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A Demand System for Input Factors when there are Technological Changes in Production

Abstract:
In a system with n input factors there are $n - 1$ independent cost shares. An often-used approach in estimating factor demand systems is to (implicitly or explicitly) assume that there is a (independent) cointegrating relationship for each of the $n - 1$ independent cost shares. However, due to technological changes there might not be as many cointegrating relationships as there are (independent) cost shares. The paper presents a flexible demand system that allows for both factor neutral technological changes as well as technological changes that affect the relative use of the different factors. The empirical tests indicate that there are fewer cointegrating relationships than usually implied by using conventional estimation approaches. This result is consistent with technological changes. I argue that since such unexplained technological changes are likely to affect input factor decisions, a demand system that allows for such changes should be preferred.

Keywords: Factor demand, technological changes, growth rates

JEL classification: C32, C52, D24

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1 Introduction

Technological improvements might change the coefficients in the production function. It is therefore too restrictive to assume that the production technology is fixed in the period the data set spans. Even to allow for a stochastic technological progress, is a too restrictive formulation given pervasive technological changes. Therefore, the present paper presents a more flexible production function where most of the coefficients in the production function are allowed to change during the estimation period.

The Cobb-Douglas function and the CES function with fixed parameters are widely used as production functions when identifying productivity growth, see Klump et al. (2007). Also in more flexible functional forms such as the translog function, the parameters are assumed fixed (see e.g. Allen and Urga, 1999). The assumption of fixed parameters implies that one imposes a functional form on the technological changes. The technological changes are often assumed to be Hicks-neutral possibly combined with some *deterministic* factor-augmenting technological changes, see Klump et al. (2007) for an overview.

In the present paper I present a more flexible production function where most of the coefficients in the production function are allowed to change during the estimation period. I allow for both factor neutral technological changes as well as technological changes that change the relative use of the different input factors; only the elasticities of scale and substitution are assumed to be constant in the estimation period.

The procedure takes into account that the data series for production and input factors are non-stationary. Therefore, a cointegrating vector autoregressive framework is used. However, the system is reformulated in order to make it possible to identify and impose restrictions on the growth rates of the different input factors. This reformulation is suggested by Hungnes (2002, 2005a) and the system can be estimated by GRaM for Ox, see Hungnes (2005b).

A technological change in the production might have a permanent effect on the use of the different input factors. However, if there is no drift in the distribution parameters in the production function, technological changes will only have a temporary effect on the growth of the input factors. In the long run, the only reason for differences in the underlying growth in the input factors is changes in the relative factor prices. Corrected for the effect of changes in relative prices, the growth rates for the different input factors should all be the same.

In the present paper a factor demand system is estimated. The system includes as many as 7 input factors. It is therefore a relative large system. In the empirical analysis we use gross investment as a proxy of the service flow of capital, and the investment price as the associated price.

The paper is organized as follows. In Section 2 I present the theoretical model and show the implication of it on the expected growth of the different input factors. Section 3 presents the results from the empirical analysis. Section 4 concludes.

2 Theoretical model

In the presentation of the theoretical model I will only consider long-term properties. For simplicity, departures from and adjustments back towards these long run relationships are ignored in this section. (However, in conjunction with the empirical analysis in the next section I will allow for such temporary effects.)

In production functions with fixed parameters based on Cobb-Douglas or CES, the technological changes are often assumed to be Hicks-neutral. Sometimes the Hicks-neutral progress is combined with some factor-augmenting technological changes, see Klump et al. (2007) for an overview. These factor-augenting technological progresses are restricted to follow a deterministic trend. Also factor demand systems based on the translog cost function implies that the factor-augmenting progress is restricted to follow a deterministic trend. However, technological improvements might not be factor neutral and factor-augmenting technological changes might not follow a deterministic trend.

In this paper I only assume that only two parameters are time invariant. The elasticity of substitution (σ) and the elasticity of scale (κ) are assumed unaltered by the technological progress. The demand for input factor i conditioned on the production level is derived based on a CES (constant elasticity of substitution) production technology where the distribution parameters are time-varying. I use low-case letters to indicate that the variables are measured in logs, hence x is logs of production (but I will refer to x as production for simplicity). The factor demand depends on the price of the input factor p_i relatively to the price of other input factors (represented by a weighted factor price) p_A. The demand function of input factor i is written as[1]

$$(1) \qquad v_{i,t} = \sigma \ln \delta_{i,t} - \sigma\left(p_{i,t} - p_{A,t}\right) - \frac{1}{\kappa}\theta_t + \frac{1}{\kappa}x_t, \qquad i = 1,\dots, n.$$

[1]See Appendix A for how the factor demand functions are derived.

In (1) $\delta_{1,t}$, ..., $\delta_{n,t}$ are time-varying distribution parameters; $\delta_{i,t} > 0$ ($\forall i, t$), $\sum_{j=1}^{n} \delta_{j,t} = 1 (\forall t)$. (With a Cobb-Douglas technology, i.e. when $\sigma = 1$, these distribution parameters express the optimal factor cost ratios.) The time-dependence of the δ's is interpreted as picking up factor-biased (or factor-augmenting) technological changes.[2] The latent stochastic variable θ_t represents the factor neutral technology level.

Generally, the expression of the weighted factor price, p_A, is rather complicated. However, if $\sigma = 1$ (i.e. with a Cobb-Douglas production function) it is simply the weighted average of the different input factors, where the weight is equal to the optimal cost share. To calculate weighted factor prices, p_A, I have used the observable cost shares in each time period. By calculating the aggregated factor price by observed cost shares, the aggregated factor price also becomes observable.

The factor demand function can be rewritten as

$$(1') \qquad v_{i,t} + p_{i,t} - p_{A,t} = \sigma \ln \delta_{i,t} + (1-\sigma)(p_{i,t} - p_{A,t}) - \frac{1}{\kappa}\theta_t + \frac{1}{\kappa} x_t .$$

The expression on the left hand side of (1') is *the real cost of factor i* or *factor i adjusted for relative factor prices*. This expression will be treated as one variable, and we will refer to it as factor i (i.e. neglecting 'adjusted for relative factor prices'.) If we assume $\sigma = 1$ the expression $(1 - \sigma)$ $(p_{i,t} - p_{A,t})$ on the right hand side of (1') disappears and the number of (effective) variables in the analysis is reduced.

The factor neutral technological level is assumed to follow a stochastic trend where γ_θ is the drift parameter:

$$(2) \qquad \theta_t = \theta_0 + \gamma_\theta t + \sum_{s=1}^{t} \varepsilon_{\theta,s} ,$$

where the error term sequence $\varepsilon_{\theta,t}$ ($t = 1, \ldots, T$) are independent identically distributed stochastic variables with a zero mean. If $\varepsilon_{\theta,t} = 0$ $\forall t$, (2) simplifies to a *deterministic* factor neutral process.

[2]However, parameter instability may also stem from other reasons, such as aggregation (over firms) effects.

The distribution parameters are also allowed to follow stochastic processes.[3]

(3) $$\ln \delta_{i,t} = \ln \delta_{i,0} + \sum_{s=1}^{t} \varepsilon_{i,s},$$

where the error term sequence $\varepsilon_{i,t}$ ($t = 1, \ldots, T$) are independent identically distributed stochastic variables with a zero mean.

There are no drift parameters in (3). The absence of a drift implies that I do not expect some input factors to become systematically more important (and others to become less important) over time.

In order to write our system in matrix notation we need some definitions: Let $\mathbf{I}_n$ be the *identity matrix* of dimension n. Furthermore, let $\mathbf{1}_{n\times m}$ be a *unity matrix* of dimension $n \times m$, i.e. a matrix where all elements are unity. Similarly, let $\mathbf{0}_{n\times m}$ be a *zero matrix* of dimension $n \times m$, i.e. a matrix of zeros.

Taking account of (2) and (3) included, the system in (1') can be written in matrix form as

(4) $$\begin{aligned} \mathbf{v}_t + \mathbf{p}_t - \mathbf{p}_{A,t} &= \sigma \cdot \ln \boldsymbol{\delta}_0 - \left(\frac{\theta_0}{\kappa} \cdot \mathbf{1}_{n\times 1} \right) + \left((1-\sigma) \cdot \mathbf{I}_n \right) \left(\mathbf{p}_t - \mathbf{p}_{A,t} \right) \\ &+ \left(\frac{1}{\kappa} \cdot \mathbf{1}_{n\times 1} \right) x_t - \left(\frac{\gamma_\theta}{\kappa} \cdot \mathbf{1}_{n\times 1} \right) t + \sum_{s=1}^{t} \left(\sigma \boldsymbol{\varepsilon}_s - \frac{1}{\kappa} \boldsymbol{\varepsilon}_{\theta,s} \right), \end{aligned}$$

where

$$\begin{aligned} \mathbf{v}_t &= \left(v_{1,t}, \ldots, v_{n,t} \right)', \\ \mathbf{p}_t &= \left(p_{1,t}, \ldots, p_{n,t} \right)', \\ \mathbf{p}_{A,t} &= p_{A,t} \cdot \mathbf{1}_{n\times 1}, \\ \boldsymbol{\delta}_0 &= \left(\delta_{1,0}, \ldots, \delta_{n,0} \right)', \\ \boldsymbol{\varepsilon}_t &= \left(\varepsilon_{1,t}, \ldots, \varepsilon_{n,t} \right)', \text{ and} \\ \boldsymbol{\varepsilon}_{\theta,t} &= \varepsilon_{\theta,t} \cdot \mathbf{1}_{n\times 1} \end{aligned}$$

If $\sigma = 1$, i.e. with a Cobb-Douglas production function, the expression in (4) simplifies to

[3]Due to the fact that the relative weights sum to unity, the errors in (2) are not independent. However, this non-linear relationship between the errors is not important in the present analysis, and is therefore not explicitly taken into account.

$$
(5) \qquad \begin{aligned} \mathbf{v}_t + \mathbf{p}_t - \mathbf{p}_{A,t} &= \ln \boldsymbol{\delta}_0 - \left(\frac{\theta_0}{\kappa} \cdot \mathbf{1}_{n\times 1} \right) \\ &+ \left(\frac{1}{\kappa} \cdot \mathbf{1}_{n\times 1} \right) x_t - \left(\frac{\gamma_\theta}{\kappa} \cdot \mathbf{1}_{n\times 1} \right) t + \sum_{s=1}^{t} \left(\boldsymbol{\varepsilon}_s - \frac{1}{\kappa} \boldsymbol{\varepsilon}_{\theta,s} \right), \end{aligned}
$$

Equations (4) and (5) include unobservable stochastic components. Below I will show how we can remove these components, depending on the type of technological progress prevailing.

2.1 Factor demand relationships

Before I present the general case, I will consider two special cases. First, consider the case where the factor neutral technological progress is deterministic and the distribution parameters in the production function are fixed:

Case 2.1 *Deterministic factor neutral technological progress:*
Defined as: $\varepsilon_{\theta,t} = 0,\ \forall t$ *and* $\varepsilon_{i,t} = 0,\ \forall i, t$. *Then*

$$
(6) \qquad \begin{aligned} \mathbf{v}_t + \mathbf{p}_t - \mathbf{p}_{A,t} &= \sigma \cdot \ln \boldsymbol{\delta}_0 - \left(\frac{\theta_0}{\kappa} \cdot \mathbf{1}_{n\times 1} \right) + \left((1-\sigma) \cdot \mathbf{I}_n (\mathbf{p}_t - \mathbf{p}_{A,t}) \right) \\ &+ \left(\frac{1}{\kappa} \cdot \mathbf{1}_{n\times 1} \right) x_t - \left(\frac{\gamma_\theta}{\kappa} \cdot \mathbf{1}_{n\times 1} \right) t, \end{aligned}
$$

expresses n relationships among observable variables.

If there are no unobserved stochastic components in the production function, the long-run factor demand for each input factor only depends on observable variables. The demand for an input factor depends on its relative price, the level of production, and a deterministic trend that represents the factor neutral technological progress.

Another special case is when there is a stochastic factor neutral drift in production, but no stochastic components in the distribution parameters.

Case 2.2 *Stochastic factor neutral technological progress:*

Defined as: $\varepsilon_{i,t} = 0,\ \forall i,\ t$. *Then it is possible to remove the stochastic components in (4) by pre-multiply by the matrix* $\mathbf{B}^{*'} = (\mathbf{I}_{n-1}, -\mathbf{1}_{(n-1)\times 1})$ *(or any matrix spanning the same space as* $\mathbf{B}^*$*), which yields*

$$\text{(7)} \qquad \mathbf{B}^{*'}[\mathbf{v}_t + \mathbf{p}_t - \mathbf{p}_{A,t}] = \mathbf{B}^{*'}(\sigma \cdot \ln \boldsymbol{\delta}_0) + \mathbf{B}^{*'}((1-\sigma)\cdot \mathbf{I}_n(\mathbf{p}_t - \mathbf{p}_{A,t})).$$

Equation (7) expresses n − 1 relationships among the observable variables where some terms have disappeared because $\mathbf{B}^{*'}\mathbf{1}_{n\times 1} = \mathbf{0}$. *(Note that* $\mathbf{B}^{*'}\mathbf{p}_{A,t} = \mathbf{0}$*, so this term could also be cancelled out.)*

Equation (7) expresses relative demand for the input factors: The demand for input factor i ($i = 1, 2, ..., n-1$) will increase relatively to factor n if the price for factor i decreases relatively to the price of factor n, i.e. $v_i - v_n = constant - \sigma\,(p_i - p_n)$. (We can of course normalize on another input factor than factor n.) In this case none of the relationships of observable variables include the production level, see (7).

Proposition 2.1 *In both Case 2.1 and Case 2.2 the expression of the cost share does not involve any unobserved stochastic processes. The log of the cost share for input factor i is given by*

$$\text{(8)} \qquad \ln \frac{\exp(p_{i,t} + v_{i,t})}{\sum_{j=1}^{n} \exp(p_{j,t} + v_{j,t})} = \sigma \ln \delta_i + (1-\sigma)\sum_{j=1}^{n} \delta_j + (1-\sigma)(p_{i,t} - p_{A,t}).$$

Otherwise, the expression for at least two of the cost ratios will involve stochastic processes.

Proof. See Appendix B for proof.

Note that if the elasticity of substitution is unity, i.e. with a Cobb-Douglas technology, δ_i is the cost ratio for input factor i.

According to Proposition 2.1, the expression for each cost ratio involves only observable variables as long as the distribution parameters, $\delta_1, \ldots, \delta_n$ are fixed. Therefore, fixed parameter production functions of the Cobb-Douglas type or CES type imply that the cost ratio is a function of observable variables, and hence follows a deterministic path. Also other common approaches imply that the cost ratios can be expressed by observable variables only. For example, with a translog cost function, the corresponding cost shares are functions of the different factor prices (plus, possibly, a deterministic trend).

In this paper I want to allow for both factor neutral and non-neutral technological progress. Even then, it is possible to find linear combinations of the equations in (6) such that the unobserved stochastic components cancel out. Such linear combination will exist if the stochastic errors are linear dependent. Assume that the errors can be written as

(9) $$\boldsymbol{\varepsilon}_t - \frac{1}{\kappa}\boldsymbol{\varepsilon}_{\theta,t} = \mathbf{A}\mathbf{e}_t,$$

where $\mathbf{A}$ is an $n \times (n - r)$ matrix with full column rank, and $\mathbf{e}_t$ is a vector (with $n - r$ elements) of errors describing the common trends in the demand system. To see how these common trends can be removed, we need to define the orthogonal complement of a matrix. Let the $n \times r$ matrix $\mathbf{B}$ be the *orthogonal complement* to $\mathbf{A}$, i.e. $\mathbf{B} = \mathbf{A}_\perp$. The orthogonal complement of the full column rank matrix $\mathbf{A}$ is written as $\mathbf{A}_\perp$ with properties such that $\mathbf{A}'_\perp\mathbf{A} = \mathbf{0}$ and $(\mathbf{A}, \mathbf{A}_\perp)$ has full rank. (The orthogonal complement of a non-singular matrix is 0, and the orthogonal complement of a zero matrix is an identity matrix with a suitable dimension.)

The matrix $\mathbf{B}$ is not unique. The matrix $\breve{\mathbf{B}} = \mathbf{B} \cdot \mathbf{Q}$, where $\mathbf{Q}$ is a non-singular matrix (of dimension $r \times r$), will also be a representation of the orthogonal complement to $\mathbf{A}$. Then $\breve{\mathbf{B}}$ and $\mathbf{B}$ are said to span the same space, and we write this as $sp(\breve{\mathbf{B}}) = sp\,(\mathbf{B})$. For our purpose, the non-uniqueness of $\mathbf{B}$ does not represent a problem since we only require that the space spanned by $\mathbf{B}$ is unique.

Let the highest number of independent linear combination of factor demand functions where the common trends are removed be r. These linear combinations can be derived by pre-multiplying (9) with the $r \times n$ matrix $\mathbf{B}' = \mathbf{A}'_\perp$, such that

(10) $$\mathbf{B}'\left(\boldsymbol{\varepsilon}_t - \frac{1}{\kappa}\boldsymbol{\varepsilon}_{\theta,t}\right) = \mathbf{0}, \forall t.$$

The system becomes

(11)
$$\mathbf{B}'\left[\mathbf{v}_t + \mathbf{p}_t - \mathbf{p}_{A,t}\right] = \mathbf{B}'(\sigma \cdot \ln \boldsymbol{\delta}_0) - \mathbf{B}'\left(\frac{\theta_0}{\kappa} \cdot \mathbf{1}_{n\times 1}\right) + \mathbf{B}'\left((1-\sigma)\cdot \mathbf{I}_n\left(\mathbf{p}_t - \mathbf{p}_{A,t}\right)\right)$$
$$+ \mathbf{B}'\left(\frac{1}{\kappa} \cdot \mathbf{1}_{n\times 1}\right)x_t - \mathbf{B}'\left(\frac{\gamma_\theta}{\kappa} \cdot \mathbf{1}_{n\times 1}\right)t.$$

Case 2.1 and Case 2.2 are both special cases of (11):

- If $r = n$, i.e. $\mathbf{B}$ has full rank, one can use $\mathbf{B} = \mathbf{I}_n$, and the system coincides with Case 2.1.
- If $r = n - 1$ and $sp\ \mathbf{B} = sp\ (\mathbf{I}_{n-1}, -\mathbf{1}_{(n-1)\times 1})'$, the system coincides with Case 2.2.

2.2 Growth and growth rates

The theoretical model presented above involves information about the growth of the different input factors. We then derive the expected growth rate for each input factor from (4) by first taking the difference and second taking the expectation. (The second step is not necessary with deterministic factor neutral technological progress, i.e. Case 2.1.) Let Δ be the difference operator and E_t the expectation operator (where the subscript indicates that the expectation is formed at the beginning of period t).

(12)
$$E_t\Delta\left[\mathbf{v}_t + \mathbf{p}_t - \mathbf{p}_{A,t}\right] = \left((1-\sigma)\cdot \mathbf{I}_n\right)E_t\Delta\left(\mathbf{p}_t - \mathbf{p}_{A,t}\right)$$
$$+ \left(\frac{1}{\kappa} \cdot \mathbf{1}_{n\times 1}\right)E_t\Delta x_t - \left(\frac{\gamma_\theta}{\kappa} \cdot \mathbf{1}_{n\times 1}\right)$$

(since $E_t\left[\sum_{s=1}^{t}\varepsilon_s - \sum_{s=1}^{t-1}\varepsilon_s\right] = E_t\varepsilon_t = 0$ and $E_t\left[\sum_{s=1}^{t}\varepsilon_{\theta,s} - \sum_{s=1}^{t-1}\varepsilon_{\theta,s}\right] = E_t\varepsilon_{\theta,t} = 0$.)

From equation (12) we see that the expected growth of the different input factors (adjusted for changes in relative input prices) depends on expected productivity growth ($E_t\, \Delta\theta_t = \gamma_\theta$), expected production growth ($E_t\, \Delta x_t$), and the scale-elasticity (κ). If $\sigma \neq 1$, the expected growth of the real input factors also depends on the expected changes in relative input prices. It also follows from (12) that the expected growth in all the factors are equal, given that the relative input prices do not change. (This stems from the assumption that no factors become more important over time in a systematic way, i.e. that there is no drift in (3).)

3 Empirical analysis

The theoretical background presented in the previous section addresses long run properties only. We should not expect the demand for the different input factors to be on its (long-run) equilibrium level in each period. However, we should expect the use of the input factors to be adjusted back towards its equilibrium level if they differ from this level.

To analyse the factor demand system I apply a cointegrated VAR model. Following Hungnes (2002), I formulate the cointegrated VAR such that the deterministic parts are interpretable. However, I extend this approach by also allowing for exogenous variables in such a way that their effect on the endogenous variables is easy to interpret.

I will make use of the following definitions: *Strict stationarity* is defined as a stochastic process whose joint distribution of observations is not a function of time, i.e. the joint distribution of $\left(x_{t_1}, x_{t_2}, \ldots, x_{t_k}\right)$ is the same as the distribution of $\left(x_{t_1+h}, x_{t_2+h}, \ldots, x_{t_k+h}\right)$ for any value of h. *Weak stationarity* (or covariance stationarity) describes a process where the first two moments are not functions of time. A stochastic process is called $I(d)$ if it is weakly stationary after differencing d times, but not weakly stationary after differencing $d-1$ times.

I will model a system with all the n input factors. Let Y_t be a vector of these variables, which are assumed to be non-stationary variables integrated of order one ($I(1)$) at most. Furthermore, let Z_t be a vector of q deterministic and exogenous variables. The coefficient matrix γ is of dimension $n \times q$. The coefficient matrices α and β are of dimension $n \times r$ (where r is the number of cointegrating vectors and - as will be shown below - corresponds to the rank of **B** in the previous section) and $\beta'\left(Y_t - \gamma Z_t\right)$ comprises r cointegrating $I(0)$ relations. Furthermore, Γ_i ($i = 1, 2, ..., p - 1$) are $n \times n$ matrices of coefficients, where p denotes the number of lags (in levels). I assume the error vector u_t to be independent and identically Gaussian white noise, $u_t \sim iidN(0, \Omega)$.

(13) $$\Delta Y_t - \gamma \Delta Z_t = \alpha\left(\beta'\left(Y_{t-1} - \gamma Z_{t-1}\right) - \mu\right) + \sum_{i=1}^{p-1} \Gamma_i \left(\Delta Y_{t-i} - \gamma \Delta Z_{t-i}\right) + u_t \, .$$

As for the matrix **B**, only the space spanned by β is identifiable.

Condition 3.1 *Assume that n − r of the roots of the characteristic polynomial*

$$A(z) = (1 - z)I_n - \alpha\beta' z - \sum_{i=1}^{p-1} \Gamma_i (1 - z) z^i$$

are equal to 1 and the remaining roots are outside the complex unit circle.

Under Condition 3.1, the system grows at the unconditional rate $E_t[\Delta Y_t] = \gamma E_t[\Delta Z_t]$ with long run (cointegrating) mean levels $E_t[\beta'(Y_t - \gamma Z_t)] = \mu$. Furthermore, under Condition 3.1, Y_t has the (moving-average) representation

(14) $$Y_t = \iota + \gamma Z_t + C\sum_{s=1}^{t} u_s + \Lambda_t ,$$

where $C = \beta_\perp (\alpha'_\perp \Gamma \beta_\perp)^{-1} \alpha'_\perp$ with $\Gamma = I_n - \sum_{i=1}^{p-1} \Gamma_i$ and the vector process Λ_t is stationary with expectation zero. The level vector ι depends on initial values in such a way that $\beta'\iota = \mu$.

Note the similarity between (14) and (4):

- Y_t corresponds to $\mathbf{v}_t + \mathbf{p}_t - \mathbf{p}_{A,t}$;
- ι corresponds to $\sigma \cdot \ln \boldsymbol{\delta}_0 - \left(\frac{\theta_0}{\kappa} \cdot \mathbf{1}_{n\times 1} \right)$;
- γZ_t corresponds to $((1-\sigma)\cdot \mathbf{I}_n)(\mathbf{p}_t - \mathbf{p}_{A,t}) + \left(\frac{1}{\kappa} \cdot \mathbf{1}_{n\times 1} \right) x_t - \left(\frac{\gamma_\theta}{\kappa} \cdot \mathbf{1}_{n\times 1} \right) t$; and
- $C\sum_{s=1}^{t} u_s$ corresponds to $\sigma \sum_{s=1}^{t} \varepsilon_s + \frac{1}{\kappa} \sum_{s=1}^{t} \varepsilon_{\theta,t}$.

Λ_t only captures short run dynamics, and was therefore ignored in the theoretical section above.

Since (14) is just another representation of (13), this shows that our empirical system is suitable for estimating and testing the hypotheses outlined in the theoretical part.

Proposition 2.1 has some important empirical implications. Let c_i be the log of the cost share of factor i. If $\sigma = 1$ then $c_i = \ln \delta_i$, and the cost share will be (weakly) stationary if δ_i is (weakly) stationary. Generally (i.e. when $\sigma > 0$) c_i will cointegrate with $p_i - p_A$ with cointegrating vector $(1,-(1 - \sigma))$ if

$\sigma \ln \delta_i + (1-\sigma)\sum_{j=1}^{n} \delta_j \ln \delta_j$ follows a stationary process. I will assume that this expression is stationary if the processes of the δ*s* are stationary.[4] Therefore, if the process of all the distribution parameters ($\delta_{1,t}, \ldots, \delta_{n,t}$) are (strictly) stationary, there will be a cointegrating relationship between each cost share and the relative factor prices. Since the cost shares sum to unity, there will be $n-1$ independent cointegrating relationships between the cost shares and the relative factor prices. Therefore, if the processes of the distribution parameters are strictly stationary, there will be (at least) $n-1$ cointegrating relationships in our analysis. (This still holds even if I am not modelling the cost shares explicitly.) If, in addition, the process of the (factor neutral) technological process is (weakly) trend stationary, there will be an additional independent cointegrating relationship in the system.

Therefore, if both θ and the δs follow stationary processes (i.e. Case 2.1), there will be one independent cointegrating relationship for each input factor and the cointegrating space is described by (6). And if the the δs are stationary and θ follows a nonstationary $I(1)$ process (i.e. Case 2.2), there are $n-1$ independent cointegrating relationships and the cointegrating space is described by (7).

If not all of the distribution parameters follow stationary processes, there will not be a cointegrating relationship for each cost share. And, at least if θ does not follow a trend stationary process, the number of independent cointegrating relationships will be less than $n-1$.

3.1 Distribution of cointegrating rank test

From the comparison of (14) and (4) we noted that the vector Z_t must include production (x_t). In the cointegrating rank test we therefore condition on the production level. Even though I condition on exogenous variables in the cointegrating rank test I can apply conventional critical values instead of critical values for partial systems.[5] I will illustrate why the conventional values should be used below.

Let Z be partitioned into exogenous and deterministic variables, i.e. $Z_t = (X'_t, D'_t)'$, where X_t is $q_X \times 1$ and D_t is $q_D \times 1$. Similarly, partition γ as (γ_X, γ_D) and ρ as (ρ_X, ρ_D) with appropriate dimensions. Furthermore, to simplify, we only include one lag here, i.e. $p = 1$. The system in (13) can be written as

(15) $$\Delta Y_t = \gamma \Delta Z_t + \alpha(\beta' Y_{t-1} - \rho Z_{t-1} - \mu) + u_t$$

[4]A non-linear function of weakly stationary processes is not necessary weakly stationary. However, a non-linear function of strictly stationary processes is strictly stationary.

[5]Thanks to Bent Nielsen and Søren Johansen for pointing out that with the formulation of the model chosen here the conventional critical values should be used.

where $\rho = \beta'\gamma$(i.e. $\rho_X = \beta'\gamma_X$ and $\rho_D = \beta'\gamma_D$).

To write the partial system we define $\rho^* = \left(\rho_X^*, \rho_D^*\right)$. The conventional formulation of a partial system can be written as[6]

(16) $$\Delta Y_t = \gamma \Delta Z_t + \alpha\left(\beta' Y_{t-1} - \rho^* Z_{t-1} - \mu\right) + u_t^*,$$

where $\rho_D^* = \beta'\gamma_D$ (which implies $\rho_D^* = \rho_D$)[7] and no restrictions between the coefficient matrixes γ_X, β and ρ_X^*. The partial system can be rewritten as

(17) $$\Delta Y_t = \gamma \Delta Z_t + \alpha\left(\beta'(Y_t - \gamma Z_{t-1}) - \left(\rho_X^* - \rho_X\right) X_{t-1} - \mu\right) + u_t^*.$$

Writing the partial system as in (17) it is easy to see why this system not only depends on the common trends, but also on the number of exogenous variables (q_X). The reason why we must adjust for exogenous variables when determining the cointegrating rank in such partial systems, is that the number of coefficients in the expression $\rho_X^* - \rho_X$ depends on the cointegrating rank as well. It is the presence of the term $\left(\rho_X^* - \rho_X\right) X_{t-1}$ that makes the convetional critical tables for determining the cointegrating rank invalid. However, since I implicitly impose the restriction $\rho_X^* = \rho_X$ in (13), this term vanishes, and I can apply the conventional critical values for the cointegrating rank tests.

To support the claim that the conventional critical values can be used to determine the rank in (13), independent of how many exogenous variables I include, I have simulated different quantiles of the distribution of the cointegrating rank test for the system formulation in both (15) and (16). The right part of Table 1 reports these simulated quantiles.

When there are no exogenous variables, the formulaltions in (15) and (16) are identical. Therefore, the simulated quantiles should be close to the asymptotic critical values. Comparing the first column in the right part of Table 1 with the corresponding row in the left part of Table 1 show that these quantiles are approximately equal.

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[6] See Harbo et al. (1998) on partial systems.

[7] To be able to rewrite the partial system like this, D_t can only include trend and seasonal dummies, see Hungnes (2005a). With other deterministic variables included, (16) is only an approximation.

Table 1: Asymptotic and simulated quantiles for systems with $D_t = t$

		Partial system, as (16)				Representation her, as (15)			
n - r	quantile	$q_X = 0$	$q_X = 1$	$q_X = 2$	$q_X = 3$	$q_X = 0$	$q_X = 1$	$q_X = 2$	$q_X = 3$
1	0.90	10.68	13.35	15.96	18.52	10.64	10.83	10.54	10.90
	0.95	12.45	15.33	18.16	20.89	12.56	13.20	12.26	12.45
	0.99	16.22	19.53	22.76	25.84	15.12	16.38	15.44	16.22
2	0.90	23.32	28.20	32.98	37.73	22.86	23.30	22.78	22.61
	0.95	25.73	30.91	35.96	40.95	25.43	26.20	25.15	25.43
	0.99	30.67	36.44	42.00	47.46	29.60	31.16	31.22	30.41
3	0.90	39.73	46.70	53.63	60.52	39.83	39.50	40.52	39.36
	0.95	42.77	50.08	57.32	64.48	42.34	42.73	43.85	42.87
	0.99	48.87	56.83	64.66	72.37	48.00	49.67	48.29	48.83
4	0.90	60.00	69.10	78.14	87.12	59.47	59.55	59.23	60.24
	0.95	63.66	73.13	82.50	91.79	63.49	63.08	63.12	63.92
	0.99	70.91	81.09	91.10	95.97	69.95	70.57	70.33	70.35

In the left part: Asymptotic quantiles (of Trace test) for systems with $D_t = t$ in (16), based on a Gamma-distribution as suggested in Doornik (1998, 2003) for different number of exogenous variables (q_X). The reported quantiles are taken from Tables 4 and 13 in Doornik (2003); the first column ($q_X = 0$) is taken from the former and the next three columns ($q_X = 1, 2, 3$) are taken from the latter. In the right part: Simulated quantiles (of Trace test) for H_l (i.e. with $D_t = t$) in (15). The number of observations is 500 and number of replications is 1000 for each combination of ($n - r$, q_X), $n - r = 1, 2, 3, 4$ & $q_X = 0, 1, 2, 3$.

However, the simulated quantiles for systems with one or more exogenous variables differ from the asymptotic quantiles reported in the left part of the table. We can see that the asymptotic quantiles increase with the number of exogenous variables, and the simulated quantiles do not. Since the simulated quantiles seem to be unaffected by including exogenous variables in (15), this indicates that we can apply the conventional asymptotical quantiles for closed systems in the cointegrating rank test.

3.2 Factor demand with Cobb-Douglas Technology ($\sigma = 1$)

To get unbiased estimates of the elasticity of substitution, relative factor prices must be (weakly) exogenous (with respect to the parameter of interest), see Richard (1980). If price changes occur due to changes in demand, the estimate of the elasticity of substitution may be (downward) biased. In the present analysis it is assumed that the substitution elasticity is unity, i.e. $\sigma = 1$. Then our analysis can be based on a Cobb-Douglas production function (with non-fixed parameters). This implies that we do not include the relative prices in the vector of exogenous variables.

Table 2: Input factors and corresponding prices

Notation	Input factor	price
L	Labour	p_L
E	Electricity	p_E
F	Fuel	p_F
M	Other materials	p_M
J_b	Buildings	$p_{J,b}$
J_{te}	Transport equipment	$p_{J,te}$
J_m	Machinery	$p_{J,m}$

In the analysis I model 7 different input factors, see Table 2. Among these are two energy inputs (electricity and fuel) and three different types of real capital (buildings, transport equipment and machinery).

For the three real capital inputs I use the investment cost as a proxy for the cost of using that particular real capital factor. Alternatively one could use the real capital stocks multiplied by the user cost of the input factors. The latter has theoretical advantages, but might be more complicated to use from a practical point of view, since the user cost is not directly observable and because there can be large measurement problems for the real capital stocks time series.

The vector of endogenous variables is therefore defined as follows:

$$Y_t = \begin{bmatrix} v_L + p_L - P_A \\ v_L + p_L - P_A \\ v_L + p_L - P_A \\ v_L + p_L - P_A \\ v_{J,b} + p_{J,b} - P_A \\ v_{J,te} + p_{J,te} - P_A \\ v_{J,m} + p_{J,m} - P_A \end{bmatrix}_t$$

The vector of exogenous variables is defined as

$$Z_t = \begin{bmatrix} x_t \\ t \end{bmatrix}.$$

In the vector Z_t I could also have included other variables, such as seasonal dummies. However, seasonal dummies are not included in this analysis.

The system in (13) combined with the choices of Y_t and Z_t implies that I indirectly impose a common factor restriction between the variables in Y_t and the production, x_t. However, inclusion of lagged differences of production (i.e. Δx_t) and/or lagged differences of endogenous variables (i.e. ΔY_t) in Z_t would remove this implied restriction. To limit the number of parameters to be estimated, we do not include such lagged differences in Z_t in this analysis.

Due to the flexibility in the description of technological changes in (13), I believe that the econometric formulation can be applied to model the factor demand system in all industry sectors. In the empirical illustration below I apply national accounts data from the Norwegian 'Building and construction' industry.

Figure 1 displays the data series I have used in the estimation. As can be seen from the plots, there are cycles in many of the time series.

Figure 2 plots the different cost shares. If the δ's follow stationary processes, the cost shares should follow stationary processes as well (when σ= 1). When σ= 1 these cost shares should be stationary around a fixed level when the δ's follow stationary processes. The graphs in Figure 2 indicate that this is not the case.

By comparing (14) and (5) we see that the theoretical part implies restrictions on the matrix of coefficients γ. Since, in the absence of changes in (relative) factor prices, the use of each input factor should grow at the same rate and react similarly to changes in production, this coefficient matrix must have the form

$$(18) \qquad \gamma = \left(\frac{1}{\kappa} \cdot \mathbf{1}_{7\times 1}, -\frac{\gamma_\theta}{\kappa} \cdot \mathbf{1}_{7\times 1} \right).$$

The matrix β in (13) corresponds to the matrix B in the theoretical part, i.e. sp (β) = sp (**B**). Therefore, it follows that these two matrixes must have the same rank, i.e. rank (β) = rank (**B**).

Remark 1: *Case 2.1 implies, in addition to that γ has the form given in (18), that β has full rank.*

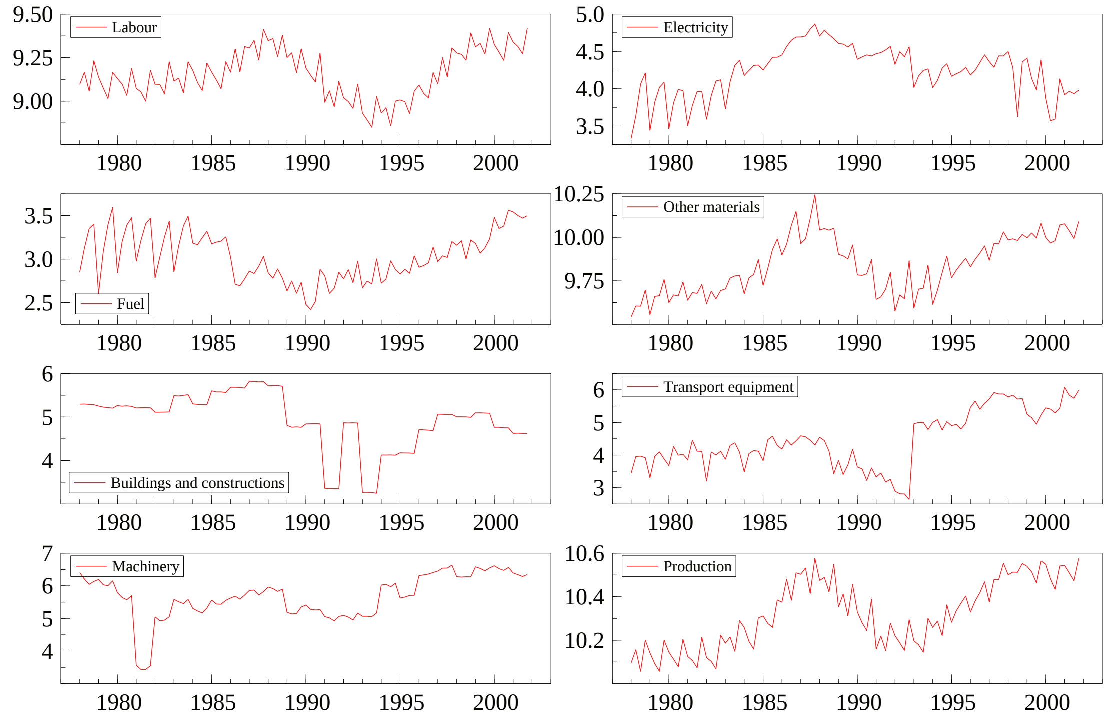

Figure 1: The data series used in the estimation. The time series for the input data are defined as $v_i + p_i - p_A$ where $i = L, E, F, M, J_b, J_{te}, J_m$. Production x (log-transformed) is reported in the lower right part of the table.

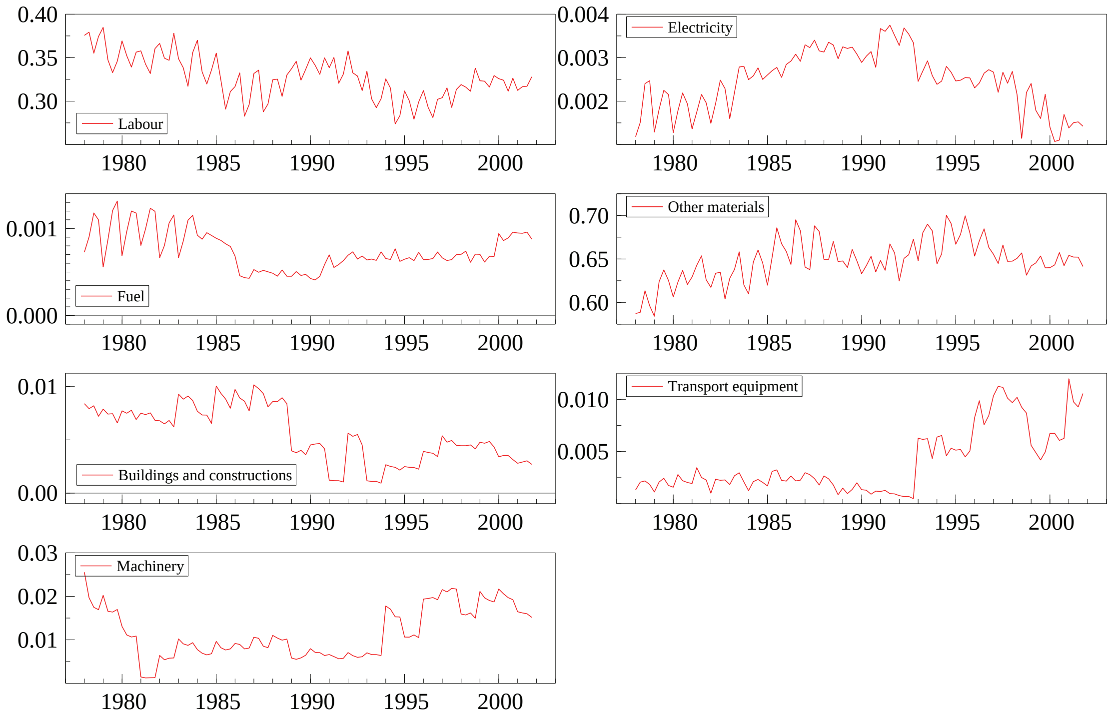

Figure 2: The cost shares for the different input factors.

Table 3: Cointegrating rank test

	log-likelihood	LR (vs $r = 7$)	p-value
$r = 7$	374.904		
$r = 6$	372.162	5.49	0.538
$r = 5$	368.066	13.68	0.688
$r = 4$	358.373	33.06	0.339
$r = 3$	346.379	57.05	0.164
$r = 2$	333.254	83.30	0.116
$r = 1$	308.215	133.38	0.003
$r = 0$	276.593	196.62	0.000

The estimation period is 1980q1 - 2002q4 (i.e. $T = 88$). The p-values are calculated based on a Γ-distribution as suggested by Doornik (1998). In the estimation 2 lags are used, i.e. $p = 2$. The estimation results are obtained by using GRaM 0.99, see Hungnes (2005b).

Remark 2: *Case 2.2 implies, in addition to that γ has the form given in (18), that rank (β) = n − 1 and furthermore that* $sp(\beta) = sp\left(\mathbf{I}_{n-1}, -\mathbf{1}_{(n-1)\times 1}\right)'$.

Table 3 reports the tests of the cointegrating rank. The reported probability values are based on a Γ-distribution, as suggested by Doornik (1998). According to these significance probabilities the hypotheses of a rank equal to 2 or higher are not rejected, whereas the hypotheses of a rank equal to 1 or 0 are rejected. Therefore, the cointegrating rank test indicates a rank of 2, so the rank is neither full nor equal to $n - 1$. Hence, the number of independent cointegrating relationships is consistent with technological changes.

Table 4 reports the results of different hypotheses on γ (tested against the system with no restrictions on γ). The first test is labelled 'Scale-elasticity'. Here I restrict all the elements in the first column of γ to be equal, a restriction which is not rejected. The reciprocal of the estimated value is the estimated κ, i.e. the scale-elasticity. The estimated scale elasticity is close to unity.

The next test, 'No drift', tests the hypothesis that there is no deterministic trend in the cost ratios. This involves restricting all elements in the second column of γ to be equal. (The sign '+' indicates that this is an additional test, i.e. I restrict both columns in γ.) This hypothesis is also not rejected. When I impose both these restrictions, the technological growth is identified. According to the estimation results the technological growth is about 0.2 per cent in annual terms.

Table 4: Likelihood ratio tests

	log-likelihood	LR	d.f.	p-value	κ	γ_θ
No restr. (r = 2)	333.254					
Scale-elasticity	328.182	10.15	6	0.119	1.062	
+ no drift	327.527	11.46	12	0.490	1.061	0.21%
+ scale-el. = 1	327.267	11.97	13	0.530	1.000	0.30%
+ no techn. growth	325.435	15.64	14	0.336	1.000	0.00%

Results from testing different restrictions on γ. LR denotes the log-likelihood ratio; d.f. the degrees of freedom; and p-value the significance value. The elasticity of scale (κ) and technological progress (γ_θ) are reported (when possible) under different restrictions. γ_θ is reported in per cent in annual terms. The estimation results are obtained by using GRaM 0.99, see Hungnes (2005b).

I also test two additional hypotheses. The first of these hypotheses is if the scale elasticity is unity. This hypothesis is not rejected, and we can conclude that the scale elasticity is not significantly different from unity. The second hypothesis is if the technological growth is zero. This hypothesis is also not rejected. Hence the technological growth is not significantly positive.

The non-rejection of the hypothesis of no technological growth implies that there are no significant trends in the data series. However, in Table 3 I have included a trend in the cointegrating space when calculating the rank test. To take account of the fact that the trends seem to be insignificant, I test the cointegrating rank without a trend variable. Table 5 reports the results of this cointegrating rank test. Also when not including the trend, the hypotheses of a rank equal to 2 or higher are rejected, whereas the hypotheses of a rank equal to 1 or 0 are rejected. Therefore, also the cointegrating rank test where the trend variable is excluded indicates a rank of 2.

Table 5: Cointegrating rank test (without trend)

	log-likelihood	LR (vs r = 7)	p-value
r = 7	365.932		
r = 6	365.088	1.69	0.830
r = 5	362.227	7.41	0.865
r = 4	357.159	17.55	0.857
r = 3	345.764	40.34	0.459
r = 2	332.387	67.09	0.226
r = 1	307.530	116.80	0.005
r = 0	276.235	179.39	0.000

Note: See Table 3.

3.3 Factor demand with CES Technology

The results in Section 3.2 indicate that there are fewer cointegrating relationships than there are (independent) cost shares. However, this result might stem from my a priori value of the elasticity of substitution. In order to test robustness of the realized rank finding, I conduct cointegrating rank tests for different (imposed) substitution-elasticities. To do this, we re-define the vector of endogenous variables to

$$Y_t = \begin{bmatrix} v_L + \sigma(p_L - P_A) \\ v_L + \sigma(p_L - P_A) \\ v_L + \sigma(p_L - P_A) \\ v_L + \sigma(p_L - P_A) \\ v_{J,b} + \sigma(p_{J,b} - P_A) \\ v_{J,te} + \sigma(p_{J,te} - P_A) \\ v_{J,m} + \sigma(p_{J,m} - P_A) \end{bmatrix}_t$$

and keep Z_t unchanged.

Table 6 reports the results of the cointegrating rank tests for different values of the substitution-elasticity (σ). Two things should be noted: First, for each rank the log-likelihood values are highest for $\sigma = 0$ or $\sigma = 0.1$. This indicates that if we had estimated the substitution-elasticity, the estimate would have been close to zero. Second, for all the chosen values of σ in Table 6, the cointegrating rank test indicates a rank of 2, confirming that the rank is substantially less than the number of (independent) cost shares.

Similarly test of restrictions on γ as those presented in Table 4 are reported in Appendix C for different values of σ.

Table 6: Cointegrating rank tests for different values of σ

Cointegrating rank test with trend

	$\sigma = 0$		$\sigma = 0.1$		$\sigma = 0.5$		$\sigma = 1$		$\sigma = 2$	
	log-lik.	p-value	log-lik.	p-value	log-lik.	p-value	log-lik.	p-value	log-lik.	p-value
$r = 7$	459.18		459.19		430.53		374.90		276.24	
$r = 6$	456.41	0.530	456.50	0.551	427.85	0.553	372.16	0.538	273.45	0.526
$r = 5$	451.22	0.508	451.18	0.499	422.91	0.561	368.07	0.688	269.90	0.764
$r = 4$	443.19	0.396	443.07	0.381	414.03	0.341	358.37	0.339	259.67	0.335
$r = 3$	430.60	0.161	430.57	0.159	401.89	0.158	346.38	0.164	248.48	0.207
$r = 2$	416.01	0.072	415.97	0.071	388.13	0.092	333.25	0.116	236.06	0.174
$r = 1$	392.25	0.003	392.01	0.002	362.83	0.002	308.22	0.003	213.52	0.013
$r = 0$	353.04	0.000	353.65	0.000	328.51	0.000	276.59	0.000	178.07	0.000

Cointegrating rank test without trend

	$\sigma = 0$		$\sigma = 0.1$		$\sigma = 0.5$		$\sigma = 1$		$\sigma = 2$	
	log-lik.	p-value	log-lik.	p-value	log-lik.	p-value	log-lik.	p-value	log-lik.	p-value
$r = 7$	445.38		446.84		421.62		365.93		257.48	
$r = 6$	443.66	0.510	445.21	0.541	420.41	0.697	365.09	0.830	256.78	0.879
$r = 5$	440.25	0.619	441.85	0.646	417.25	0.761	362.23	0.865	253.85	0.875
$r = 4$	433.92	0.538	435.68	0.577	411.81	0.750	357.16	0.857	249.17	0.895
$r = 3$	422.76	0.244	424.32	0.251	400.21	0.342	345.76	0.459	239.04	0.634
$r = 2$	408.44	0.083	410.12	0.089	386.72	0.156	332.39	0.226	225.63	0.336
$r = 1$	390.54	0.018	390.56	0.011	362.089	0.003	307.53	0.005	205.00	0.041
$r = 0$	351.83	0.000	352.61	0.000	327.904	0.000	276.24	0.000	177.50	0.001

4 Conclusions

The present paper suggests an approach for estimating factor demand systems with technological changes. The approach allows for both factor neutral technological progress as well as technological changes that change the relative use of the different input factors.

The estimation approach makes it possible to impose the restriction that the expected growth in all input factors is equal. This corresponds to assuming that all changes in relative use of input factors not explained by changes in relative input prices, are unpredictable based on the given information set.

By applying the estimation approach elaborated here we can estimate the scale elasticity and the expected technological growth. The identification of these properties is important for detecting if the system has reliable long run properties (such as growth rates), especially if one aims to use the system for forecasting.

In the present paper the focus is on estimating a factor demand system. The reduced cointegrating rank is consistent with technological changes. Conventional estimation approaches might undermine the extent of technological changes.

The approach presented here can also be used to analyse consumer demand (because tastes might change). In a system with 9 consumer groups, Raknerud et al. (2003) show that there are only 6 cointegrating vectors. The reduced number of rank (compared to the number of goods) indicates that there have been changes in the utility function. These changes in the utility function may stem from changes in tastes. However the lack of a stable utility function may be the result of changes in income distribution. Nevertheless, independent of the reason, it may be important to allow for such instability when estimating demand systems.

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Deriving the factor demand equations

In this appendix I show how to derive the factor demand equations. In this appendix I will use untransformed variables, not log transformed variables as in the main part of the paper. The level variables are $X = \exp\{x\}$ for production, $V_i = \exp\{v_i\}$ for input factor i, $P_i = \exp\{p_i\}$ for the price of input factor i, and $P_A = \exp\{p_A\}$ expressing the aggregated input factor price. For simplicity, the time subscript for time is dropped. The production function, with substitution elasticity σ is

$$X = \begin{cases} \Theta^* \left(\sum_{j=1}^{n} \delta_j V_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}\kappa} & \text{for } \sigma > 0, \sigma \neq 1, \\ \Theta^* \left(\prod_{j=1}^{n} V_j^{\delta_j}\right)^{\kappa} & \text{for } \sigma = 1. \end{cases}$$

Furthermore, define

$$\Theta = \Theta^* \left(\prod_{j=1}^{n} \delta_j^{\delta_j}\right)^{\kappa\sigma}.$$

When $\sigma \neq 1$ the cost minimizing problem is given by

$$\min_{V_1,\ldots,V_n} \left\{ \sum_{j=1}^{n} P_j V_j \text{ s.t. } X^{\frac{\sigma-1}{\sigma}\frac{1}{\kappa}} = \left(\Theta^*\right)^{\frac{\sigma-1}{\sigma}\frac{1}{\kappa}} \sum_{j=1}^{n} \delta_j V_j^{\frac{\sigma-1}{\sigma}} \right\},$$

which yields the first order conditions

$$P_i - \lambda\left(\Theta^*\right)^{\frac{\sigma-1}{\sigma}\frac{1}{\kappa}} \frac{\sigma-1}{\sigma} \delta_i V_i^{-\frac{1}{\sigma}} = 0, \ (i = 1,\ldots, n),$$

where λ is the Lagrange multiplier. Solving these equations for the input factors yields

$$V_i = \left(\frac{\lambda\left(\Theta^*\right)^{\frac{\sigma-1}{\sigma}\frac{1}{\kappa}} \frac{\sigma-1}{\sigma} \delta_i}{P_i} \right)^{\sigma}, \ (i = 1,\ldots, n)$$

These expressions can be used to insert for the input factors in the 'transformed' production function

$$X^{\frac{\sigma-1}{\sigma}\frac{1}{\kappa}} = \left(\Theta^*\right)^{\frac{\sigma-1}{\sigma}\frac{1}{\kappa}} \sum_{j=1}^{n} \delta_j \left(\left(\frac{\lambda\left(\Theta^*\right)^{\frac{\sigma-1}{\sigma}\frac{1}{\kappa}} \frac{\sigma-1}{\sigma}\delta_j}{P_j} \right)^{\sigma} \right)^{\frac{\sigma-1}{\sigma}}$$

$$= \left(\Theta^*\right)^{\frac{\sigma-1}{\sigma}} \lambda^{\sigma-1} \left(\frac{\sigma-1}{\sigma} \right)^{\sigma-1} \left(\sum_{j=1}^{n} \delta_j^{\ \sigma} P_j^{\ 1-\sigma} \right).$$

Solving for the Lagrange multiplier yields

$$\lambda = \left(\Theta^*\right)^{-\frac{1}{\kappa}} \left(X^{-\frac{\sigma-1}{\sigma}\frac{1}{\kappa}} \right)^{\frac{1}{1-\sigma}} \left(\frac{\sigma-1}{\sigma} \right)^{-1} \left(\sum_{j=1}^{n} \delta_j^{\ \sigma} P_j^{\ 1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

$$= \left(\Theta^*\right)^{-\frac{1}{\kappa}} \left(\prod_{j=1}^{n} \delta_j^{\ \delta_j} \right)^{-1} \frac{\sigma}{\sigma-1} X^{\frac{1}{\sigma\kappa}} P_A,$$

where we have defined the weighted factor price as

$$(19) \qquad P_A = \left(\prod_{j=1}^{n} \delta_j^{\ \delta_j} \right) \left(\sum_{j=1}^{n} \delta_j^{\ \sigma} P_j^{\ 1-\sigma} \right)^{\frac{1}{1-\sigma}}.$$

Inserting the expression for the Lagrange parameter into the expression for the factor demand yields

$$V_i = \left(\frac{\left[\left(\Theta^*\right)^{\frac{1}{\kappa}} \left(\sum_{j=1}^{n} \delta_j^{\ \delta_j} \right)^{-1} \frac{\sigma}{\sigma-1} X^{\frac{1}{\sigma\kappa}} P_A \right] \left(\Theta^*\right)^{\frac{\sigma-1}{\sigma}\frac{1}{\kappa}} \frac{\sigma-1}{\sigma} \delta_i}{P_i} \right)^{\sigma}$$

$$= \left(\delta_i \left(\Theta^*\right) \left[\left(\prod_{j=1}^{n} \delta_j^{\ \delta_j} \right)^{\sigma} \right]^{-\frac{1}{\sigma\kappa}} X^{\frac{1}{\sigma\kappa}} \frac{P_A}{P_i} \right)^{\sigma}$$

$$= \delta_i^{\ \sigma} \Theta^{-\frac{1}{\kappa}} X^{\frac{1}{\kappa}} \left(\frac{P_A}{P_i} \right)^{\sigma}.$$

This is the conditional demand function for the input factor when assuming a CES production function. It can be shown (following the same approach as above) that this expression also applies when the substitution elasticity equals unity, i.e. with Cobb-Douglas technology.

The expression for the weighted factor price in (19) is only valid when $\sigma \neq 1$. Here we will show the expression for the aggregated factor price when s = 1. Taking logs of (19) yields

$$\ln(P_A) = \sum_{j=1}^{n} \delta_j \ln(\delta_j) + \frac{\ln\left(\sum_{j=1}^{n} \delta_j^{\sigma} P_j^{1-\sigma}\right)}{1-\sigma}$$

Both the nominator and the denominator in the last part of the expression above approach zero when $\sigma \to 1$. Therefore we apply L'Hopital's rule:[8]

$$\begin{aligned}
\lim_{\sigma\to 1} \frac{\ln\left(\sum_{j=1}^{n} \delta_j^{\sigma} P_j^{1-\sigma}\right)}{1-\sigma} &= \lim_{\sigma\to 1} \frac{\frac{\partial}{\partial\sigma}\ln\left(\sum_{j=1}^{n} \delta_j^{\sigma} P_j^{1-\sigma}\right)}{\frac{\partial}{\partial\sigma}(1-\sigma)} \\
&= \lim_{\sigma\to 1} \frac{\left[\sum_{j=1}^{n} \delta_j^{\sigma} P_j^{1-\sigma}\right]^{-1} \frac{\partial}{\partial\sigma}\left(\sum_{j=1}^{n} \delta_j^{\sigma} P_j^{1-\sigma}\right)}{-1} \\
&= -\lim_{\sigma\to 1}\left\{\left[\sum_{j=1}^{n} \delta_j^{\sigma} P_j^{1-\sigma}\right]^{-1}\left[\sum_{j=1}^{n} \delta_j^{\sigma} P_j^{1-\sigma}\left(\ln\delta_j - \ln P_j\right)\right]\right\} \\
&= -\left[\sum_{j=1}^{n} \delta_j\right]\cdot\left[\sum_{j=1}^{n} \delta_j\left(\ln\delta_j - \ln P_j\right)\right] \\
&= -\sum_{j=1}^{n} \delta_j\left(\ln\delta_j - \ln P_j\right)
\end{aligned}$$

This yields $P_A = \prod_{j=1}^{n} P_j^{\delta_j}$ when σ = 1, or generally

$$P_A = \begin{cases} \left(\prod_{j=1}^{n} \delta_j^{\delta_j}\right)\left(\sum_{j=1}^{n} \delta_j^{\sigma} P_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}} & \text{for } \sigma > 0, \sigma \neq 1, \\ \prod_{j=1}^{n} P_j^{\delta_j} & \text{for } \sigma = 1. \end{cases}$$

[8] Thanks to Terje Skjerpen for showing how to prove that the expression converges to $\prod_{j=1}^{n} P_j^{\delta_j}$ when $\sigma \to 1$.

Proof of Proposition 2.1

Proof. From Appendix A we have the expression for the demand of input factor *i*:

$$V_i = \delta_i^{\sigma} \Theta^{-\frac{1}{\kappa}} X^{\frac{1}{\kappa}} \left(\frac{P_A}{P_i} \right)^{\sigma}.$$

Therefore, the cost of factor *i* is

$$P_i V_i = \delta_i^{\sigma} \Theta^{-\frac{1}{\kappa}} X^{\frac{1}{\kappa}} P_A^{\sigma} P_i^{1-\sigma}.$$

and the cost ratio is

$$\begin{aligned}
\frac{P_i V_i}{\sum_{j=1}^{n} P_j V_j} &= \frac{\Theta^{-\frac{1}{\kappa}} X^{\frac{1}{\kappa}} P_A^{\sigma} \delta_i^{\sigma} P_i^{1-\sigma}}{\Theta^{-\frac{1}{\kappa}} X^{\frac{1}{\kappa}} P_A^{\sigma} \sum_{j=1}^{n} \delta_j^{\sigma} P_j^{1-\sigma}} \\
&= \frac{\delta_i^{\sigma} P_i^{1-\sigma}}{\sum_{j=1}^{n} \delta_j^{\sigma} P_j^{1-\sigma}} \\
&= \frac{\left(\prod_{j=1}^{n} \delta_j^{\delta_j} \right)^{1-\sigma} \delta_i^{\sigma} P_i^{1-\sigma}}{\left[\prod_{j=1}^{n} \delta_j^{\delta_j} \left(\sum_{j=1}^{n} \delta_j^{\sigma} P_j^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \right]^{1-\sigma}}.
\end{aligned}$$

Applying that the expression in the square brackets is equal to P_A yields (8).

Appendix C

CES Production Function - Empirical results

In Section 3 I only tested the restictions on γ in the Cobb-Douglas case (i.e. $\sigma = 1$). Here I repreat these tests for different values of the substitution-elasticity (i.e. σ). From Table B-1 we can see that the estimate of the scale-elasticity is close to unity and the estimate of the technological growth is close to zero for all choices of σ. However, for $\sigma = 2$ the estimate of the technological growth is significantly positive.

Table C-1: Likelihood ratio tests for different choices of σ

$\sigma = 0$	log-likelihood	LR	d.f.	p-value	κ	γ_θ
No restr. (r = 2)	416.012					
Scale-elasticity	413.330	5.36	6	0.498	1.055	
+ no drift	405.208	21.61	12	0.042	1.055	-0.22%
+ scale-el. = 1	404.797	22.43	13	0.049	1.000	-0.04%
+ no techn. growth	404.789	22.45	14	0.070	1.000	0.00%
$\sigma = 0.1$	log-likelihood	LR	d.f.	p-value	κ	γ_θ
No restr. (r = 2)	415.974					
Scale-elasticity	413.309	5.33	6	0.502	1.079	
+ no drift	405.842	20.26	12	0.062	1.066	-0.21%
+ scale-el. = 1	405.319	21.31	13	0.067	1.000	-0.01%
+ no techn. growth	405.318	21.31	14	0.094	1.000	0.00%
$\sigma = 0.5$	log-likelihood	LR	d.f.	p-value	κ	γ_θ
No restr. (r = 2)	388.126					
Scale-elasticity	384.543	7.17	6	0.306	1.112	
+ no drift	379.448	17.36	12	0.137	1.081	-0.03%
+ scale-el. = 1	378.874	18.50	13	0.139	1.000	0.14%
+ no techn. growth	378.687	18.88	14	0.170	1.000	0.00%
$\sigma = 2$	log-likelihood	LR	d.f.	p-value	κ	γ_θ
No restr. (r = 2)	236.056					
Scale-elasticity	228.941	14.23	6	0.027	0.975	
+ no drift	225.669	20.77	12	0.054	0.963	0.08%
+ scale-el. = 1	225.540	21.03	13	0.072	1.000	0.84%
+ no techn. growth	216.337	39.44	14	0.000	1.000	0.00%

Hypotheses testing on γ for different values of σ. For $\sigma = 1$ see Table 4.