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The substitution bias of the consumer price index

Abstract:
The paper uses elementary consumer theory to propose an inflation independent ratio definition of the substitution bias of the Laspeyres consumer price index, and derives an approximate substitution bias which depends on the size of the price change as measured by a norm in the Laspeyres plane and on the elasticity of substitution in the direction of the price change. This norm or distance measure can be interpreted as a price substitution index which yields useful information about the movements of relative prices. Norwegian CPI data are used to quantify these relationships.

Keywords: consumer price index (CPI), substitution bias, elasticity of substitution.

JEL classification: classification code: C43,D12.

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1. Introduction

Economists have, at least since the days of Konüs (1924), been well aware of the tendency of the Laspeyres consumer price index to overstate the change in the cost of living. Konüs, himself, Bowley (1928), Frisch (1936), and Wald (1939) developed methodologies and indices which would better approximate the true index. It was the desire to "construct a cost-of-living index which depends only upon observable prices and properties of demand functions" that led to the Linear Expenditure System of Klein and Rubin (1948). And it was such demand systems that Tran Van Hoa (1969), Noe and von Furstenberg (1972), Goldberger and Gamaletsos (1970) used to provide the first numerical estimates of the substitution bias.

The substitution bias has, since these early attempts, generally been defined in an *ad hoc* manner either as a difference or as a percentage difference between the Laspeyres index and some approximation to the true index. Apparently no attempt has been made to justify this measure on theoretical grounds. The introduction by Diewert (1976) of the superlative indices provided us with new methods of approximating the true CPI and of evaluating the bias, but no new methods for defining it. Instead, authors like Braithwait (1980), Manser and McDonald (1988), and Aizcorbe and Jackman (1993) have proceeded, using the old measure, to discuss the factors which might determine the size of the bias, such as the magnitude of the price change, the rate of inflation, the ease of commodity substitution, and the level of aggregation. Another important factor affecting the size of the bias is the frequency with which the base year of the index is changed. It is Braithwait (1980), with his index of price dispersion and his attempt to evaluate the relative importance of relative price changes and commodity substitution, that seems to come closest to the approach of the present paper.

More recently, the exact CES index, introduced by Lloyd (1975) and Moulton (1996), has been promoted by Shapiro and Wilcox (1997), Schultze and Mackie (2002, pp. 6, 60–1, 92) and ILO (2004, pp. 327–8) as a method of constructing in a timely manner a CPI which may in some way allow for the curvature of the preferences. A more general Taylor approximation approach has been proposed by Diewert (1998), Schultze and Mackie (2002, p. 91) and ILO (2004, pp. 330–2), but the method generally requires excessive information about the second order properties of the preferences.

This paper proposes a novel, and essentially geometric, approach to the analysis of the substitution bias of the consumer price index. Using elementary consumer theory it defines the bias as the ratio between the Laspeyres index and the true cost of living index. The definition makes the bias independent of the rate of inflation and emphasizes its 'real' character. The bias thus defined can be considered an alternative, discrete measure of the curvature of the underlying preference as reflected in the level surface of the true cost of living index. This measure supplements the elasticity of substitution, which is regarded as a local measure of the curvature.

The frame of reference for the bias measure is the level set of the Laspeyres index and we give this Laspeyres plane a geometry and an associated norm which makes it possible to measure the size of the substitution inducing price change, or equivalently to define a price substitution index which measures the component of the price change which induces commodity substitution by the consumers. This further makes it possible to decompose the price change into an inflationary component and a substitution inducing component and to evaluate their relative importance. The geometry of the price change is described in section 2, along with some illustrations using Norwegian CPI data which indicate that the substitution inducing price change was on the average slightly larger than the inflationary component over the period 1990–1998.

Section 3 defines the substitution bias and describes some of its properties. It also presents the directional shadow elasticity of substitution (DSES) originally introduced in Frenger (1978). The latter is the local measure of the curvature of the level surface of the true price index and depends on the direction in the Laspeyres plane along which it is measured. Frenger (2005) developed a method for computing the DSES implicit in the use of superlative price indices such as the Fisher or the Törnqvist indices based on data historically available for the preparation of the consumer price index.

It is in general infeasible to measure the substitution bias from its proposed definition since this requires knowledge of the true cost of living index. Section 4 therefore develops a second order approximation to the true substitution bias, which can be used in practice to determine the magnitude of the effect. This approximate measure also decomposes the bias into a price substitution effect, the magnitude of which is the size, or norm, of the price change in the Laspeyres plane, and a curvature or substitution effect, which is measured by the directional shadow elasticity of substitution (DSES), a decomposition which quantifies the relationship suggested by Braithwait (1980). The approximate bias allows us to compute an approximate index which may be considered an alternative to the Lloyd-Moulton-Shapiro-Wilcox approach mentioned above. The paper concludes, in section 5, with some comments and ideas for further work.

2. The distance measure

In the literature there is a great deal of discussion of the substitution bias, but apparently few attempts to define it explicitly or to quantify it. The general tendency seems to be to introduce an index P^*, which is assumed to be the true cost of living index, and another index P^A which is known and is assumed to approximate the true index. The approximating index could f.ex. be a Laspeyres index or Fisher's Ideal index. One then proceeds to discuss the error or bias, however measured, which the use of the approximating index entails. I will suggest a formal definition of the substitution bias in the next section. The magnitude of this bias will however crucially depend on the size of the change in relative prices, and so we will first analyze the magnitude of the price change, an aspect of the problem which is independent of the curvature of the preferences and the definition of the bias.

This section develops a measure of the size of the substitution inducing price change. We start by briefly reviewing, mainly for notational purposes, some basic elements of the theory of consumer preferences and expenditure functions (subsection 2.1), and then introduce the tangent plane to the level surface of the expenditure function (subsection 2.2): it consists of all the price changes which will leave the Laspeyres price index unchanged and will for that reason be termed the Laspeyres plane. It is also where the substitution inducing price change lies.

In subsection 2.3 I introduce the distance measure in the form of a mathematical norm on the Laspeyres plane, calling it the elasticity norm. It will provide us with the desired measure of the size of any price change in the Laspeyres plane. In subsection 2.4 I consider the decomposition of an arbitrary price change into a substitution inducing and an inflationary component, and then present a numerical example illustrating the computation of the norm.

It should be noted that the whole analysis of this section is based on the information provided by the base period price and quantity data and the comparison period price, information that is typically available for the computation of price indices in real time. Further, we are only dealing with first order properties of the expenditure function. The discussion of curvature of the preferences, commodity substitution, and index bias has to await the next section.

2.1. Utility theory

Assume that the preferences of the consumer can be represented dually by the expenditure function $C(u,p)$, where u is a utility indicator and p is the vector of commodity prices lying in the *price space* $R^n_+ = \{p = (p_1, \dots, p_n) | \, p_i > 0 \, , \; i = 1, \dots, n\}$. We will, as appears to be standard practice in the CPI literature,[1] assume that the preferences are homothetic so that the expenditure function can be decomposed into the product

$$C(u,p) = u \, c(p) \; , \tag{1}$$

where c is the *unit expenditure function*. In the following we will normalize the utility level to $u = 1$, and use almost exclusively the unit expenditure function, which is assumed to satisfy the standard neoclassical regularity conditions, in particular homogeneity, concavity, and sufficient differentiability on R^n_+.

Let $x = (x_1, \dots, x_n)$ denote the vector of goods consumed at prices $p = (p_1, \dots, p_n)$. By Shephard's lemma $x = c_p(p)$ or $x_i = c_i(p) = \partial c(p)/\partial p_i, \; i = 1, \dots, n$, as we are restricting ourself to the normalized unit utility level. The value shares are given by

$$s_i(p) \; = \; \frac{p_i \, c_i(p)}{c(p)} \; , \qquad\qquad i = 1, \dots, n. \tag{2}$$

The *true (or Konüs) cost of living index* in period t with period t_0 as a base is[2]

$$P^*(p, p^0) \; = \; \frac{C(u,p)}{C(u,p^0)} \; = \; \frac{c(p)}{c(p^0)} \; , \tag{3}$$

[1]See f.ex. Schultze and Mackie (2002, p. 82) or ILO (2004, ch. 17, p. 316). The principal reason for the assumption is that the "quantity" data generally come from a survey of consumer expenditures, which provide budget data in the form of expenditure shares.

[2]See Konüs (1924, p. 10), Deaton and Muellbauer (1980, p. 170), or Rødseth (1997, p. 66).

where p and p^0 are the price vectors for the comparison period t and the base period t_0, respectively. The true index is just a renormalization of the unit expenditure function $c(p)$ by the base year expenditure $c(p^0)$. Thus $P^*(p, p^0)$, considered as a function of p, has all the properties of a unit expenditure function. In particular it is linear homogeneous and concave in p.

Consider the level surface

$$M \;=\; \{\, p \mid c(p) = c(p^0) \,\} \;=\; \{\, p \mid P^*(p, p^0) = 1 \,\} \;. \tag{4}$$

It is both the $c(p^0)$ level surface of the unit expenditure function, and the unit level surface of the true cost of living index $P^*(p, p^0)$ given the base price p^0. Any price $p \in M$ leaves the consumer equally well off since at p with expenditure $c(p)$ he could still afford to buy a commodity basket which would yield the same utility level as the expenditure $c(p^0)$ did at p^0. And by definition the true cost of living is unchanged.

2.2. The Laspeyres plane

The tangent plane L_0 to the level surface M at p^0,[3]

$$L_0 \;=\; \{\, p \in R^n_+ \mid x^{0\prime}(p - p^0) = 0 \,\} \;, \tag{5}$$

consists in all those comparisons prices p which leave expenditures unchanged. The "prime" on x^0 denotes transposition of the vector. The defining condition may alternatively be written

$$\sum_{k=1}^{n} s_k(p^0)\left(\frac{p_k}{p_k^0} - 1\right) \;=\; 0 \;, \tag{6}$$

indicating that the share weighted average of the relative price changes is zero. The condition also implies that the Laspeyres index

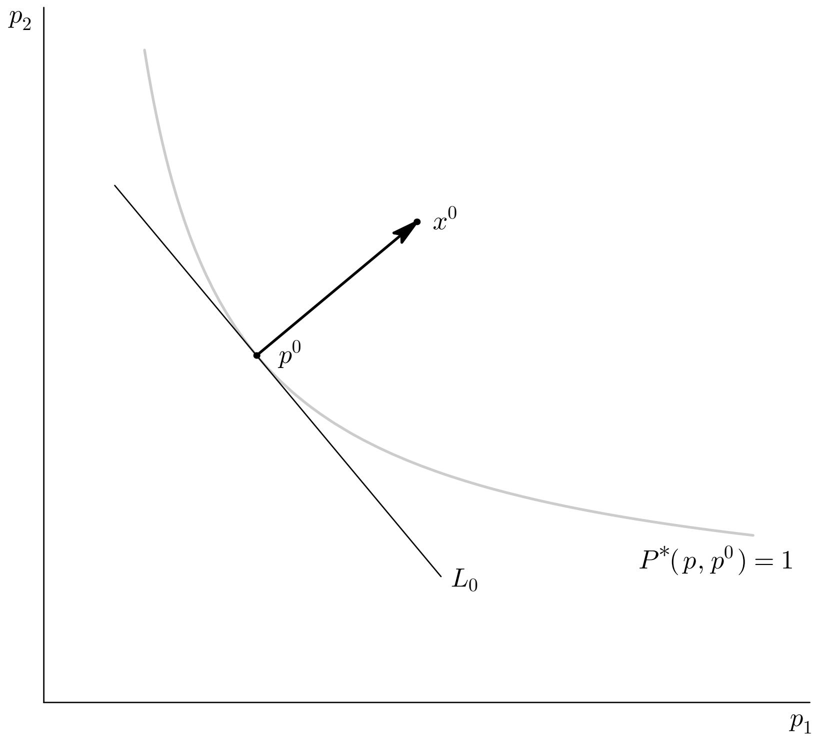
$$P^L(p, p^0) \;=\; \frac{\sum_{i=1}^n p_i x_i^0}{\sum_{i=1}^n p_i^0 x_i^0} \;=\; \frac{p' x^0}{c(p^0)} \;, \tag{7}$$

will be 1 for any comparison price p which lies in L_0. In the present index number context it seems appropriate to term the tangent plane L_0 the *Laspeyres plane*.

The situation is illustrated in figure 1 which emphasizes the kind of information typically available for CPI construction. In this case $n = 2$ and we are given the base period price or point p^0 in the price space R^2_+. At p^0 the unit expenditure is $c(p^0)$ and the commodity demand vector is $x^0 = c_p(p^0)$. Knowing the pair (p^0, x^0) allows us to draw the

[3] L_0 is an affine subspace of the price space. It is different from the tangent space $T_{p^0}M$ used in the proper definition of the DSES, since $T_{p^0}M$ is a subspace of $p^0 \times R^n$. In the present context, however, it may be just as well to identify the one with the other, and I will do so! There is however a problem with using the price change vector $v = p - p^0$ since v will not lie in the affine plane L_0, but rather in a linear subspace, which by definition goes through the origin.

Figure 1: The base year data and the Laspeyres plane.



Laspeyres plane L_0 which is normal to x^0 at p^0. We do not know the expenditure function. Expenditure minimization, alternatively the concavity of the expenditure function, shows that it must lie above the plane L_0 and that the level set M must be tangent to L_0 at p^0. Lack of knowledge of c suggests drawing the level set gray.

2.3. The elasticity norm

One of the primary purposes of this paper is to develop a measure of the magnitude of the change in relative prices between two periods, i.e. we want a measure of that part of the price change which leads to a change in the proportions in which commodities are consumed. It is not obvious how we should measure the change in relative prices when there are more than two commodities. The approach of this paper is to define a formal mathematical norm on the Laspeyres plane L_0.

To define a norm on L_0 entails that to every price vector $p \in L_0$ we consider the associated price change vector $v = p - p^0$ and assign it the length

$$\|v\|_0 \;=\; \|p-p^0\|_0 \;=\; \left[\sum_{i=1}^{n} s_i(p^0)\left(\frac{p_i}{p_i^0}-1\right)^2\right]^{\frac{1}{2}}, \qquad p \in L_0\,. \tag{8}$$

It is readily seen that (8) defines a proper norm on L_0 if $s_i(p^0) > 0$ for $i = 1, \ldots, n$,[4] and we call it the *elasticity norm.* The subscript on the norm indicates that it is a norm on L_0, the Laspeyres plane for the base period t_0. The norm was originally suggested by the denominator of the expression for the directional shadow elasticity of substitution presented in (23) below, but should here be considered independently of its historical origin.[5]

This length or size of the change in relative prices is a weighted mean of order 2 of the absolute value of the individual relative price changes with their value shares as weights. Let $\varepsilon_i = p_i/p_i^0 - 1$ denote the relative price change of the i'th commodity. The expression (8) for the metric may then be written

$$m(\varepsilon) \;=\; \left[\sum_{i=1}^{n} s_i^0\, \varepsilon_i^2\right]^{\frac{1}{2}}.$$

Let $\underline{\varepsilon} = \min\{|\varepsilon_1|, \ldots, |\varepsilon_n|\}$ and $\overline{\varepsilon} = \max\{|\varepsilon_1|, \ldots, |\varepsilon_n|\}$, then

$$\underline{\varepsilon} \;\le\; m(\varepsilon) \;\le\; \overline{\varepsilon}\,.$$

Thus the length of the price change must lie between the smallest and the largest relative price change, measured in absolute value.

In effect $\|p - p^0\|$ becomes a "price substitution index" or a "price distortion index", a noninflationary measure of the change in relative prices. We could formalize this approach by defining an index $P^S(p, p^0) = \|p - p^0\|_0$, though its domain is limited to L_0 and the base period value of this index is $P^S(p^0, p^0) = \|p^0 - p^0\|_0 = 0$ rather than one. Thus we cannot talk about percentage changes in the index. This also constitutes a presentational disadvantage since its values tend to be of the same order of magnitude as the percentage changes in an ordinary price index.

A basic property of the norm is that it is linear in the distance from the base point p^0. Let $\check{p}$ be an arbitrary price vector in L_0 and consider the price vector p,

$$p \;=\; \theta\,\check{p} \;+\; (1-\theta)\,p^0\,, \qquad \theta > 0\,, \tag{9}$$

which is an affine combination of p^0 and $\check{p}$. The price vector $p \in L_0$ since $c_p(p^0)'\,(p - p^0) = 0$. As θ changes p moves along a ray in L_0 with "origin" at p^0. For $\theta = 0$, $p = p^0$, and for $\theta = 1$, $p = \check{p}$. And in particular

$$\|p-p^0\|_0 \;=\; \|\theta\,\check{p} + (1-\theta)\,p^0 - p^0\|_0 \;=\; \theta\,\|\check{p} - p^0\|_0\,, \qquad p \in L_0\,. \tag{10}$$

[4] According to Royden (1968, p. 181), a nonnegative real-valued function $\|\;\|$ defined on a vector space is called a *norm* if: (i) $\|x\| = 0$ if and only if $x = 0$, (ii) $\|\alpha\, x\| = |\alpha|\,\|x\|$ for real α, and (iii) $\|x+y\| \le \|x\|+\|y\|$. The last property is the triangle inequality. P. Berck and K. Sydsæter, *Economists' Mathematical Manual*, Springer-Verlag, 1991, describe a norm on p. 90.

[5] In particular since the development of the denominator in (23) was a rather tortuous process. The original version is found in Frenger (1978, pp. 292–4).

The length of $p - p^0$ is proportional to θ. This is essentially property (ii) of the norm.[6]

Though we are primarily interested in the Laspeyres plane L_0 at the base period price p^0, there is a separate Laspeyres plane L_p for each p in the price space. And each L_p has associated with it a different norm $\| \, \|_p$. The definition (8) implies that this norm is homogeneous of degree zero in p in the sense that $\|\lambda \, v_{\lambda p}\| = \|v_p\|$.

In his attempt to assess the relative importance of price change and substitutability Braithwait (1980, p. 71) introduces "a useful measure of the dispersion of relative prices",[7]

$$D(p,p^0) \;=\; \frac{1}{2}\sum_{i=1}^{n} s_i(p^0)\left(\frac{p_i}{p_i^0} \;-\; P^L(p,p^0)\right)^2 . \tag{11}$$

Note that his p need not lie in L_0 and he thus fails to deflate the comparison price, but instead subtracts the Laspeyres inflation.

Manser and McDonald (1988, p. 909) suggest that there had been a "higher variability in relative prices" in the post-1973 inflationary period than in the period included in previous studies, and state that: "We have confirmed this fact for the U.S. data analyzed in this paper." They do not specify how they carried out this analysis.[8]

Diewert (1998, p. 50) assumes that the preferences are Cobb-Douglas with equal value shares $s_i = 1/n$, $i = 1, \dots, n$. He defines the "residual price variations" $\varepsilon_i = p_i^1/(1+\iota)p_i^0 - 1$, ι being the Laspeyres rate of inflation. He proceeds to computes the Taylor expansion of the substitution bias with respect to the ε_i, and in the process derives an expression for "the variance of the inflation-adjusted percentage changes in prices",

$$\mathrm{Var}(p) \;=\; \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2 \;=\; \frac{1}{n}\sum_{i=1}^{n}\left(\frac{p_i^1/(1+\iota)}{p_i^0} \;-\; 1\right)^2 , \tag{12}$$

which is in fact $\|p^1/(1+\iota) - p^0\|_0^2$. The restrictive assumptions obscure the important role of the value shares as weights in the expression for the variance and the metric.

To get a better understanding of the Laspeyres plane it may be useful to get acquainted with some of its "inhabitants". Changing a single price p_k will generally not lead to a price change vector in L_0 since expenditures will not remain constant. If instead we change p_k and then alter all other prices proportionately so as to keep expenditure constant we obtain the *Hicks price change vectors*

$$v^{k*} \;=\; \left(-\frac{p_1^0}{1-s_k^0}, \dots, -\frac{p_{k-1}^0}{1-s_k^0}, \frac{p_k^0}{s_k^0}, -\frac{p_{k+1}^0}{1-s_k^0}, \dots, -\frac{p_n^0}{1-s_k^0}\right), \qquad k = 1, \dots, n, \tag{13}$$

[6] See footnote 4, p. 8.

[7] He states that measure is a special case of an approximation to the substitution bias derived by Paulus (1974), though I have trouble seeing the relationship in part, probably, because I don't understand Paulus's measure which is the 'first term on the right' in his equation (7). Braithwait's dispersion measure is related to my distance measure (8) by

$$D(p,p^0) \;=\; \frac{1}{2}\left[P^L(p,p^0)\right]^2 \|\tilde p - p^0\|_0^2 , \qquad \tilde p_i \;=\; \frac{p_i/P^L(p,p^0)}{p_i^0} \;\in\; L_0 .$$

[8] A brief discussion of the possible role of relative prices is also found in Noe and von Furstenberg (1972), but no formal analysis is presented.

which lie in L_0. The Hicks vectors correspond in a natural way to the Euclidean basis vectors, but since there are n such vectors in L_0 they are not linearly independent. Another useful set of vectors in L_0 is obtained by allowing only two prices, f.ex. the prices of the i'th and the j'th goods, to change while leaving all other prices and expenditures unchanged. This construction defines the *ratio price change vectors*

$$v^{ij} = \left(0, 0, \ldots, \frac{p_i^0}{s_i^0}, \ldots, -\frac{p_j^0}{s_j^0}, \ldots, 0\right), \qquad i, j = 1, \ldots, n, \; i \neq j. \tag{14}$$

These are the directions in which the shadow elasticities of substitution between goods i and j are defined.[9] The normalization of the vectors v^{k*} and v^{ij} is somewhat arbitrary: it is their direction and not their length which is of primary importance.

Associated with the elasticity norm defined in (8) we have the elasticity inner product

$$\langle v^1, v^2 \rangle_0 = \sum_{i=1}^{n} s_i(p^0) \frac{v_i^1}{p_i^0} \frac{v_i^2}{p_i^0}, \qquad p^1, p^2 \in L_0, \tag{15}$$

where $v^1 = p^1 - p^0$ and $v^2 = p^2 - p^0$ are two price change vectors, $p^1, p^2 \in L_0$ We can show that:

Lemma 1. The vectors v^{ij}, $i, j = 1, \ldots, n$, $i \neq j$, and the vectors v^{k*}, $k = 1, \ldots, n$, are orthogonal with respect to the elasticity metric for $k \neq i \neq j$.

This inner product has several interesting implications. In this paper I use the orthogonality concept only in the following illustrations of the norm.[10]

Figure 2 shows the unit "sphere" of two different metrics at $p^0 = (1, 1, 1)$ and $c(p^0) = 1$. The price space is 3-dimensional, and thus the Laspeyres plane L_0 is 2-dimensional. In case (a) the value shares are equal, $s_1 = s_2 = s_3 = 1/3$, while in case (b) $s_1 = s_2 = 0.45$ and $s_3 = 0.1$. Thus we are dealing with two different sets of preferences with different commodity demand at p^0 and different Laspeyres planes. In drawing the figure we are confronted with several problems. We cannot use the standard Euclidean vectors, representing changes in the individual prices p_i, $i = 1, 2, 3$, to form the basis since these vectors do not lie in L_0. We will instead use the Hicks vector v^{1*} as the abscissa [see (13)]. As ordinate we use the orthogonal complement of v^{1*} in L_0 under the Euclidean metric. And we renormalize both vectors so as to obtain a set of orthonormal basis vectors (f^1, f^2) for L_0 in the Euclidean metric. We do so because we want to represent the elasticity metric as seen "with Euclidean eyes".[11]

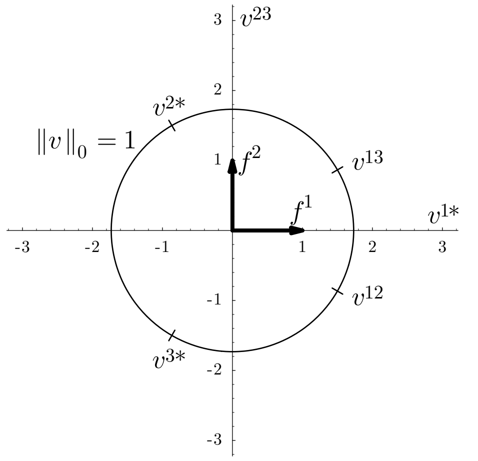
In case (a) the value shares are all equal, with the result that the elasticity metric is proportional to the Euclidean metric. In this case the vectors v^{1*} and v^{23}, which we

[9] See (24) below for a definition of the shadow elasticity of substitution.

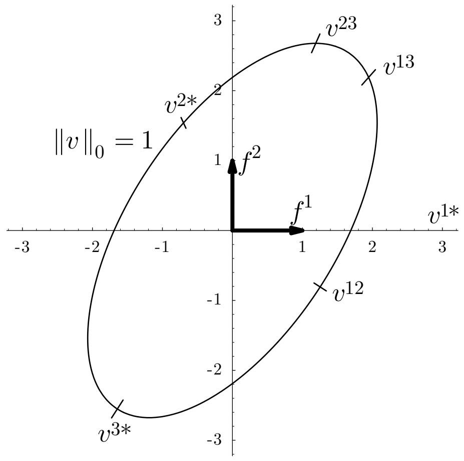
[10] We can adapt the correlation analysis of price changes which Frisch (1936, p. 9) introduced to study the drift of chain indices, and use the inner product to define the angle (or direction cosine) between two price change vectors. This gives us a tool for describing the direction of change of prices over time.

[11] We could draw the figures in terms of an orthogonal basis under the elasticity metric formed f.ex. by v^{1*} and v^{23}, and renormalize these vectors so they are of unit length in the elasticity metric. But then all units "circles" would just be round and we would not be able to show the "distorting" effect of the elasticity metric.

Figure 2: The unit spheres of two elasticity metrics



a) $s_1 = 1/3$, $s_2 = 1/3$, $s_3 = 1/3$,



b) $s_1 = 0.45$, $s_2 = 0.45$, $s_3 = 0.10$.

know to be orthogonal under the elasticity metric (lemma 1), are also orthogonal under the Euclidean metric and v^{23} is proportional to f^2. The unit sphere is round also when seen with Euclidean eyes. In addition to the Hicks vector v^{1*}, which is proportional to f^1, I have represented the ratio vectors v^{12} and v^{13} and the Hicks vectors v^{2*} and v^{3*} by short line segments crossing the unit sphere, while v^{23}, which we know to be orthogonal to v^{1*} by lemma 1, is in this case proportional to f^2.

In case (b) the three value shares are different, and the unit sphere of the elasticity metric becomes an ellipse when seen with Euclidean eyes. The vector v^{23} is not proportional to f^2. The ellipse is narrowest in the direction v^{12} involving price changes in the two goods 1 and 2 with the largest value shares. It is widest in the direction v^{3*} representing a change in good 3 with the smallest value share, and an offsetting proportionate change in the prices of the other two goods. The directions v^{12} and v^{3*} are in fact the minor and the major axes of the ellipse.

2.4. Decomposition of the price change

Consider a change to a new price p^1 in the comparison period t_1. The new price will in general not lie in the Laspeyres plane L_0. But for any p^1 we can find a $\tilde{p}^1$ which lies in L_0 and is proportional to p^1. Let λ_1 denote the proportionality factor, then

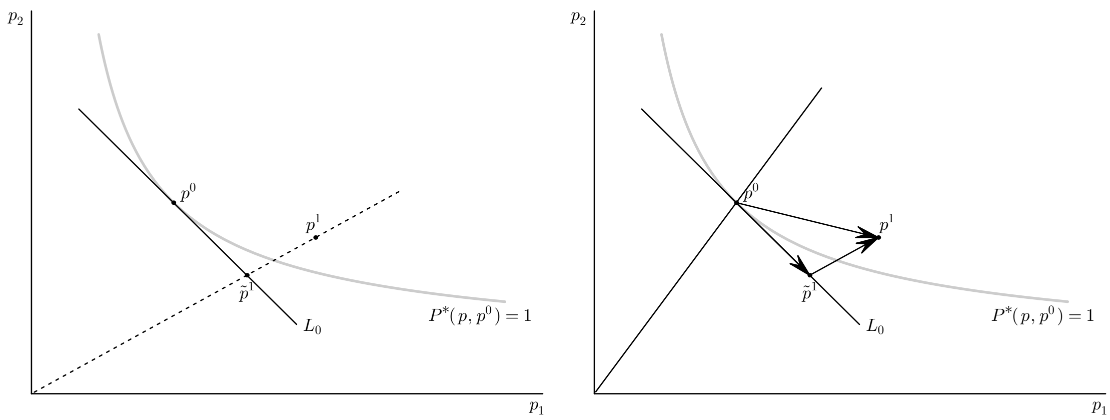
$$\tilde{p}^1 = \frac{p^1}{\lambda_1} , \tag{16}$$

with each price changing in the same proportion, $p_i^1/\tilde{p}_i^1 = \lambda_1$, $i = 1, \ldots, n$. The proportionality factor λ_1 is in fact equal to the value of the Laspeyres index (7) since

$$P^L(p^1, p^0) \;=\; \frac{\sum_i c_i(p^0)\, p_i^1}{\sum_i c_i(p^0)\, p_i^0} \;=\; \frac{\sum_i c_i(p^0)\, \lambda_1 \tilde{p}_i^1}{\sum_i c_i(p^0)\, \tilde{p}_i^1} \;=\; \lambda_1 \; . \tag{17}$$

The second step relies on the fact that both p^0 and $\tilde{p}^1$ lie in the Laspeyres plane L_0. This deflation process is illustrated in part (a) of figure 3.

Figure 3: The decomposition of the price change.



a) Deflating onto the Laspeyres plane, b) Decomposition of the price change.

We separate the price change $p^1 - p^0$ into two components

$$p^1 - p^0 \;=\; \overbrace{(\tilde{p}^1 - p^0)}^{\substack{\text{substitution}\\ \text{inducing}}} \;+\; \overbrace{(p^1 - \tilde{p}^1)}^{\text{inflation}} \; , \tag{18}$$

as illustrated in figure 3, part (b). The first component lies in L_0 and the second is proportional to p^1. This would appear to be the picture one generally has in mind when discussing inflation measures. We will measure the length of $p^1 - \tilde{p}^1$ by the rate of inflation, and consider it a measure of length by on an equal footing with the norm $\|\;\|$ in L_0. But while lengths are by definition nonnegative, we will preserve the sign of the rate of inflation as it provides us with additional information.

2.5. A numerical example

To illustrate the distance measure or norm we use information from the data base of the Norwegian consumer price index.[12] In the present case the price space has dimension

[12]See section 4.4 below for more details.

140 and the Laspeyres plane L_0 has dimension 139. We have observations for the period 1990–1998, and any one of those years can be a base year and all of the remaining ones can be comparison years. Thus we need a slightly more general notation. Let $T = \{1990, \ldots, 1998\}$ denote the period for which we have observations, and let $t, \tau \in T$, $\tau \neq t$. In practice t will be considered the base year and τ the comparison year of the binary comparison between the years t and τ.

The Laspeyres inflation (7) between period t and period τ is

$$\lambda_\tau \;=\; P^L(p^\tau, p^t) \;=\; \frac{p^{\tau\prime}\, x^t}{p^{t\prime}\, x^t}\,, \qquad\qquad t, \tau \in T\,,\; \tau \neq t\,.$$

We deflate p^τ in order to obtain $\tilde{p}^\tau \in L_t$ [see (16) and figure 3, part (a)],

$$\tilde{p}^\tau \;=\; \frac{p^\tau}{\lambda_\tau} \;=\; \frac{p^\tau}{P^L(p^\tau, p^t)}\,, \qquad\qquad \tilde{p}^\tau \in L_t\,. \tag{19}$$

The size of the substitution inducing price change between period t and period τ is given by the norm [see (8)]

$$\|\tilde{p}^\tau - p^t\|_t \;=\; \left[\sum_{i=1}^{n} s_i(p^t)\left(\frac{\tilde{p}_i^\tau}{p_i^t} - 1\right)^2\right]^{\frac{1}{2}}\,, \qquad\qquad \tilde{p}^\tau \in L_t\,,$$

where the subscript t on the metric is a reminder that the distance is measured in the period t Laspeyres plane L_t.

In table 1 we present the length of the deflated price changes in the Laspeyres planes associated with each base year in T. The rows represent the base periods and the columns

Table 1: Size of substitution inducing price changes, $100 \cdot \|\tilde{p}^\tau - p^t\|_t\,.^{a)}$

		comparison year								
		1990	1991	1992	1993	1994	1995	1996	1997	1998
base year	1990	–	4.6	6.9	8.9	10.4	12.4	13.2	15.0	17.4
	1991	5.2	–	3.2	5.5	7.5	9.7	10.4	12.2	14.8
	1992	7.8	3.2	–	3.3	5.4	8.1	8.8	10.5	12.9
	1993	10.4	5.7	3.4	–	3.8	7.0	7.5	9.4	11.8
	1994	12.3	7.7	5.5	3.3	–	3.4	5.1	7.0	9.2
	1995	14.8	10.0	8.0	5.8	3.3	–	3.2	4.9	7.5
	1996	16.3	11.3	9.2	7.3	5.3	3.3	–	2.7	5.7
	1997	18.7	13.3	11.2	9.2	7.3	5.2	2.7	–	4.2
	1998	21.3	15.7	13.5	11.6	9.8	7.9	5.8	4.2	–

a) The table entries represent average non-inflationary percentage change in prices measured in % units. See text.

designate the comparison periods. The first row measures the distance from p^{1990}, the distances being measured in the 1990 tangent plane T_{1990}. Thus the second column of the first row shows that $\|\tilde{p}^{1991} - p^{1990}\|_{1990} = 0.046 = 4.6\%$, measured in the 1990 norm $\| \|_{1990}$.[13] On the other hand the first column in the second row shows that $\|\tilde{p}^{1990} - p^{1991}\|_{1991} = 5.2\%$. The two distances are different because the weights s^{1990} and s^{1991} in the two norms are different, and because the distances are measured in two different tangent planes. Returning to the first row of the table we note that the distance $\|\tilde{p}^{\tau} - p^{1990}\|_{1990}$ increases uniformly with τ. This is the case for all the years in the table, and is valid for changes in both directions. In the table I have chosen to multiply all lengths by 100 so as to avoid a lot of zeros and to facilitate their association with percentage changes. I also abuse the percentage notation "%" by using it to designate 1/100'th of a unit of length in the norm $\| \|_t$.

The third row of table 2 summarizes the results of table 1 by taking the average of all lengths measured over the same time difference. Thus the average length of the price

Table 2: Average substitution inducing price changes and Laspeyres inflation. $^{a)}$

years difference	1	2	3	4	5	6	7	8
number of occurrences	16	14	12	10	8	6	4	2
average length in L_0	3.6	5.8	7.9	9.8	11.8	13.6	16.1	19.3
average Laspeyres inflation	2.1	4.1	6.1	8.1	10.1	12.2	14.7	17.5

a) Rows three and four are averages of the data in table 1 and table 3 respectively.

change between the 16 adjacent time periods is 3.6%. The average increases with the time difference. There is of course no *a priori* reason for the length to increase with the time difference, but I would guess it to be a dominant pattern of most annual CPI data.

Table 3 presents the traditional measure of inflation computed as the percentage change in the Laspeyres index $P^L(p^\tau, p^t)$. On the average the annual change in the price substitution measure is consistently larger than the Laspeyres inflation, thus changes in relative prices have tended to be larger that the inflation rate.

The analysis thus far has been entirely in terms of first order effects! Nothing has been said about substitution, curvature, or second order effects, but this is about to change.

[13] Strictly speaking ought $\tilde{p}^{1991}$ to have a subscript to indicate which tangent plane it lies in, but this should be clear from the metric used, in this case the 1990 metric, or from the context. The alternative is to write $\tilde{p}^{1991}_{1990}$, but that appears rather cumbersome.

Table 3: Laspeyres inflation, $P^L(p^\tau, p^t) - 1$.

base year	comparison year 1990	1991	1992	1993	1994	1995	1996	1997	1998
1990	–	3.4	5.8	8.4	10.1	13.0	14.2	17.3	20.1
1991	−3.1	–	2.3	4.7	6.3	9.0	10.2	13.2	16.0
1992	−5.2	−2.2	–	2.2	3.7	6.3	7.6	10.5	13.2
1993	−7.3	−4.3	−2.1	–	1.4	3.9	5.2	8.0	10.7
1994	−8.5	−5.6	−3.4	−1.3	–	2.4	3.7	6.5	9.0
1995	−10.5	−7.7	−5.5	−3.4	−2.3	–	1.2	3.8	6.3
1996	−11.4	−8.7	−6.6	−4.6	−3.4	−1.1	–	2.5	5.0
1997	−13.3	−10.8	−8.8	−6.8	−5.6	−3.4	−2.4	–	2.3
1998	−14.8	−12.4	−10.6	−8.6	−7.3	−5.1	−4.3	−2.0	–

3. The substitution bias

The previous section has been concerned with the geometry of the price change, finding appropriate measures for the size of the substitution inducing price change and for the inflation. We will now turn to the second order effects and the curvature of the preferences, and start by looking more closely at the definition of the substitution bias. The primary purpose of this section is to propose a ratio definition of the substitution bias and to describe some of its properties, which we do in subsection 3.1. Subsection 3.2 then reviews the definition of the directional elasticity of substitution, which will play a key role in the definition of the approximate bias. The approach of this paper is to interpret the elasticity of substitution as a local measure of the bias, while the substitution bias is a discrete measure.

3.1. The definition of the substitution bias

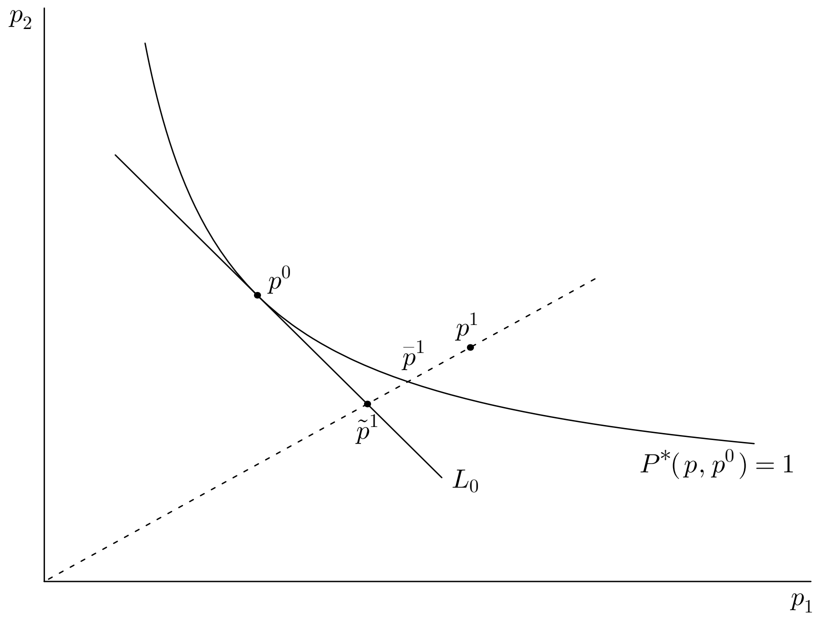
In order to define the substitution bias we must assume that the preferences and the associated expenditure function c are known. In figure 4 I draw the unit level surface for the true cost of living index $P^*(p, p^0)$ [see (4)],

$$M \;=\; \{\, p \mid P^*(p, p^0) = 1 \,\} \;.$$

Assume that the price p changes to p^1. Then there is a $\bar{p}^1$ in M and a $\mu > 0$ such that $p^1 = \mu\,\bar{p}^1$. This μ is a measure of the change in the price level, and is in fact the true cost of living index since

$$P^*(p^1, p^0) \;=\; \frac{c(p^1)}{c(p^0)} \;=\; \frac{c(\mu\,\bar{p}^1)}{c(p^0)} \;=\; \mu\,\frac{c(\bar{p}^1)}{c(p^0)} \;=\; \mu \;.$$

Figure 4: The definition of the substitution bias



We must also settle on which inflation measure we are trying to determine the bias of. We could f.ex. use a geometric (Jevons) index or Fisher's ideal index. We will however restrict ourself to the Laspeyres index which is represented by the Laspeyres plane in figure 4. As we did above, we can also define a $\tilde{p}^1 \in L_0$ such that $p^1 = \lambda \tilde{p}^1$ [see (16)]. By construction μ and λ are the common proportionality factors

$$\mu = \frac{p_i^1}{\bar{p}_i^1}\,, \qquad \text{and} \qquad \lambda = \frac{p_i^1}{\tilde{p}_i^1}\,, \qquad i = 1, \ldots, n,$$

with $\mu \leq \lambda$.

Returning to figure 4 we see that the difference between $\bar{p}^1$ and p^1 is a measure of the change in the true cost of living while the difference between $\tilde{p}^1$ and $\bar{p}^1$ measures the size of the bias resulting from the use of the Laspeyres index. Let us define the *substitution bias* of the Laspeyres price index at p^1 with p^0 as base period price as the ratio of the Laspeyres index to the true index,

$$\beta^L(p^1, p^0) \equiv \frac{P^L(p^1, p^0)}{P^*(p^1, p^0)} = \frac{\sum_i \bar{p}_i^1 \, x_i^0}{\sum_i \tilde{p}_i^1 \, x_i^0} = \frac{\lambda}{\mu}\,. \tag{20}$$

The superscript L is intended to emphasize that it is the bias of the Laspeyres index.

We may note that

$$\beta^L(p, p^0) = \frac{P^L(p, p^0)}{P^*(p, p^0)} = \frac{p' x^0}{c(p)} = \frac{\tilde{p}' x^0}{c(\tilde{p})}\,. \tag{21}$$

The expression $p'x^0/c(p)$ indicates that the bias $\beta^L(p,p^0)$ is homogeneous of degree 0 in prices and thus fully determined by the values it takes on L_0 as emphasized by the last expression. Or, said another way, the substitution bias of the Laspeyres index does not depend on the rate of inflation.

Lemma 2. *Properties of the substitution bias β^L:*

i) $\beta^L(p,p^0) \geq 1$ for all $p \in R^n_+$.

ii) $\beta^L(p,p^0)$ is homogeneous of degree zero in p for all $p \in R^n_+$.

iii) Let $\tilde{p}(\theta) = \theta\,\check{p} + (1-\theta)\,p^0$ be a line segment in L_0 with $\check{p} \in L_0$ and $\theta > 0$. Then $\beta^L(\tilde{p}(\theta_2),p^0) \geq \beta^L(\tilde{p}(\theta_1),p^0)$ for $\theta_2 \geq \theta_1$.

Let us also briefly consider the relationship between β^L and the bias measured as difference,

$$b^L(p,p^0) \;=\; P^L(p,p^0) \,-\, P^*(p,p^0) \;=\; P^L(p,p^0)\left[1 \,-\, \frac{1}{\beta^L(p,p^0)}\right] . \qquad (22)$$

In particular we note that b^L is homogeneous of degree one in prices.

3.2. The directional elasticity of substitution

Any price change $v = p - p^0$, $p \in L_0$, will by definition leave unit expenditures unchanged. We define the *directional shadow elasticity of substitution* (DSES) of the unit expenditure function c at p^0 in the direction v by[14]

$$\mathrm{DSES}_{p^0}(p-p^0) \;=\; -\,\frac{\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{c_{ij}(p^0)}{c(p^0)}\,(p_i-p_i^0)(p_j-p_j^0)}{\displaystyle\sum_{i=1}^{n} s_i(p^0)\left(\frac{p_i-p_i^0}{p_i^0}\right)^2}\,, \qquad \begin{array}{l} p \in L_0, \\ p \neq p^0 \,. \end{array} \qquad (23)$$

It measures the curvature of the factor price frontier at p^0 in the direction v. It will in general be a function both of the point p^0 at which it is evaluated and of the direction v. Note however that the length of the price change v is irrelevant since the DSES is homogeneous of degree zero in v. It follows from concavity of the expenditure function that $\mathrm{DSES}_{p^0}(v) \geq 0$.

[14]The DSES was introduced in Frenger (1978), which also presents the empirical application which motivated the definition. See also Frenger (1985), which uses the DSES to test for the concavity of the underlying cost function, and Frenger (1992) for a general presentation. The terminology is borrowed from McFadden (1963), who defined the shadow elasticity of substitution. There is of course also a directional direct elasticity of substitution (DDES) defined in the quantity space.

If c is a CES function with substitution parameter σ, then $\mathrm{DSES}_{p^0}(v) = \sigma$ at all points p^0 and in all directions v. The shadow elasticity of substitution σ_{ij} is obtained as a special case of $\mathrm{DSES}_{p^0}(v)$ by choosing for v the ratio vector v^{ij} introduced in (14),

$$\sigma_{ij} \;=\; \mathrm{DSES}_{p^0}(v^{ij}) \;=\; \frac{-\dfrac{c_{ii}}{c_i^2} \;+\; 2\,\dfrac{c_{ij}}{c_i\,c_j} \;-\; \dfrac{c_{jj}}{c_j^2}}{\dfrac{1}{p_i\,c_i} \;+\; \dfrac{1}{p_j\,c_j}}\,, \qquad \begin{array}{l} i,j = 1,\ldots,n, \\ i \neq j\,. \end{array} \tag{24}$$

Similarly we can consider the Hicks vector v^{i*} [see (13)], and define the Hicks-Samuelson elasticity of substitution

$$\sigma_{i*} \;=\; \mathrm{DSES}_{p^0}(v^{i*}) \;=\; -\,\frac{s_i}{1-s_i}\,\frac{c_{ii}\,c}{c_i^2}\,, \qquad\qquad i = 1,\ldots,n.$$

It has been proposed by both Hicks (1963, pp. 339, 379)) and Samuelson (1968, p. 468). The second equality shows that is is a renormalization of the "own" Allen-Uzawa (or partial) 'elasticity of substitution'.

The Allen-Uzawa elasticity, on the other hand, is essentially a renormalization of the elasticity of the Hicksian (or compensated) demand for the i'th good with respect to the j'th price. It implies a change in the j'th price only, and the associated price change vector v does not lie in L_0. It is thus not a special case of the DSES, and in the opinion of the author not a proper elasticity of substitution.

The main advantage of the DSES is that it is defined for an arbitrary price change in L_0. In the context of price indices and homothetic preference it will allow us to measure the elasticity of substitution in the direction of the actual price change from the base period to comparison period.

We recognize the denominator in (23) as the square of the elasticity norm of the price change $p - p^0$ introduced in (8), allowing us to rewrite the DSES as

$$\mathrm{DSES}_{p^0}(p-p^0) \;=\; -\,\frac{\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{c_{ij}(p^0)}{c(p^0)}\,(p_i - p_i^0)(p_j - p_j^0)}{\|p - p^0\|_0^2}\,, \qquad \begin{array}{l} p \in L_0, \\ p \neq p^0\,. \end{array} \tag{25}$$

It is in fact this denominator which originally suggested the elasticity norm introduced in section 2.3.

The determination of the elasticity of substitution is in general an empirical question requiring the estimation of the parameters of the expenditure function. Thus information about the DSES is rarely available to the compiler of price indices. It turns out, however, that if we are using the superlative Törnqvist or quadratic mean of order r indices and are willing to assume that these indices are exact for the true preferences, then the usual CPI data (p^0, x^0) and (p^1, x^1) are all we need to compute the DSES in the direction of the price change $p^1 - p^0$.

If we use a Törnqvist (or translog) index, defined by

$$P^0(p^1, x^1; p^0, x^0) \;=\; \prod_{i=1}^{n} \left(\frac{p_i^1}{p_i^0}\right)^{\frac{1}{2}(s_i^0 + s_i^1)}\,,$$

and the true unit expenditure function is a translog function, then the implicit DSES at p^0 in the direction $v = p - p^0$, $p \in L_0$, is

$$\mathrm{DSES}_{p^0}(p - p^0) \;=\; 1 \;-\; \frac{\displaystyle\sum_{i=1}^{n} (s_i^1 - s_i^0)\,(\ln p_i^1 - \ln p_i^0)}{\displaystyle\sum_{i=1}^{n} s_i^0 \left(\ln \frac{p_i^1}{p_i^0} \;-\; \sum_{k=1}^{n} s_k^0 \,\ln \frac{p_k^1}{p_k^0}\right)^2} \;. \tag{26}$$

Thus the DSES at p^0 in the direction v of the observed price change is fully determined by the observations (p^0, x^0) and (p^1, x^1) for the given choice of superlative index.[15]

The bias $\beta^L(p, p^0)$ becomes in essence a measure of the curvature of the unit expenditure function c. While the directional elasticity of substitution $\mathrm{DSES}_{p^0}(v)$ is a local (or infinitesimal) measure of the curvature of c at p^0 in the direction v, the bias $\beta^L(p, p^0)$ is an overall measure of the curvature of c over the price change $p - p^0$. The units of measurement of the two curvatures are also entirely different. While the DSES is measured as an elasticity (a percentage response to a percentage change), the β^L is measured as a proportionality factor, or as a distance measure in the sense of a distance (or gauge) function.

4. An approximate substitution bias

The definition (20) of the substitution bias β^L presented in section 3.1 is not very useful in practice since we generally do not know the true price index P^*. But having introduced a definition of the length of the price change in section 2 and an appropriate concept of the curvature of the expenditure function in section 3.2, we are ready to define an approximate substitution bias which can be computed on the basis of the data generally available for the construction of consumer price indices.

In this section we will first consider the Taylor expansion of the true price index and then take the Taylor expansion of the substitution bias β^L in section 4.2 defining both an approximate substitution bias and an approximate true price index. The section then concludes with an illustration of the proposed measures using data from the Norwegian CPI data base.

4.1. Approximating the true index

We start by considering the second order Taylor series expansions of the true price index $P^*(p, p^0)$ and the Laspeyres indices $P^L(p, p^0)$. The Taylor expansion of the true

[15] See Frenger (2005, eq. 23), which also derives analogous expressions for the implicit DSES for the mean order r index and the special case of the Fisher index ($r = 2$).

index around the base period price p^0 is[16]

$$P^*(p,p^0) \;\simeq\; 1 \,+\, \sum_{k=1}^{n} \frac{c_k(p^0)}{c(p^0)}\,(p_k - p_k^0) \,+\, \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} \frac{c_{kl}(p^0)}{c(p^0)}\,(p_k - p_k^0)(p_l - p_l^0)\;, \qquad (27)$$

an expression which is valid for all $p \in R_+^n$.

Let us restrict the price p to lie in the Laspeyres plane L_0. This causes the first order term in (27) to vanish. Further it allows us to introduce the norm (8) and apply the alternate expression (25) for the DSES, and makes it possible to rewrite the Taylor approximation to the true index (27) as[17]

$$\begin{aligned} P^*(\tilde p,p^0) \;&\simeq\; 1 \,+\, \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} \frac{c_{kl}(p^0)}{c(p^0)}\,(\tilde p_k - p_k^0)(\tilde p_l - p_l^0) \qquad\qquad \tilde p \in L_0\,, \\ &=\; 1 \,+\, \frac{1}{2}\,\frac{\sum_{k=1}^{n}\sum_{l=1}^{n} \frac{c_{kl}(p^0)}{c(p^0)}\,(\tilde p_k - p_k^0)(\tilde p_l - p_l^0)}{\|\tilde p - p^0\|_0^2}\,\|\tilde p - p^0\|_0^2 \\ &=\; 1 \,-\, \frac{1}{2}\,\mathrm{DSES}_{p^0}(\tilde p - p^0)\,\|\tilde p - p^0\|_0^2\;, \end{aligned} \qquad (28)$$

an expression which looks essentially like a Taylor expansion in a single variable with the DSES as the second derivative and $\|\tilde p - p^0\|_0$ as the step size.

Let us briefly consider the Taylor expansion of the Laspeyres index (7). It is is just a linear function in the comparison price p, and the expansion reduces to

$$P^L(p,p^0) \;\simeq\; P^L(p^0,p^0) \,+\, \sum_{k=1}^{n} \frac{s_k^0}{p_k^0}\,(p_k - p_k^0) \;=\; \sum_{k=1}^{n} s_k^0\,\frac{p_k}{p_k^0}\;. \qquad (29)$$

If in fact $p \in L_0$, then the Taylor approximation (and the index) is exact and reduces to

$$P^L(\tilde p,p^0) \;=\; 1\;, \qquad\qquad \tilde p \in L_0\,. \qquad (30)$$

The Laspeyres index is essentially independent of the true form of the preferences!

4.2. The approximate substitution bias

Let us now consider the ratio definition of the substitution bias [see (20)],

$$\beta^L(p,p^0) \;=\; \frac{P^L(p,p^0)}{P^*(p,p^0)}\;.$$

[16]See Schultze and Mackie (2002), eqn. (45) on p. 91.

[17]The second term in (28) is not defined for $p = p^0$ since $\mathrm{DSES}_{p^0}(p^0 - p^0)$ is not defined. In practice it is convenient to set it equal to 0.

Taking its Taylor expansion around p^0 gives, after some omitted steps,

$$\beta^L(p,p^0) \;\simeq\; 1 \;-\; \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{c_{kl}(p^0)}{c(p^0)}\,(p_k-p_k^0)\,(p_l-p_l^0)\;, \tag{31}$$

an expression which is valid for all p.

We start by restricting the definition of the approximate bias to L_0 so as to be able to apply both the norm and the directional elasticity of substitution. Thus we define the approximate substitution bias $\hat\beta^L$ on L_0 as the above Taylor approximation wrt. p (31) with p restricted to lie in the Laspeyres plane,

$$\begin{aligned}\hat\beta^L(\tilde p,p^0) \;&=\; 1 \;-\; \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{c_{kl}(p^0)}{c(p^0)}\,(\tilde p_k-p_k^0)\,(\tilde p_l-p_l^0) \qquad \tilde p\in L_0\,,\\ &=\; 1 \;+\; \frac{1}{2}\,\mathrm{DSES}_{p^0}(\tilde p-p^0)\,\|\tilde p-p^0\|_0^2\;. \end{aligned}\tag{32}$$

Next, we extend this definition to the whole price space by appealing to the homogeneity of degree zero of $\beta^L(p,p^0)$ in p,[18] and define the *approximate substitution bias* for an arbitrary p by setting it equal to the bias of the canonical $\tilde p \in L_0$,

$$\begin{aligned}\hat\beta^L(p,p^0) \;&=\; \hat\beta^L(\tilde p,p^0) \qquad\qquad \tilde p \;=\; \frac{p}{P^L(p,p^0)}\;,\\ &=\; 1 \;+\; \frac{1}{2}\,\mathrm{DSES}_{p^0}(\tilde p-p^0)\,\|\tilde p-p^0\|_0^2\;. \end{aligned}\tag{33}$$

In this way we impose homogeneity of degree zero in prices upon the approximate bias $\hat\beta^L(p,p^0)$, a property which was considered important in the definition of the true bias β^L and which it is desirable for the approximating definition also to possess. The expression for the approximate bias shows that, as stated by Manser and McDonald (1988, p. 909), "the size of the substitution bias is expected to be positively related both to the degree of commodity substitutability and to the amount of relative price change." They did not, however, have the means to measure either the 'degree of substitutability' or the 'amount of relative price change'.

The approximate substitution bias then has the same properties as the true bias β^L presented in lemma 2:

Corollary 3. Properties of the approximate substitution bias $\hat\beta^L$:

i) $\hat\beta^L(p,p^0) \geq 1$ for all $p \in R^n_+$.

ii) $\hat\beta^L(p,p^0)$ is homogeneous of degree zero in p for all $p \in R^n_+$.

iii) Let $\tilde p(\theta) = \theta\,\check p + (1-\theta)\,p^0$ be a line segment in L_0 with $\check p \in L_0$ and $\theta > 0$. Then $\hat\beta^L(\tilde p(\theta_2),p^0) \geq \hat\beta^L(\tilde p(\theta_1),p^0)$ for $\theta_2 \geq \theta_1$.

[18] See lemma 2-ii.

Equation (32) gives the relationship between the DSES and the substitution bias, and emphasizes the local character of the DSES as a measure of the bias. The determination of either one of the measures implicitly determines the other, but neither one is in general readily determined. The link between the two measures is the length $\|\tilde{p} - p^0\|_0$ and this is readily computed from data available to statistical agencies. It is also just a first order parameter, while both the DSES and the substitution bias are second order parameters.

The approximate bias allows us also to define the following approximate true price index,

$$\hat{P}(p,p^0) \;\equiv\; \frac{P^L(p,p^0)}{\hat{\beta}^L(\tilde{p},p^0)} \;=\; \frac{P^L(p,p^0)}{1 + \dfrac{1}{2}\,\mathrm{DSES}_{p^0}(\tilde{p}-p^0)\,\|\tilde{p}-p^0\|_0^2}\,, \qquad \tilde{p} \;=\; \frac{p}{P^L(p,p^0)}\,. \tag{34}$$

This essentially reverses the procedure used in the definition (20) of the bias β^L of the true index. By construction, the approximate index is linearly homogeneous in the comparison price p. This approximate index may be computed solely on the basis of the base year information (p^0, x^0), the comparison price p, and some estimate of the DSES. It thus represents an alternative to the Lloyd-Moulton index procedure of Shapiro and Wilcox (1997).

4.3. The approximate substitution bias measured as a difference

We may also briefly consider the Taylor expansion of the substitution bias, measured as a difference [see (22)], which becomes [see (30) and (28)]

$$\begin{aligned} b^L(\tilde{p},p^0) \;=\; P^L(\tilde{p},p^0) \,-\, P^*(\tilde{p},p^0) \;&\simeq\; -\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{c_{kl}(p^0)}{c(p^0)}\,(\tilde{p}_k - p_k^0)(\tilde{p}_l - p_l^0) \qquad p \in L_0\,, \\ &=\; \frac{1}{2}\,\mathrm{DSES}_{p^0}(\tilde{p}-p^0)\,\|\tilde{p}-p^0\|_0^2 \qquad\qquad (35) \\ &\equiv\; \hat{b}^L(\tilde{p},p^0)\,. \end{aligned}$$

The last equality defines $\hat{b}^L(\tilde{p},p^0)$, the approximate substitution bias when the bias is defined as a difference.[19] We can again extend the definition to all p by associating with it the canonical $\tilde{p} \in L_0$. In addition, in accordance with general practice, we multiply $\hat{b}^L$ by the Laspeyres inflation, defining

$$\hat{b}^L(p,p^0) \;=\; P^L(p,p^0)\,\hat{b}^L(\tilde{p},p^0)\,, \qquad \tilde{p} \;=\; \frac{p}{P^L(p,p^0)}\,, \qquad p \in R_+^n\,. \tag{36}$$

[19] In the pre-substitution-bias days von Bortkiewicz (1923, p. 374–7) analyzed of the difference between the Laspeyres and the Paasche indices. Let P^P denote the Paasche price index and let Q^L and Q^P denote the Lapeyres and the Paasche quantity indices, then the von Bortkiewicz δ becomes approximately

$$\delta \;=\; \frac{P^P - P^L}{P^L} \;=\; \frac{Q^P - Q^L}{Q^L} \;\simeq\; -\,\mathrm{DSES}_{p^0}(\tilde{p}-p^0)\,\|\tilde{p}-p^0\|_0^2\,, \qquad p \in L_0\,.$$

The sign reflects the negative correlation which exists between price and quantity changes in the economic theory of index numbers.

We see that the ratio and the difference definitions are related by

$$\hat{b}^L(\tilde{p}, p^0) \;=\; \hat{\beta}^L(\tilde{p}, p^0) \,-\, 1\,, \qquad\qquad \tilde{p} \in L_0\,.$$

Thus the Taylor approximations of $\hat{\beta}^L$ and $\hat{b}^L$ give essentially the same result, except for the 1, at least for $\tilde{p} \in L_0$. It is for comparison prices that are not in L_0 that the difference arises: the ratio definition $\hat{\beta}^L$ is homogeneous of degree 0 in prices while the difference definition $\hat{b}^L$ is homogeneous of degree 1.

When the true function is CES with substitution parameter σ, then we know that $\mathrm{DSES}_{p^0}(\tilde{p} - p^0) = \sigma$ for all $\tilde{p} \in L_0$, and the approximate bias defined as a difference (35) reduces to

$$\hat{b}^L(p, p^0) \;\simeq\; \frac{1}{2} P^L(p, p^0)\, \sigma \|\tilde{p} - p^0\|_0^2 \;=\; \frac{1}{2} P^L(p, p^0)\, \sigma \sum_{i=1}^n s_i(p^0) \left(\frac{\tilde{p}_i}{p_i^0} - 1\right)^2 , \qquad \tilde{p} \;=\; \frac{p}{P^L(p, p^0)}\,.$$

This approximation reduces further to the expression obtained by Diewert (1998, p. 50) under the restrictive assumption of Cobb-Douglas preference ($\sigma = 1$) and equal value shares [the "variance" term $V(p)$ was introduces in (12) above],[20]

$$B_E \;\equiv\; P^L \,-\, P^F \;\simeq\; \frac{1}{2}\,(1 + \iota)\,\mathrm{Var}(p)\,.$$

As stated earlier, Braithwait (1980) tried to assess the relative importance of price change and substitutability and for that purpose he supplemented his price dispersion measure D [see (11)] with a measure of the curvature, and he found that: "A concise measure of the degree of commodity substitution is the expenditure share-weighted average of income-compensated own-price elasticities." Let $\varepsilon_{ii} = p_i c_{ii}/x_i$ be the own Hicksian price elasticity of demand. Then his measure may be written $\bar{\varepsilon} = \sum_i s_i \varepsilon_{ii}$.[21] Thus Braithwait has "in some form" the three elements appearing in (35), but he does not have the formal relationship! Braithwait applied his analysis also to 16 subaggregates of the CPI.

Schultze and Mackie (2002, p. 91) assume arbitrary (non-homothetic) preferences represented by an expenditure function C from which they derive the associated true price index P^*. By taking the Taylor expansion with respect to an arbitrary price change they show that the bias is approximately

$$b^L \;\equiv\; P^L(p^1, p^0) \,-\, P^*(p^1, p^0) \;\simeq\; -\,\frac{1}{2}\,\frac{(p^1 - p^0)'\, C_{pp}(p^0)\,(p^1 - p^0)}{p^{0\prime}\, x^0}\,,$$

where p^0 and p^1 are base period and comparison period prices and C_{pp} is the Hessian of the expenditure function. This is essentially the first line of (35) without deflating the comparison price or equivalently restricting it to lie in the Laspeyres plane. They conclude

[20] Diewert's substitution bias is defined as the difference between the Laspeyres index and the Fisher index P^F, the latter being a proxy for the true cost of living index.

[21] His measure of the degree of commodity substitution is independent of the direction of the price change and must be considered as some kind of average measure. It is in my opinion not a proper measure of the curvature of the preferences, and does not give the right result for a CES function since in that case $\bar{\varepsilon} = -\,\sigma\,(1 - \sum_i s_i^2)$.

that: "Thus, the difference between the base period cost-of-living index and the Laspeyres price index is zero to the first order so that the Laspeyres is a first-order approximation to the base period cost of living. The approximate difference between them, depends on how much substitution is possible, which is represented by the matrix S^0 $[C_{pp}(p^0)]$ as well as by the size of the difference between the base and current price vectors." Their expression provides no assistance in computing the bias, and the approximation seems rather useless for practical purposes! ILO (2004, ch. 17, pp. 330–2) use the same kind of second order approximations to analyze the bias of the Lowe index.

4.4. Application to the Norwegian CPI data

We will now use the procedure of section 4.2 to compute the approximate substitution bias $\hat{\beta}^L$ and the approximate true index $\hat{P}$ for the Norwegian CPI. But before we do it may be convenient to summarize the steps of the construction procedure. Again we let t designate an arbitrary base year and let τ denote a comparison year (different from t). The data base for the CPI provides us with information on prices and quantities consumed (p^t, x^t) for each year t in the sample period T. In step (iv) we need information on the DSES, which can either be computed from the CPI data base using the procedure of Frenger (2005) or obtained from some other source. I have divided the computations into the following 6 steps:

i) The Laspeyres index: $P^L(p^\tau, p^t)$ [see (7)].

ii) Deflated price $\tilde{p}^\tau$ [see (19)]

$$\tilde{p}^\tau \;=\; \frac{p^\tau}{P^L(p^\tau, p^t)}\;. \tag{37}$$

iii) Size of price change [see (8)]

$$\|\tilde{p}^\tau - p^t\|_0 \;=\; \left[\sum_{k=1}^{n} s_k(p^t)\left(\frac{\tilde{p}_k^\tau}{p_k^t} - 1\right)^2\right]^{\frac{1}{2}}\,, \qquad\qquad \tilde{p}^\tau \in L_t\,. \tag{38}$$

iv) Determination of the appropriate directional shadow elasticity of substitution (DSES),

   a) compute the implicit DSES for some mean order r price index, f.ex. using (26). This method requires knowledge of x^τ.[22]

   b) obtain the DSES from some other source.

v) Then using (32), define the quadratic approximation $\hat{\beta}^L$ to the Laspeyres substitution bias β^L,

$$\hat{\beta}^L(\tilde{p}^\tau, p^t) \;=\; 1 \;+\; \frac{1}{2}\,\mathrm{DSES}_{p^t}(\tilde{p}^\tau - p^t)\;\|\tilde{p}^\tau - p^t\|_0^2\;. \tag{39}$$

[22] If one uses the implicit DSES for some superlative price index, then one would in practice probably use the associated price index instead of the approximate index derived in step (vi).

vi) And calculate the following approximation to the true index [see (34)]

$$\hat{P}(p^\tau, p^t) \;=\; \frac{P^L(p^\tau, p^t)}{\hat{\beta}^L(\tilde{p}^\tau, p^t)} \;=\; \frac{P^L(p^\tau, p^t)}{1 + \dfrac{1}{2}\,\mathrm{DSES}_{p^t}(\tilde{p}^\tau - p^t)\,\|\tilde{p}^\tau - p^t\|_0^2}\,. \qquad (40)$$

Returning to the application, the data are taken from the Norwegian CPI data base for the lowest level for which the Survey of Consumer Expenditures provides the necessary budget shares. At this level there are 148 commodities, but only 140 provide complete data for all the years 1990–1998. Thus our price space has dimension 140 and the Laspeyres plane L_0 has dimension 139. The Norwegian CPI is a chained Laspeyres index with weights changing annually.

The results of the computations are presented in table 4. The first column presents the size of the price step $\|\tilde{p}^{t+1} - p^t\|_t$ in the tangent plane at p^t [see (38)]. These are the same data that appear above the empty diagonal in table 1. Column two presents the

Table 4: The substitution bias and approximate indices.

year	length $^{a)}$ % $\|\tilde{p}^{t+1} - p^t\|_t$ (38)	DSES $^{b)}$ (26)	approx. substitution bias (%) $\hat{\beta}^L - 1$ (39)	Laspeyres index $^{c)}$ % $P^L(p^{t+1}, p^t)$ (7)	approx. Törnqvist $^{d)}$ index (%) $\hat{P}(p^{t+1}, p^t)$ (40)
1990	4.6	0.6760	0.07	3.36	3.28
1991	3.2	0.3467	0.02	2.31	2.29
1992	3.3	0.6702	0.04	2.25	2.21
1993	3.8	0.9794	0.07	1.39	1.32
1994	3.4	0.7941	0.05	2.42	2.37
1995	3.2	1.2739	0.07	1.24	1.17
1996	2.7	1.0808	0.04	2.55	2.51
1997	4.2	1.2552	0.11	2.28	2.16
aver.	3.6	0.8845	0.06 $^{e)}$	2.22	2.16

a) See above the empty diagonal in table 1.

b) See last column of table 1 in Frenger (2005, p. 17).

c) See above the empty diagonal in table 3.

d) The name Törnqvist has been added as a reminder that a DSES computed from (26) was used.

e) The same average bias was computed in Frenger (2005, p. 17) as the difference between the chained Laspeyres index and the Törnqvist index.

implicit DSES of the Törnqvist price index computed from the CPI data using (26). We see f.ex. that the 1990 DSES measured in the direction of the 1990–1991 price change is

0.6760. The average DSES over the period is 0.8845. In their much quoted study Shapiro and Wilcox (1997) use a CES function and find that a $\sigma = 0.7$ fits their U.S. CPI data best, a result which is fairly close to the values derived above.

The approximate substitution bias is then computes using (39). This bias is on the whole rather small, averaging only 0.06% over the period, and this seems mainly to be due to the small step size. We may note that the DSES's are all positive, as required by the theory, and thus the biases are all larger than unity. We then use use the expression (34) to compute the approximate true index, taking the substitution bias $\hat{\beta}^L(\tilde{p}^{t+1}, p^t)$ from the third column and the Laspeyres index $P^L(p^{t+1}, p^t)$ from the fourth. The approximate index is then presented in column five. This approximate index differs from the proper Törnqvist index (not presented above) by at most one digit at the fourth decimal place.

We may note that in his numerical example of the upper level substitution bias Diewert (1998, p. 50) uses a step size of 7.1%. Combining this with his implicit elasticity of substitution of unity and an inflation rate of 2% he gets a bias of 0.26%.[23] Diewert is the only author I know of that has utilized the step size in the computation of the bias, but this is a statistic that is rather easy to compute. Ignoring the inflation term, which is quantitatively rather unimportant, we see that Diewert's ballpark figure is almost four times the average presented in table 4.[24]

On the other hand I have already stated that the Norwegian CPI is rebased and chained annually, which means that $\tau = t + 1$ and that the distance between $\tilde{p}^\tau$ and the base price p^t is small. If the Laspeyres index had a fixed base year for the entire period these price changes would be substantially larger. To illustrate the point return to table 1 and consider $\|\tilde{p}^{1995} - p^{1990}\|_{1990} = 12.4\%$. If the index had not been rebased each year, this would have given us, with the 1990 DSES, a bias $\hat{\beta}^L(p^{1995}, p^{1990})$ of 0.52%.

In the computations above we have both price and quantities for the comparison period. Shapiro and Wilcox (1997) confront the challenge of constructing a superlative index in real-time, the primary problem being that of not having comparison period quantity or budget share data. Their solution is to use an exact CES index to construct a real-time consumer price index which they claim is "substantially free of across-strata [substitution] bias". The advantage of the CES function is that is can be fully parametrized by the base period data for any substitution parameter σ. One is then free to specify σ *a priori*, and they find that the value $\sigma = 0.7$ allows the CES index to grow, on the average, at almost exactly the same rate as the Törnqvist index over the sample period 1988–1995.[25]

[23] Diewert uses $\mathrm{Var}(p) = \|p - p^0\|^2 = 0.005^{1/2}$. He also adds to the upper level bias an elementary substitution bias of about the same size.

[24] Braithwait (1980) [see (11) above] computed his dispersion index, but did not integrate it in an expression for the computation of the bias.

[25] Diewert (1998, p. 54) states that "The most significant new development is the application by Shapiro and Wilcox (1997) of an index number formula that was independently proposed by Lloyd (1975) and Moulton (1996). This formula offers the promise of overcoming the major practical difficulty of using a superlative index like the Fisher ideal index: that information on quantities currently being consumed is typically not available for a lag of a year or more — which clearly makes it unsuitable for producing a monthly estimate like the Consumer Price Index."

In the "Executive Summary" Schultze and Mackie (2002) conclude that: "The BLS should publish, contemporaneously with the real-time CPI, an advance estimate of the superlative index, utilizing either

As an alternative to their procedure, as I have alluded to earlier, one could use the method outlined in (37) – (40) using $\mathrm{DSES}_{p^t}(\tilde{p}^{t+1} - p^t) = \sigma$. One could then construct the approximate index $\hat{P}(p^{t+1}, p^t)$. I believe it would give a good approximation and provide results which are very close to those obtained by Shapiro and Wilcox.

The proposed method has two main advantages. First, it never postulates that the true function is a CES function, only that the directional elasticity of substitution at p^t in the direction of the $\tilde{p}^{t+1} - p^t$ price change is σ. The proposed approximation is flexible in that it can provide a second order approximation to an arbitrary index at any point and in any direction. And the resulting index $\hat{P}$ could even claim to be superlative. But the argument is essentially formal, since Shapiro and Wilcox could contend [they do not] that all they do is to apply the CES index in the one direction of the observed price change.[26] The second, and more practical, advantage of the method is that it provides us with consistent information on the length $\|p_{t+1} - p_t\|_t$ and on the approximate bias $\hat{\beta}^L$, which would appear to be useful information in evaluating the index construction. Note also that the norm $\| \ \|_t$ only depends on information which is available in real-time.

5. Concluding comments

The paper has developed a consistent procedure for measuring the bias of the Laspeyres price index as summarized by the equations (37) – (40). In particular, it brings together the distance measure, the elasticity of substitution, and the approximate bias in an intuitive and appealing way as formalized by (32). But the most novel and perhaps the most useful result of the paper might be the introduction of the metric in the Laspeyres plane with its associated interpretation as a price substitution index. The numerical examples illustrates its key role in determining the size of the substitution effect, and thus of the quantitative importance of rebasing the consumer price index.

These numerical results are based on a rather limited sample of Norwegian data, which emphasizes the importance of extending these calculations to other countries. The Eurostat database for the Harmonized Indices of Consumer Prices appears to be a good source of data and an fruitful area of application of the present approach. Further it might be interesting to reexamine some of the earlier studies on substitution bias in order to determine what role the size of the substitution inducing price change has played in them.

The paper limits its definition and analysis of the substitution bias resulting from the use of the Laspeyres price index. This simplifies the analysis and the geometric interpretation, but excludes the study of the bias of, f.ex., the Paasche index, the Fisher index, and the recently 'rediscovered' Lowe index popularized by Balk and Diewert in ILO (2004, pp. 329–32)). I believe that many of the ideas developed above do carry over to the more general setting, but at the cost of loss of simplicity and intuitiveness. Empirical

a constant-elasticity-of-substitution method or some other technique." (Recommendation 7-1).

[26]Their method for estimating σ by determining the value which best approximates a superlative index stretches the argument a bit further.

applications may also impose much heavier claims on data.

A topic not addressed is the effect of the level of aggregation on the estimation of the substitution bias. It is, however, clear that both the method and level of aggregation have important effects on the resulting measures of the second order properties of the preferences, and it would appear that the line of inquiry presented in this paper lends itself well to such an analysis.

List of symbols

symbol	explanation	page	eqn. nr.
$b^L(p,p^0)$	substitution bias of the Laspeyres index (measured as a difference), $b^L(p,p^0) = P^L(p,p^0) - P^*(p,p^0)$	17	22
$c(p)$	unit expenditure function	5	1
$\mathrm{DSES}_{p^0}(v)$	directional shadow elasticity of substitution at p in the direction $v = p - p^0$, $p \in L_0$	17	23
L_0	Laspeyres plane or equiv. the tangent plane to M at p^0	6	5
M	level surface of true price index, $M = \{\, p \mid P^*(p,p^0) = 1 \,\}$	6	4
$p \in R^n_+$	price vector, $p = (p_1, \ldots, p_n)$		
$\tilde{p}$	price in L_0		
P	a price index		
$P^*(p,p^0)$	the true (Konüs) price index	5	3
$\hat{P}(p,p^0)$	approximate true price index	22	34
$P^L(p,p^0)$	the Laspeyres price index	6	7
R^n_+	price space, $R^n_+ = \{p = (p_1, \ldots, p_n) \mid p_i > 0\,,\ i = 1, \ldots, n\}$	5	
$s(p)$	value shares, $s = (s_1, \ldots, s_n)$, $s_i = p_i x_i / c$	5	2
v	price change vector, $v = (v_1, \ldots, v_n) = v_p$. The price change vector is "attached" at p.		
x	commodity vector, $x = (x_1, \ldots, x_n)$, $x = c_p(p)$		
$\beta^L(p,p^0)$	substitution bias of the Laspeyres index (measured as a ratio), $\beta^L(p,p^0) = P^L(p,p^0)/P^*(p,p^0)$	16	20
$\hat{\beta}^L(p,p^0)$	approximate substitution bias of the Laspeyres index	21	32
$\Vert p - p_0 \Vert_0$	length of price change $p - p_0$, $p \in L_0$, in the elasticity norm	8	8
%	percentage change, but also 1/100 'th of a unit of length in the elasticity norm	14	

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