_{i}\alpha_{ij}e^{-\omega_{ij}}\varphi_{i}^{S}(0) = M_{j}\beta_{ji}e^{\omega_{ij}}\varphi_{j}^{D}(0)$$

for all (i, j), which implies that

(C.8) $$e^{\omega_{ij}} = \left(\frac{N_{i}\alpha_{ij}\varphi_{i}^{S}(0)}{M_{j}\beta_{ji}\varphi_{j}^{D}(0)}\right)^{1/2}.$$

Furthermore, when the formula for ω_{ij} given in (C.8) is inserted into the first equation in (C.4) we obtain that

(C.9) $$\varphi_{i}^{S}(0) = \frac{1}{1+\sum_{k}\alpha_{ik}\left(\frac{M_{k}\beta_{ki}\varphi_{k}^{D}(0)}{N_{i}\alpha_{ik}\varphi_{i}^{S}(0)}\right)^{1/2}} = \frac{\sqrt{\varphi_{i}^{S}(0)}}{\sqrt{\varphi_{i}^{S}(0)}+\sum_{k}\sqrt{\alpha_{ik}\beta_{ki}\varphi_{k}^{D}(0)M_{k}/N_{i}}}.$$

A similar expression holds for $\varphi_j^D(0)$. Hence, it follows that the equilibrium choice probabilities are determined by

(C.10a) $$\frac{1}{\sqrt{\varphi_i^S(0)}} - \sqrt{\varphi_i^S(0)} = \sum_k \sqrt{\alpha_{ik}\beta_{ki}\varphi_k^D(0)\lambda_k^D \kappa / \lambda_i^S}\ ,$$

(C.10b) $$\frac{1}{\sqrt{\varphi_j^D(0)}} - \sqrt{\varphi_j^D(0)} = \sum_k \sqrt{\alpha_{ik}\beta_{ki}\varphi_k^S(0)\lambda_k^S / \lambda_j^D \kappa}$$

and

(C.11) $$\varphi_i^S(j) = \sqrt{\alpha_{ik}\beta_{ki}\varphi_i^S(0)\varphi_j^D(0)\lambda_k^D \kappa / \lambda_i^S}.$$

Provided a unique solution of (C.10a,b) exists it follows that the contract ω_{ij} is determined by (C.8) and the choice probability in (C.11) is uniquely determined.

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