SAMFUNNSØKONOMISKE STUDIER

52

ECONOMETRICS OF INCOMPLETE CROSS-SECTION/TIME-SERIES DATA:

CONSUMER DEMAND IN NORWEGIAN HOUSEHOLDS 1975—1977

ØKONOMETRISK ANALYSE AV UFULLSTENDIGE TVERRSNITTS/TIDSSERIE-DATA:

KONSUMETTERSPØRSELEN I NORSKE HUSHOLDNINGER 1975—1977

BY/AV
ERIK BIØRN AND EILEV S. JANSEN

STATISTISK SENTRALBYRÅ
CENTRAL BUREAU OF STATISTICS OF NORWAY

ECONOMETRICS OF INCOMPLETE CROSS-SECTION/TIME-SERIES DATA:

CONSUMER DEMAND IN NORWEGIAN HOUSEHOLDS 1975—1977

ØKONOMETRISK ANALYSE AV UFULLSTENDIGE TVERRSNITTS/TIDSSERIE-DATA:

KONSUMETTERSPØRSELEN I NORSKE HUSHOLDNINGER 1975—1977

SAMFUNNSØKONOMISKE STUDIER NR. 52

ECONOMETRICS OF INCOMPLETE CROSS-SECTION/TIME-SERIES DATA:

CONSUMER DEMAND IN NORWEGIAN HOUSEHOLDS 1975—1977

ØKONOMETRISK ANALYSE AV UFULLSTENDIGE TVERRSNITTS/TIDSSERIE-DATA: KONSUMETTERSPØRSELEN I NORSKE HUSHOLDNINGER 1975—1977

BY/AV
ERIK BIØRN AND EILEV S. JANSEN

STATISTISK SENTRALBYRÅ
Oslo—Kongsvinger 1982

ISBN 82-537-1782-2
ISSN 0085-4344

PREFACE

This is an econometric study of consumer demand in Norwegian households based on selected data from the annual Surveys of Consumer Expenditure 1975 - 1977. It is mainly concerned with simultaneous estimation of a system of demand functions for five aggregated commodity groups, but additional results for 28 disaggregated commodities are also reported.

The authors provide several interesting interpretations of the underlying theoretical model and their empirical findings at both levels of aggregation. The results for the disaggregated commodities give a detailed picture of the structure of consumption in Norwegian households. This information is of value, inter alia, for the current work with macro-economic planning models and tax incidence models in the Central Bureau of Statistics.

The study also demonstrates the fruitfulness of a co-operation between econometricians and data producers within a statistical office. The way in which the authors exploit the particular sampling design with partly overlapping (rotating) panels followed in the Surveys of Consumer Expenditure gives evidence of this. Emphasis is laid on the development of advanced estimation methods which take explicit account of the fact that the data consist of households which participate in the surveys in two consecutive years (1975 - 1976 and 1976 - 1977, respectively). The analysis shows that this approach yields valuable information on individual differences in consumption behaviour that could not have been obtained if the samples had been non-overlapping.

Central Bureau of Statistics, Oslo, 27 May 1982

Arne Øien

FORORD

Dette er en økonometrisk studie av konsumetterspørselen i norske husholdninger basert på utvalgte data fra Statistisk Sentralbyrås løpende forbruksundersøkelser for årene 1975 - 1977. Størstedelen av analysen er viet simultan estimering av et system av etterspørselsfunksjoner for fem aggregerte varegrupper, men den inneholder også supplerende resultater for 28 disaggregerte varegrupper.

Forfatterne gir flere interessante tolkninger av den underliggende teoretiske modell og de empiriske resultater på begge aggregeringsnivåer. Resultatene for de disaggregerte gruppene gir et detaljert bilde av forbruksmønsteret i norske husholdninger. Disse er nyttige blant annet for arbeidet med videreutvikling av de makroøkonomiske planleggingsmodellene og skatteinsidensmodellene i Statistisk Sentralbyrå.

Studien er også et eksempel på et fruktbart samarbeid mellom økonometrikere og dataprodusenter i Byrået. Dette kommer særlig til uttrykk ved at forfatterne utnytter den spesielle utvalgsplanen med roterende utvalg som benyttes i forbruksundersøkelsene. Det er lagt spesiell vekt på å utvikle avanserte estimeringsmetoder for å få tatt hensyn til at datamaterialet består av husholdninger som har deltatt i forbruksundersøkelsene i to år på rad (henholdsvis 1975 - 1976 og 1976 - 1977). Analysen viser at dette gir verdifull informasjon om individuelle forskjeller i forbruksmønsteret som ikke kunne ha blitt avdekket om utvalgene var blitt trukket på nytt hvert år.

Statistisk Sentralbyrå, Oslo, 27. mai 1982

Arne Øien

ABSTRACT

This is an econometric study of a complete system of consumer demand functions based on combined cross-section/time-series data. The theoretical basis of the model is the Fourgeaud-Nataf specification of demand systems (Econometrica, 1959), converted to budget shares. Its disturbance vector is decomposed into an individual component and a remainder. The empirical basis is reports from 418 Norwegian households, one half observed in 1975 and 1976, the other half in 1976 and 1977, i.e. the data are incomplete cross-section/time-series data. Two different levels of aggregation of commodities are considered, specifying 28 and 5 groups, respectively. Type of household is represented parametrically by the number of household members and the age of its main income earner. FIML estimates of the coefficients along with the implied estimates of Engel and Cournot elasticities and the income flexibility (the Frisch parameter) are reported. A main conclusion is that a significant part of the disturbance variance can be ascribed to unobserved individual differences for the large majority of commodity groups.

AUTHORS' PREFACE

In this book, we present the result of an econometric exploration of an original data source. The project has a history which deserves mention:

In the beginning of the 1970's, a plan was worked out in the Central Bureau of Statistics for a substantial revision of the design of the surveys of consumer expenditures. Its essence was to replace the large national surveys, performed at intervals of several years, by a more or less continuous registration of household expenditures according to a unified plan. In 1974, when the annual surveys had just started, Arne Amundsen and the first author - recognizing the interesting perspectives for econometric research that these data opened up - advised the Bureau that a fraction of the participating households in each year should be asked to report once again in the following year. This idea was accepted, and the principle of 'rotation' came into effect for 25 per cent of the respondents each year from 1975 onwards.

Four years later, in 1978, when time was ripe for econometric treatment of the data files which had accumulated, Erik Biørn had worked out a couple of preliminary methodological papers on the utilization of such data. This work - supplemented by some ideas on the econometric treatment and computer strategies from Eilev S. Jansen - provided the starting point for the authors' joint work. Since then, we have worked together in very close co-operation at all stages of the project. Our individual contributions are somewhat entangled, but the broad lines are that Erik Biørn has had the main responsibility for the formulation of the theoretical and econometric framework, whereas Eilev S. Jansen has mainly made his contributions in the applied econometric department (inter alia, by elaborating the econometrics of FIML estimation and directing the computer work). A further indication of the division of labour between us is that Erik Biørn has written first drafts to chapters I, II, III, sections 6.1, 6.5, 6.6, 7.4, 7.5, and appendices A, B, D, E, and F, and Eilev S. Jansen the first drafts to chapters IV, V, VIII, sections 6.2-6.4, 7.1-7.3, 7.6, and appendices C, G, and H. The final study is, of course, the result of the authors' common efforts.

Previous versions of this study have been presented to several audiences outside the Central Bureau of Statistics, including: an econometric seminar held at Stockholm School of Economics (May 1980), the 4th World Congress of the Econometric Society in Aix-en-Provence, France (September 1980), a staff seminar at INSEE, Paris (October 1980), the

Winter Meeting of the Econometric Society in Lyngby, Denmark (January 1981), and staff seminars at the Institute of Economics, University of Oslo (February 1981). Inspiration apart, we have benefited from comments put forward on these occasions and we want in particular to express our gratitude to: Arne Amundsen, Pietro Balestra, Anton Barten, Olav Bjerkholt, Søren Blomquist, Anders Klevmarken, Jan Kmenta, Jacques Mairesse, and Pascal Mazodier.

At the Central Bureau of Statistics, we have received valuable programming support from Øystein Amland, Tore Kristoffersen, Jørgen Ouren, and Anne Sagsveen of the Programming Division. Tore Kristoffersen assisted us in the initial stages, i.e. during the estimation of the single equation versions of the model. Later, Anne Sagsveen has been a keen and expedient support while we struggled with the computational problems of simultaneous estimation of the complete demand model. She is also co-author of the example program in the program annex.

Last, but not least, we would like to thank members of the typing staff of the Bureau, especially Elin Berntzen, Kari Jensen and Solveig Wiig for their impressive ability to transform our incomprehensible manuscripts into a final form.

Oslo, 23 April 1982

E.B. E.S.J.

CONTENTS

Page

INNHOLD

INDEX OF FIGURES

Page

INDEX OF TABLES

I. ECONOMETRICS OF INCOMPLETE CROSS-SECTION/TIME-SERIES DATA: THE CASE OF DEMAND ANALYSIS

During years, econometric aspects of the household demand for consumption commodities have inspired a continuous flow of research, and a lot of empirical results with relevance to econometric modelling and policy analysis have emerged. The present work is a methodological and empirical treatise on the *micro*-econometrics of household demand. To the authors' knowledge, it is the first study of its kind: Its novelty lies primarily in the use of genuine micro data from household budget surveys and in the application of an error components model in that context. Moreover, the parametric form assumed for the demand functions - based on a model originally proposed by Fourgeaud and Nataf (1959) - is one which has not received much attention from empirical researchers, in spite of its virtues for analyzing micro data.

The study's methodological contributions are mainly to the econometric treatment of combined cross-section/time-series data (referred to as CS/TS data in the sequel). In econometric research, a growing attention has been devoted to such data in recent years, for two principal reasons: They are now more accessible to researchers than before, and it is recognized that they give a far wider scope for analyzing individual behaviour and differences in behaviour than more conventional data types. On the other hand, the problem of modelling adequately the stochastic mechanism generating the data becomes more exacting.

The theoretical contributions to the econometric literature on CS/TS data usually assume that the same individuals are observed in all the time periods under consideration, i.e. *complete* CS/TS data.[1] From the point of view of analytical simplicity, the assumption that complete time series exist for all individuals under observation, is obviously very convenient. In many practical cases, however, it is not satisfied, or cannot be made to be satisfied unless one is willing to discard a substantial part of the observations. The data available to the researcher are *incomplete* CS/TS data.[2]

This state of affairs is a main starting point for this study. It is concerned with estimation of consumer demand functions on the basis of *individual* data from a sample of 418 Norwegian households. The data cover a period of three years, but no household is observed more than twice, i.e. we have incomplete CS/TS data. This 'incompleteness'

1) Examples are Balestra and Nerlove (1966), Wallace and Hussain (1969), Nerlove (1971), Maddala (1971), Mazodier (1971), and Mundlak (1978).
2) See Biørn (1981a, 1981b), for a formal and fairly general treatment of such data structures and their relation to complete CS/TS data.

reflects the difficulties in persuading randomly selected households to engage in repetitive reporting of consumption expenditures. Usually, a data collecting agency has no legal means to force an individual to participate in a sampling survey. Attempts to construct a data set in the complete CS/TS format for household budget surveys could be predicted to fail; the panel would almost certainly be subjected to serious 'attrition' after a few years. This, of course, is due to the fact that the reporting of consumption expenses is a time-consuming activity - at least when detailed book-keeping is involved, as is the case in the Norwegian Surveys of Consumer Expenditures. In these surveys, the book-keeping period is only two weeks, but even for those households which are asked to report only once, the rate of non-response is as high as 30 per cent.[3)]

A second starting point for the study is a desire to explore the importance of unobservable individual differences in consumption habits. From casual observation, we all know that different people buy different commodities, and even if differences in income, family size, and other demographic characteristics are accounted for, substantial 'individual factors' reflecting differences in tastes, attitudes, experiences, etc. seem to be left. Regardless of the number of observable variables we specify in our model, we cannot expect to explain all these differences. Thus, they will become part of the model's disturbances. Since our data are individual data with repeated observation of each individual, it is possible to identify these factors and make them subject to a formal econometric analysis.

Why do we want information on the unobservable individual component of consumption expenditures? A decomposition of the disturbance

---

3) In recent years, attention has been devoted to one class of models for incomplete CS/TS data in the context of the so-called *selectivity bias* (or self-selection bias) problem. See e.g. Hausman and Wise (1977), Griliches, Hall, and Hausman (1978), and Maddala (1978). Briefly, their problem is that of estimating regression models efficiently from CS/TS data in situations where the panel has been subjected to 'attrition' or 'accretion' over time as the result of a *systematically changing* degree of non-response: The missing of observations does not take place at random, but is the result of individual choice in such a way that the probability of belonging to the panel in any given period is related to variables which are endogenously determined in the model. Then the standard methods of estimation will not produce consistent coefficient estimates. Such approaches are obviously relevant when dealing with for instance problems of work participation (confer the examples given in Maddala (1978, section II)). In the present study, we shall not be concerned with the selectivity bias problem, as we believe it is of minor importance in connection with consumption data. It may, however, be present for some commodities. If, for instance, a household with a high consumption of liquor is more likely to be a non-respondent than other households, selectivity bias will be present in our data set.

variances is, of course, interesting by itself, but it is not an issue of academic interest only. Estimates of variance components may provide valuable information when micro data on consumption are used to forecast changes in *aggregate* consumption and for estimating the precision of such forecasts. It may also be useful for assessing the potential gain to be obtained from a revision of the sampling design.

The third motivation for this study is, as already declared, a desire to explore the problems of estimating a complete system of consumer demand functions from individual household data by means of advanced econometric methodology. To the authors' knowledge, no previous study has treated this topic. A few researchers have dealt with the estimation of complete demand systems from time series of household budget surveys, viz. Pollak and Wales (1978, 1980, 1981) and Salvas-Bronsard (1978), based on data from U.K. and France, respectively. However, these studies differ from the present one in one crucial respect: instead of using observations from the individual households, *group means* (cell means) calculated from the individual reports are considered as the basic data. This aggregation, of course, implies that all variation of consumption expenditures and their explanatory variables within the different groups is swept under the carpet. With our use of the primary data, we incur no such loss of information - this is a crucial point since our model, like those considered by Pollak and Wales, is non-linear.[4)] Another basic advantage with our data is that they permit identification of individual disturbance components.[5)]

The study is organized as follows: The theoretical model is set out in chapter II and its main properties are examined. Chapter III is concerned with the econometric specification of the model, with particular emphasis on the decomposition of its error terms. A fairly detailed account of an iterative, step-wise algorithm for estimation of the parameters of the model is given in chapter IV . The algorithm is outlined both for the case of simultaneous estimation of the complete demand system and for the simpler problem of single equation estimation. Chapter V gives a brief survey of data. Empirical results for the simultaneous version of the demand system are given in chapter VI, additional single equation results, at a more disaggregated level of commodity classification, are presented in chapter VII. In the concluding chapter, chapter

4) It is also essential for the calculation of the standard errors of the estimated coefficients.

5) Salvas-Bronsard (1978) adopts a disturbance components specification (with an individual, a time specific, and a remainder component) for her group means data. It is difficult to recognise the interpretation of the "individual" disturbance component in her analysis, however, since the basic individual data underlying her group means are not complete CS/TS data. In fact, the different panels of households are non-overlapping.

VIII, we survey the main findings, point out some unsolved problems, and suggest topics for further research. In addition to its text section, the study also includes 8 appendices, dealing with specific, more technical, problems; a table annex, recording all the relevant empirical findings; and a particular annex presenting an example computer program.

II. THE STRUCTURAL MODEL

2.1. Point of departure

There exists a large arsenal of specifications of complete systems of consumer demand functions in the literature.[1] The various models differ not only with respect to flexibility of functional forms, number of free parameters, claims to data etc.; their relation to formal axiomatic theory of consumer's choice is also widely different. We shall focus attention on *static* demand systems and we tacitly assume that the functional form is identical for all the commodities in the system.[2] We may then distinguish between three broad classes of demand systems.

The first class contains those systems which can be derived by maximizing a static utility function of known parametric form, subject to the condition that the sum of the expenses on the different commodities exhausts a given budget. Secondly, we have the systems which satisfy the general restrictions of the consumer demand theory (adding-up, homogeneity, and symmetry of the Slutsky substitution matrix), without being derivable from maximization of a (direct) utility function of known parametric form. An important subclass consists of the models for which the parametric form of the corresponding *indirect* utility function has been established, but whose basic *direct* utility function is still unknown.[3] The third class contains the systems which do not satisfy all the theoretical constraints, or satisfy them only approximately.

It is not easy to find demand systems which agree with all the theoretical constraints and at the same time exhibit the flexibility of functional form which is necessary for confrontation with micro data. For example, demand functions which are linear in total expenditure (i.e. have linear Engel curves), belong to the 'theoretically admissible' class of models, as functions with this property can be derived from parametric utility functions, e.g. the Stone-Geary or the quadratic utility functions.[4] However, strict linearity is not supported by micro data. If this hypothesis is tested, e.g. by running regressions on cross-section data from individual households, it is almost universally rejected in

1) Barten (1977) gives a useful survey.
2) I.e., we do not pursue the suggestions made in Johansen (1981) to combine different functional forms within the same demand system.
3) See e.g. Diewert (1974, 1982), Lau (1977), and Jansen (1978).
4) The class of utility functions which imply linear expenditure functions has been formally examined by Gorman (1961), Pollak (1971), and Somermeyer (1974).

favour of more general specifications.[5)] A first order Taylor expansion of the underlying 'true' demand functions about the sample mean is empirically inadequate when observation units with widely different income levels are represented in the data set. We should search for a more flexible parametrization.

An examination of the literature reveals, however, that the class of complete demand systems which both satisfy the theoretical constraints and admit non-linear Engel curves is very limited. Three of its members are the Indirect Translog System (Christensen, Jorgenson, and Lau (1975)), Carlevaro's generalization of the Stone Linear Expenditure System (LES) (Carlevaro (1976)), and the Quadratic Expenditure System (QES) (Pollak and Wales (1978, 1980)), which is another generalization of the LES. A fourth member is the AIDS (i.e. Almost Ideal Demand Systems) - a class of demand systems proposed recently by Deaton and Muellbauer (1980a), (1980b, section 3.4). The AIDS, however, gives, in spite of its name, a rather rigid representation of the income response: when all prices are constant, the demand functions expressed as budget shares are simply linear functions of the logarithm of total expenditure.

From an econometric point of view, it is a common feature of these four models that the number of parameters to be estimated is considerable, even with a moderate number of commodity groups involved. Furthermore, the majority of parameters are introduced to give a refined representation of the price response. This, of course, may place heavy claims on the variability of the relative prices if serious problems of collinearity are to be avoided. For these reasons, we have found neither of the above mentioned models particularly attractive as a basis for the present analysis.

We have chosen a specification based on a model proposed by Fourgeaud and Nataf more than 20 years ago (Fourgeaud and Nataf (1959)). This model both satisfies the general restrictions of the consumer demand theory and admits non-linear Engel curves, but is, nevertheless, considerably more parsimonious in terms of the number of free parameters than any of its four 'competitors' mentioned above. In spite of this, it has been largely neglected in empirical work; to our knowledge, only one previous econometric analysis of the Fourgeaud-Nataf model exists (Nasse (1973)).

5) An analysis of Norwegian household data based on a complete system of demand functions specified as cubics in income is presented in Biørn (1978). This model belongs to the third, 'theoretically inadmissible' class of models mentioned above.

The main limitation of this model seems to be that it gives a fairly restrictive parametrization of the price response. As Anton Barten says in his 1977 survey article,

> "The strong point of the Fourgeaud-Nataf approach is the flexibility it allows for introducing the influence of total expenditure ....., its weak point being the constraint imposed on the price effects." (Barten (1977, p. 42).)

Angus Deaton also expresses some scepticism towards the applicability of the Fourgeaud-Nataf model to estimate the price responses:

> "..... in spite of [its] apparent generality, the income and price elasticities of the Fourgeaud and Nataf model, like those derived from direct and indirect additivity, lie approximately along a straight line. Consequently such a model cannot be used for the independent measurement of income and own-price elasticities." (Deaton (1975b, p. 263).)[6]

The meaning of the term 'independent measurement' in the last sentence is not, however, clear to us: When estimating price and income elasticities within the framework of a *complete* system of demand equations restricted by utility maximizing behaviour, the estimates will, of course, *always* be statistically dependent. This holds irrespective of the flexibility of the functional form used; the difference between the models in this respect is only one of degree.

It should be noted, however, that the Fourgeaud-Nataf model gives a less restrictive modelization of the price response than for instance the LES model (Stone (1954)), still one of the most popular demand models in the econometric literature. We shall elaborate this point in section 2.3 below.

The main features of the Fourgeaud-Nataf model are presented in section 2.2. In section 2.3, we discuss its most important implications and attempt to state its relation to some other complete systems of consumer demand functions. Section 2.4 is concerned with its implied demand elasticities.

2.2. The Fourgeaud-Nataf model: General properties

In their article, Fourgeaud and Nataf posed the following interesting problem: Let x_i be the volume of demand for the i'th commodity, p_i its price $(i=1,\dots,N)$, y the total value of expenditure,

---

6) Deaton also remarks that the Fourgeaud-Nataf model, as well as models based on direct or indirect additivity, are special cases of Pollak's concept 'generalized additive separability' (Pollak (1972)).

$$(2.1) \qquad y = \sum_{i=1}^{N} p_i x_i ,$$

and $P = P(p_1,\dots,p_N)$ a price index function homogeneous of the first degree. Restrict the demand function for each commodity to be a *function of two variables only*, its *'real price'* p_i/P and *'total real expenditure'* y/P, i.e. let

$$(2.2) \qquad x_i = F_i \left(\frac{y}{P}, \frac{p_i}{P}\right) \qquad (i=1,2,\dots,N).$$

Which restrictions should be imposed on the N+1 functions $F_1,\dots,F_N$ and P to make this specification compatible with the constraints implied by static utility maximizing behaviour?

From this starting-point, Fourgeaud and Nataf were led to study a class of *demand functions* which can be written in the following form:[7)]

$$(2.3) \qquad x_i = t_i \left(\frac{p_i}{P}\right)^{\beta-1} C\left(\frac{y}{P}\right) + \frac{s_i}{p_i/P}\left[\frac{y}{P} - C\left(\frac{y}{P}\right)\right] \qquad (i=1,\dots,N),$$

with the corresponding *price index function*

$$(2.4) \qquad P = \left(\sum_{j=1}^{N} t_j p_j^{\beta}\right)^{\frac{1}{\beta}} ,$$

where t_i, s_i, and β are constants and $C(y/P)$ is a (so far) unspecified function. We observe that C is the crucial element in the characterization of the income response in the Fourgeaud-Nataf model. All specifications in which C is non-linear in y/P, imply non-linear Engel curves, and conversely: linearity of C also implies linearity of the Engel curves.

For this model to be consistent with utility maximization, $C(y/P)$ should obey certain regularity conditions (see below), and the coefficients s_i should add to unity:[8)]

$$(2.5) \qquad \sum_{i=1}^{N} s_i = 1.$$

Moreover, as is apparent from (2.3) and (2.4), the coefficients t_i can, without loss of generality, be normalized to add to unity:

---

7) To be precise, (2.3)-(2.4) delimit the *main* class of models examined in Fourgeaud-Nataf's article.
8) See Fourgeaud and Nataf (1959) and Nasse (1973). Our function C is equivalent to the function (-K) in Fourgeaud-Nataf's article and in the first part of Nasse's article.

$$(2.6) \qquad \sum_{i=1}^{N} t_i = 1.$$

Or stated in econometric terms: this normalization is required to make all the coefficients of the model identifiable.

The parametric form of the (direct) utility function corresponding to the Fourgeaud-Nataf model has not yet, in general, been established. A class of *indirect utility functions* which generate demand functions of the form (2.3)-(2.4) is, however, known. According to Fourgeaud and Nataf (1959, p. 346) and Nasse (1973, p. 1140), it has the form

$$(2.7) \qquad V^*(y,p_1,\ldots,p_N) \equiv W^*(u,v_1,\ldots,v_N) = \Phi(u) \prod_{i=1}^{N} v_i^{-s_i},$$

where u is 'total real expenditure',

$$(2.8) \qquad u = \frac{y}{P},$$

v_i is the 'real price' of commodity i,

$$(2.9) \qquad v_i = \frac{p_i}{P} \qquad (i=1,\ldots,N),$$

and Φ is a function implicitly defined in terms of C as follows:

$$(2.10) \qquad \frac{d \log \phi(u)}{du} \equiv \frac{\Phi'(u)}{\Phi(u)} = \frac{1}{u-C(u)}.$$

Considering utility as ordinally measurable, we can, however, take any positive monotonic transformation of V* (or W*) as an equally valid representation of the consumers' preferences. Let us, in particular, consider the *logarithmic* transformation, i.e. $V = \log V^*$ ($W = \log W^*$), which leads to the indirect utility indicator

$$(2.11) \qquad V(y,p_1,\ldots,p_N) \equiv W(u,v_1,\ldots,v_N) = \psi(u) - \sum_{i=1}^{N} s_i \log v_i,$$

where $\psi(u) = \log \Phi(u)$, i.e.

$$(2.12) \qquad \psi'(u) = \frac{1}{u-C(u)}.$$

Thus, the Fourgeaud-Nataf model admits an indirect utility function which is *additive* in real prices and total real expenditure, in the sense that the class of indirect utility functions, which is consistent with the model contains (at least) one member of this form.

It is easy to verify that the demand system (2.3)-(2.4) is compatible with the indirect utility function (2.11)-(2.12). According to Roy's identity, written in absolute form, we have[9)]

$$(2.13)\quad x_i = -\frac{\partial V/\partial p_i}{\partial V/\partial y} \qquad (i=1,\ldots,N).$$

From (2.11) and (2.12), while making use of (2.5), (2.6), (2.8) and (2.9), we find

$$(2.14)\quad \frac{\partial V}{\partial y} = \frac{1}{P\{u-C(u)\}},$$

$$(2.15)\quad \frac{\partial V}{\partial p_i} = -\frac{1}{P}\left\{\frac{s_i}{v_i} + \frac{C(u)}{u-C(u)} \cdot t_i v_i^{\beta-1}\right\} \qquad (i=1,\ldots,N).$$

The demand equations (2.3) follow straightly by inserting (2.14) and (2.15) in (2.13).

A basic implication of utility maximizing behaviour is that an increase in total expenditure (income), or a decrease in one of the prices will always lead to an increase in maximal obtainable utility. From (2.14) we see that positive marginal utility of (real or nominal) income implies that the following inequality should be satisfied:

$$(2.16)\quad C(u) < u \qquad \text{for all } u.$$

Likewise, negative marginal utility of the i'th *real* price ($\partial W/\partial v_i<0$) implies

$$(2.17)\quad s_i > 0 \qquad (i=1,\ldots,N).$$

We note from (2.15) that these two inequalities also ensure negative marginal utility of the i'th *nominal* price ($\partial V/\partial p_i<0$) if $C(u)\geqq 0$ and $t_i \geqq 0$. The above conclusions are valid irrespective of which monotonic transformation of (2.7) we take to represent the consumers' preferences; they are not confined to the additive form (2.11).

9) See Roy (1943). Lau (1969, p.375) claims to be the first to write the identity in this form, which is *not* correct as it can be found in e.g. Houthakker (1960, p.250); confer Jansen (1978, p.11).

Implicit in the above discussion is the interpretation of P as an index of the 'level' of the consumer prices. We are not, however, forced to adopt this interpretation. *Formally*, (2.4) is a linear homogeneous function of the prices belonging to the constant elasticity of substitution (CES) family. We can thus characterize the behaviour of the P function - and hence the model's representation of price responses - by referring to well-established properties of CES functions.[10] From (2.4) and (2.8) it follows that the elasticity of P with respect to the j'th price is equal to

$$(2.18)\quad \pi_j = \frac{\partial P}{\partial p_j}\frac{p_j}{P} = t_j v_j^{\beta}.$$

If we stick to the interpretation of P as a price index function, it is reasonable to require that this elasticity be non-negative, which implies

$$(2.19)\quad t_j \geqq 0 \qquad\qquad (j=1,\ldots,N).$$

Then, t_j in fact represents the *weight* assigned to the j'th price in the index function in its 'basic year' (i.e. when $p_1=\ldots=p_N=P=1$). If we relax this interpretation, however, it is not obvious that non-negativity of all t_j's should be imposed as an *a priori* constraint.

In this study, we have chosen to base our econometric specification of the model not on the demand functions written in the quantity form, (2.3), but on the corresponding functions expressed as budget shares. The reasons for this choice are set out in section 3.2 and further elaborated in appendix B. Let $w_i=p_ix_i/y$ be the budget share of commodity i in *value* terms (the corresponding volume budget share may be defined as $x_i/u = w_iP/p_i$). From (2.3), (2.8), and (2.9) it follows that the *budget share demand function* of commodity i can be expressed in terms of the total real expenditure u and the real price v_i as follows:

10) If $\beta>1$, (2.4) is formally equivalent to a single input/multiple output CES function (P and p_j being analogous to the input and the j'th output, respectively), with concave production possibility surfaces and with an overall elasticity of substitution on the output side equal to $1/(\beta-1)$. Conversely, if $\beta<1$, it is formally equivalent to a multiple input/single output CES function (p_j and P being analogous to the j'th input and the output, respectively), with convex indifference surfaces and with an overall elasticity of substitution on the input side equal to $1/(1-\beta)$. The limiting case $\beta=1$ characterizes "perfect substitutability", i.e. a linear P function, whereas $\beta\to 0$ implies a log-linear (Cobb-Douglas) P function. Finally, when $\beta\to\infty$, the function degenerates to $P=\max_i(p_i)$, while $\beta\to-\infty$ leads to $P=\min_i(p_i)$. For a further discussion, see e.g. Hasenkamp (1978).

$$(2.20)\quad w_i = \frac{p_i x_i}{y} = \frac{v_i x_i}{u} = s_i + (t_i v_i^{\beta} - s_i)\,\frac{C(u)}{u} \qquad (i=1,\ldots,N).$$

This formulation of the model will be the basis of the following analysis.

2.3. Implications of the Fourgeaud-Nataf model. Its relationship to other demand systems

The Fourgeaud-Nataf specification (2.3)-(2.4) contains several interesting demand systems as special cases, depending on the particular form chosen for the function $C(u)$ and the value assigned to the coefficient β. Let us look briefly at three members of this family, one of which will be the basis for the present study.

CLASS I: $C(u) = Au + B$.

As noted above, this parametrization results in demand functions which are linear in total (real or nominal) expenditure. Moreover, since $\psi'(u) = 1/(u-Au-B)$ implies $\psi(u) = \log\{(1-A)u - B\}/(1-A)$, the indirect utility function, defined by (2.11) and (2.12), can in this case be written as an explicit function of u and v_i:

$$W(u,v_1,\ldots,v_N) = \frac{1}{1-A}\log\{(1-A)u-B\} - \sum_{i=1}^{N} s_i \log v_i .$$

In this class of models, let us in particular consider the cases A=0 and B=0.

$A=0$, $u>B$.

In this case, C is a constant, and the demand functions (2.3) take the form

$$(2.21)\quad x_i = Bt_i\left(\frac{p_i}{P}\right)^{\beta-1} + \frac{s_i}{p_i}\left[y-B\left(\sum_{j=1}^{N} t_j p_j^{\beta}\right)^{\frac{1}{\beta}}\right] \qquad (i=1,\ldots,N).$$

If moreover $\beta=1$, the model degenerates to the *Linear Expenditure System* (LES) of Stone (1954): Bt_i corresponds to the "minimum consumption" of the i'th commodity, and $y-B(\Sigma_j t_j p_j) = y-BP$ represents the "supernumerary income".[11)]

---

11) This interpretation of the Fourgeaud-Nataf model has also been noted by Johansen (1969, p.37) and Nasse (1973, p.1 144).

B=0, A<1.

In this case, C is proportional to u, and the budget share demand functions (2.20) degenerate to

$$(2.22)\quad w_i = s_i + (t_i v_i^{\beta} - s_i)A \qquad (i=1,\dots,N),$$

i.e. all budget shares are independent of total expenditure. If moreover $\beta=0$, the budget shares are also price independent; they are constants equal to $(1-A)s_i + At_i$ $(i=1,\dots,N)$, i.e. a weighted average of s_i and t_i. This confirms that the s and t coefficients in the Fourgeaud-Nataf model have a 'budget share dimension'.

If A converges towards 1 from below (recall the restriction (2.16)), we get the following budget share function at the limit:[12)]

$$w_i = t_i v_i^{\beta} = \frac{t_i p_i^{\beta}}{\Sigma_j t_j p_j^{\beta}} \qquad (i=1,\dots,N).$$

Also in this case, the budget shares are income independent.

CLASS II: C(u) = a log u + b.

This is the specification of the C function applied by Nasse (1973) in his empirical study of the Fourgeaud-Nataf model. The resulting demand functions are mixed linear and log-linear functions of total expenditure. This parametrization thus gives a more flexible representation of the income response than the models in class I, with the same number of structural parameters. The resulting budget share demand functions are

$$(2.23)\quad w_i = s_i + (t_i v_i^{\beta} - s_i)\left(a\frac{\log u}{u} + b\frac{1}{u}\right) \qquad (i=1,\dots,N).$$

Since $\lim_{u\to\infty} \log u/u = 0$, this specification of the C function implies that

$$\lim_{u\to\infty} w_i = s_i,$$

i.e. s_i may be interpreted as the asymptotic budget share of commodity i in this case. This holds regardless of the value of a, b, β and t_i. For

12) Demand functions of this form have been discussed by Samuelson (1965, section 6), who calls them 'the Bergson demand functions'. See also Johansen (1969, p.39).

the models in class I, the asymptotic budget shares will not be price independent unless either A=0 or β=0.

CLASS III: $C(u) = \bar{a}\, u \log u + \bar{b}\, u$.

With this parametrization of the C function, we obtain budget share functions which are linear in the logarithm of real expenditure:

$$(2.24)\quad w_i = s_i + (t_i v_i^{\beta} - s_i)(\bar{a} \log u + \bar{b}) \qquad (i=1,\ldots,N).$$

The model thus exhibits the same form of income response as that proposed by Working (1943) and later also used by Leser (1963). Contrary to the models in classes I and II, asymptotic budget shares do not in general exist for any commodity in this case. Only the particular members of class III models in which $\beta=0$ and either $\bar{a}=0$ or $s_i=t_i$ will have this property, and then the budget shares are constants equal to $\bar{b}t_i + (1-\bar{b})s_i$ for *all* constellations of income and prices.

The limiting case $\beta=0$ (with $t_i \neq s_i$) is of particular interest. When $\beta \to 0$, the price index function degenerates to the log-linear index [13)]

$$\log P = \sum_{j=1}^{N} t_j \log p_j .$$

Inserting this into (2.24) and arranging terms, we get

$$(2.25)\quad w_i = \bar{b}t_i + (1-\bar{b})s_i + \bar{a}(t_i - s_i)\{\log y - \Sigma_j t_j \log p_j\} \qquad (i=1,\ldots,N),$$

i.e. all budget shares are log-linear functions not only of total expenditure, but also of prices. This model is a member of the AIDS class of demand systems recently proposed by Deaton and Muellbauer (1980a). The general expression for the AIDS functions, written in budget share form, is[14)]

$$(2.26)\quad w_i = \alpha_i - \alpha_o\beta_i + \sum_j \gamma_{ij} \log p_j + \beta_i\{\log y - \sum_k \alpha_k \log p_k - \tfrac{1}{2}\sum_{jk}\sum \gamma_{kj} \log p_k \cdot \log p_j\}$$

$$(i=1,\ldots,N),$$

13) Confer footnote 10 above.
14) See Deaton and Muellbauer (1980a, eqs. (8) and (9)). They present the AIDS model as a generalization of the logarithmic budget shares model of Working-Leser, referred to above.

with the following restrictions:

$$\sum_i \alpha_i = 1, \quad \sum_i \beta_i = \sum_i \gamma_{ij} = 0, \quad \gamma_{ji} = \gamma_{ij} \quad \text{for all } i \text{ and } j.$$

One observes that (2.25) is equivalent to (2.26) when $\gamma_{ij} = 0$ for all i and j. Thus, the Fourgeaud-Nataf class and the AIDS class contain the log-linear budget share functions (2.25) as a common member.

A more general parametrization which comprises class I, II, and III as special cases would be

$$C(u) = a_1 u + a_2 \log u + a_3 u \log u + b,$$

class I corresponding to the restrictions $a_2 = a_3 = 0$, class II corresponding to $a_1 = a_3 = 0$, and class III corresponding to $a_2 = b = 0$. This is thus a flexible and econometrically interesting parametrization of the C(u) function that might be well worth empirical investigation. In this study, we shall, however, follow Nasse (1973) and confine attention to *class II*. The resulting budget share functions, (2.23), i.e.

$$w_i = s_i + (t_i v_i^{\beta} - s_i)\left(a\frac{\log u}{u} + b\frac{1}{u}\right) \qquad (i=1,\ldots,N),$$

both admit non-linear Engel curves - including linearity as a special case - and imply a 'saturation effect' in the expenditure pattern, since the coefficients s_i can be interpreted as asymptotic budget shares. Still, it is rather parsimonious in terms of the number of free parameters, as the demand system contains only 2N+1 independent coefficients[15]: N-1 s coefficients, N-1 t coefficients, plus β, a, and b.

2.4. The demand elasticities

The essential properties of any system of demand functions are reflected in the kind of restrictions it imposes on the price and income elasticities. We terminate this presentation of our structural model by briefly referring the expressions for the most important of these elasticities. The Engel elasticity of commodity i is

15) When no background variables are included. See section 3.1.

$$(2.27)\quad E_i = \frac{s_i u + (t_i v_i^{\beta} - s_i)a}{s_i u + (t_i v_i^{\beta} - s_i)(a \log u + b)} \qquad (i=1,\ldots,N).$$

The *Cournot and Slutsky elasticities* of commodity i with respect to the price of commodity j are, respectively

$$(2.28)\quad e_{ij} = S^{*}_{ii}\,(\delta_{ij} - \pi_j) - E_i \pi_j,$$

$$(2.29)\quad S_{ij} = S^{*}_{ii}\,(\delta_{ij} - \pi_j) + E_i (w_j - \pi_j) \qquad (i=1,\ldots,N;j=1,\ldots,N),$$

where π_j is defined in (2.18), $\delta_{ij}=1$ for j=i and 0 otherwise, and

$$(2.30)\quad S^{*}_{ii} = \frac{(\beta-1)t_i v_i^{\beta}(a \log u+b) - s_i\,(u-a \log u-b)}{s_i u + (t_i v_i^{\beta} - s_i)(a \log u+b)} \qquad (i=1,\ldots,N).$$

These expressions are derived and commented upon in appendix A (see in particular eqs. (A.10), (A.11), (A.14) and (A.15)).

The *income flexibility* (or more precisely, the flexibility of the marginal utility of income) - or its inverse - often called 'the Frisch parameter'[16) is another elasticity which has been frequently used to characterize a complete system of demand functions, inspired *inter alia* by the famous work of Frisch in this area (Frisch (1932, 1959)).[17) Frisch in fact proposed its use as an individual welfare indicator (Frisch (1959, p.189)).

We show in appendix A that the expression for the income flexibility corresponding to the demand system (2.23) and the indirect utility indicator (2.11) is simply

$$(2.31)\quad \omega = -\frac{u[1-C'(u)]}{u-C(u)} = -\frac{u-a}{u-a \log u-b}.$$

As Frisch showed, the income flexibility, as well as the marginal utility of income from which it is derived, is *not an ordinal concept* - i.e. it is not generally invariant with respect to the particular choice of utility

16) See e.g. Lluch and Powell (1975, p.278). Cf. also Goldberger (1967, pp.54-56) and Goldberger and Gamaletsos (1970, p.396-397).
17) Frisch called it the 'money flexibility' (Frisch (1932, p.15)), whereas Houthakker (1960, p.248)) suggested the better term 'income flexibility'. Houthakker, however, defined the income flexibility as the *inverse* of the Frisch concept money flexibility. In this study, we shall use 'income flexibility' as synonymous with 'money flexibility'.

indicator we use to represent the consumer's preferences.[18] So the interpretation of (2.31) as the income flexibility depends in a crucial way on our additive representation of the utility indicator, (2.11). With this reservation, we note that the income flexibility in the Fourgeaud-Nataf model is a function of the real income u, but is independent of the real prices. This follows from the additivity of the utility indicator. The Engel, Slutsky, and Cournot elasticities, on the other hand, are functions of both the real income u and the real price(s) of the commodity (commodities) to which the elasticity refer.[19] All elasticities are, however, via eqs. (2.4), (2.8), and (2.9), functions of the *nominal* income y and *all* the nominal prices p_i.

18) It is, however, invariant with respect to a positive *linear* transformation of the utility indicator (Frisch (1937, p.377)).
19) Cf. eqs. (2.27)-(2.30).

III. ECONOMETRIC SPECIFICATION OF THE MODEL

3.1. Background variables

The model outlined in chapter II implies that all the coefficients of the demand system, which characterize the underlying preferences - i.e. s_i, t_i, a, b, and β - have the same values for all observation units. Since our data set covers households with widely different demographic characteristics[1], this is a restrictive hypothesis; systematic differences in preferences according to 'type of household' should be allowed for. We could solve this problem in two ways: (i) we could split our data into homogeneous subsamples to be analyzed separately, or (ii) we could postulate that some of the coefficients depend parametrically on observed characteristics of the household and treat the complete data set as one single body.

In this study, we choose the latter approach, and to preserve a parsimonious parametrization, we let type of household be indicated by two quantitative variables only: the *number of persons in the household, n*, and the *age of its head, A*. This is, of course, a rather rough and approximative way of representing the multitude of demographic characteristics which may affect the composition of consumption. The number of children and their ages and the number of aged persons are, for example, not specified. However, in practice, one will often find that when the age of the head person (main income earner) of a family and the number of family members are given, then the number of children and their ages may be 'predicted' with a reasonable degree of accuracy, i.e. n and A may be said to represent a *complex* of background variables. And restricting ourselves to choosing a common set of background variables for all commodity groups, it is hard to find an equally parsimonious selection which gives a better overall representation of these factors.

We assume that the coefficients t_i and β have the same values for all households. This implies that we do not allow for household specific differences in the price index function (2.4). Or stated otherwise: all households which report in the same period are assumed to face the same level of consumer prices, P, and the same set of real prices, $v_i = p_i/P$. The coefficients s_i - which may be interpreted as asymptotic budget shares, cf.section 2.3 - and the coefficients a and b - i.e. the coefficients in the C(u) function - are assumed to depend *linearly* on n and A:[2]

1) Confer Biørn and Jansen (1980, section 2.1) and chapter V below.
2) Pollak and Wales (1978, 1980, 1981), who in a series of papers have considered alternative ways of incorporating demographic variables into complete systems of consumer demand functions, call this approach 'linear demographic translating'.

$$
\begin{aligned}
& s_i = s_{io} + s_{in}\, n + s_{iA}\, A \qquad\qquad (i=1,\ldots,N),\\
(3.1)\quad & a = a_o + a_n\, n + a_A\, A,\\
& b = b_o + b_n\, n + b_A\, A.
\end{aligned}
$$

This parametrization increases the total number of independent coefficients in the model from 2N + 1 to 4N + 3, viz. 3(N-1) s coefficients, N-1 t coefficients (recall the adding-up conditions (2.5) and (2.6)), 3 a coefficients, 3 b coefficients, and β.

3.2. Structure of disturbances

Even when the number of persons in the household (n) and the age of its head (A) are included as structural explanatory variables there may be substantial individual factors left. These represent *individual differences* in tastes, habits, expectations, experiences, and other unobservable (or unobserved) variables affecting the composition of consumption. If these variables are omitted, their effect will, of course, be "thrown into" into disturbances. In this section, we discuss the specification of the disturbances adopted in this study to take care of such differences, while paying regard to the particular structure of our combined cross-section/time-series data.

There exist, in principle, two alternative strategies for incorporating unobserved individual factors in a regression equation. The first is to define a set of *binary (dummy) variables*, one for each individual, to take care of individual shifts in some of the coefficients of the equation - usually its constant term. The second strategy is to decompose the disturbance term of the equation into independent additive components, one of which representing the individual differences. The salient feature of this *disturbance component (error component) specification* is that the individual components are considered as stochastic variables generated by a common distribution - or weaker, that they have the same expectation and variance.[3] The specification with binary variables, on the other hand, treats the individual differences as non-stochastic.[4]

---

3) The 'random coefficients model', in which the structural coefficients are also considered as stochastic variables, is another possible generalization. See e.g. Swamy (1970, 1974), Kelejian (1974), and Wansbeek and Kapteyn (1981a).

4) Formally, the disturbance component specification is equivalent to a specification with dummy shifts in the constant term and with the coefficients of the dummy variables considered as stochastic variables generated by the same distribution. For a further elaboration of the similarities and differences between the two approaches, see Wallace and Hussain (1969), Maddala (1971), and Mazodier (1971).

A second significant difference between the two approaches, which is closely related to the first, is the difference in the number of unknown parameters to be estimated. Since the binary variables specification occupies one new parameter for each individual included, it will usually involve a substantial loss of degrees of freedom. In our case, with only two observations of each individual, this loss would be almost prohibitive since the number of unknown parameters would be more than half the total number of observation points. The disturbance component specification, on the other hand, only introduces the second order moments of the disturbance components as additional parameters. This is our main reason for preferring this specification in the present case. A secondary argument is that the disturbance component specification gives relevant information for several interesting applications of the results. (Confer the remarks in chapter I.)

Assume that our data set consists of reports from M different households. Each household reports twice, so that the total number of observations is 2M. Define

ε_{iht}: Disturbance in the budget share function (2.20) for the i'th commodity, household h, report t $(i=1,\dots,N;\ h=1,\dots,M;\ t=1,2)$,

with the following decomposition:

$$(3.2) \qquad \varepsilon_{iht} = \mu_{ih} + \nu_{iht} \qquad \begin{pmatrix} i=1,\dots,N \\ h=1,\dots,M \\ t=1,2 \end{pmatrix},$$

where μ_{ih} is the component in the disturbance of commodity i which is specific to household h and ν_{iht} is the remainder. We disregard, for analytical simplicity[5], time specific effects, which are also frequently included in error components models for *complete* cross-section/time-series data. (See e.g. Avery (1977).) Although the three component specification is attractive for theoretical (and aesthetical) reasons, not least because of its symmetry, time specific effects are, arguably, in practice less important than the individual ones. This is in particular the

5) Some aspects of the problems which arise when including time specific components in models for *incomplete* CS/TS data are discussed in Biørn (1981a, section 4) for the single equation case and in Biørn (1981b, section 4.1) for a seemingly unrelated regressions model. Wansbeek and Kapteyn (1981b) have recently outlined a procedure for Maximum Likelihood estimation of a three component single equation model from rotating panels, which generalizes a similar procedure for the two component specification considered in Biørn (1981a). Their formulae, however, turn out to be rather complicated.

case when the number of individuals is large and the time span covered by the data is short. It thus does not seem to be worth the extra effort to try to identify possible time specific effects from our data, which cover only three years. So we will include such effects in the remainder component, ν_{iht}.

We assume that the two components are independently distributed, with zero expectations,

$$(3.3) \qquad E(\mu_{ih}) = E(\nu_{iht}) = 0 \qquad \begin{pmatrix} i=1,\dots,N \\ h=1,\dots,M \\ t=1,2 \end{pmatrix},$$

and that their second order moments satisfy

$$(3.4) \qquad \begin{aligned} &(a)\ E(\mu_{ih}\ \mu_{jk}) = \delta_{hk}\ \sigma^{\mu}_{ij}, \\ &(b)\ E(\nu_{iht}\ \nu_{jks}) = \delta_{hk}\ \delta_{ts}\ \sigma^{\nu}_{ij}, \\ &(c)\ E(\mu_{ih}\ \nu_{jks}) = 0 \end{aligned} \qquad \begin{pmatrix} i,j=1,\dots,N \\ h,k=1,\dots,M \\ t,s=1,2 \end{pmatrix},$$

where δ_{hk} and δ_{ts} are Kronecker deltas (δ_{hk}=1 for k=h, and 0 for k$\neq$h; δ_{ts}=1 for s=t, and 0 for s$\neq$t). These assumptions imply

$$(3.5) \qquad E(\varepsilon_{iht}) = 0,$$

$$(3.6) \qquad E(\varepsilon_{iht}\ \varepsilon_{jks}) = \begin{cases} \sigma^{\mu}_{ij}+\sigma^{\nu}_{ij} & \text{for } k=h,\ s=t \\ \sigma^{\mu}_{ij} & \text{for } k=h,\ s\neq t \\ 0 & \text{otherwise} \end{cases} \qquad \begin{pmatrix} i,j=1,\dots,N \\ h,k=1,\dots,M \\ t,s=1,2 \end{pmatrix}.$$

We note that (3.6) implies that all disturbances from different households - whether relating to the same or to different commodity groups - are uncorrelated.

A basic implication of (3.6) is that the disturbance variances and covariances of the demand functions are constant when the model is written in the form of budget shares:

$$\begin{aligned} var(\varepsilon_{iht}) &= \sigma_{ii} = \sigma^{\mu}_{ii} + \sigma^{\nu}_{ii} \\ cov(\varepsilon_{iht},\varepsilon_{jht}) &= \sigma_{ij} = \sigma^{\mu}_{ij} + \sigma^{\nu}_{ij} \end{aligned} \qquad \begin{pmatrix} i,j=1,\dots,N \\ h=1,\dots,M \\ t=1,2 \end{pmatrix}.$$

The rationale for this hypothesis is thoroughly discussed in appendix B, and we shall only recapitulate the main points here. First, homoscedasticity of the disturbances in (2.20) seems *a priori* to be a more reasonable hypothesis than it would be to assume homoscedasticity of the disturbances in the ordinary quantity demand functions (2.3), since the scope for individual variations in consumption habits is probably larger the larger is the household (real) income. Second, our transformation to budget shares ensures that a proportional change of all prices and total expenditure will leave the second order moments of the disturbances in the corresponding quantity demand functions unaffected. We show in appendix B that the budget shares transformation is in fact the simplest transformation of the demand model which takes both these moments into consideration. Third, an analysis of the *marginal* distributions of the budget shares and the distributions of the corresponding expenditures, based on the skewness and the kurtosis, indicates that the former are, with a few exceptions, closer to normality than the latter.[6] A sensible conjecture would be that similar results hold for the corresponding distributions of disturbances.[7] This is a major point, since our estimation and testing procedures in chapters IV, VI, and VII rely on the assumption that the disturbances follow a normal distribution.

Since both the structural parts of the budget share functions, (2.20), and the observed budget shares add to one identically, the disturbances satisfy

$$(3.7) \qquad \sum_{i=1}^{N} \varepsilon_{iht} = 0 \qquad\qquad \begin{pmatrix} h=1,\dots,M \\ t=1,2 \end{pmatrix}.$$

We show in appendix B, paragraphs 5 and 6 (see, in particular, Proposition 3(i) and the final part of paragraph 6), that this adding-up constraint implies the following restrictions on the second order moments of the μ's and the ν's:

$$(3.8) \qquad \sum_{i=1}^{N} \sigma_{ij}^{\mu} = \sum_{i=1}^{N} \sigma_{ij}^{\nu} = 0 \qquad\qquad (j=1,\dots,N).$$

Non-singularity of the matrices (σ_{ij}^{ν}) and $(\sigma_{ij}) = (\sigma_{ij}^{\mu} + \sigma_{ij}^{\nu})$ - which is a prerequisite for Maximum Likelihood estimation to be feasible - can

6) See Biørn and Jansen (1980, subsection 3.3.2).
7) A formal analysis of distributional properties of the disturbances of stochastic equations written in the form of shares is given in Woodland (1979).

thus be assured only if we omit (at least) one commodity group from consideration.

In specifying the second order moments of the disturbances, we have followed the tradition in the literature on disturbance components models by assuming that the components of ε_{iht} are uncorrelated, cf. (3.4c). A remark on this assumption is in order.[8] Let $\sigma^{\mu\nu}_{ij}$ be the covariance between μ_{ih} and ν_{jht} and replace (3.4c) by

$$(3.4c')\quad E(\mu_{ih}\,\nu_{jks}) = \delta_{hk}\,\sigma^{\mu\nu}_{ij} \qquad \begin{pmatrix} i,j=1,\dots,N \\ h,k=1,\dots,M \\ s=1,2 \end{pmatrix}.$$

Eq. (3.6) would then change to

$$(3.6')\quad E(\varepsilon_{iht}\,\varepsilon_{jks}) = \begin{cases} \sigma^{\mu}_{ij} + \sigma^{\nu}_{ij} + \sigma^{\mu\nu}_{ij} + \sigma^{\mu\nu}_{ji} & \text{for } k=h,\ s=t \\ \sigma^{\mu}_{ij} + \sigma^{\mu\nu}_{ij} + \sigma^{\mu\nu}_{ji} & \text{for } k=h,\ s\neq t \\ 0 & \text{otherwise} \end{cases} \qquad \begin{pmatrix} i,j=1,\dots,N \\ h,k=1,\dots,M \\ t,s=1,2 \end{pmatrix}.$$

With this generalization, a problem of identification would arise: only σ^{ν}_{ij} and $\sigma^{\mu}_{ij} + \sigma^{\mu\nu}_{ij} + \sigma^{\mu\nu}_{ji}$ could be identified, but not the three components of the latter. Or stated otherwise: eqs. (3.6) and (3.6') are *observationally equivalent structures* in the Koopmans sense (Koopmans (1953, p.36)).

Still, this alternative specification has some notable implications for the interpretation of the model and its estimation results. Consider the formula for the covariance between ε_{ih1} and ε_{ih2}:

$$E(\varepsilon_{ih1}\,\varepsilon_{ih2}) = \sigma^{\mu}_{ii} + 2\sigma^{\mu\nu}_{ii}\ ,$$

which follows straightly from (3.6'). Since σ^{μ}_{ii} (= var μ_{ih}) is non-negative, the covariance between ε_{ih1} and ε_{ih2} is always non-negative when hypothesis (3.4c) is imposed, but it may be negative when the generalization (3.4c') is applied, due to the fact that $\sigma^{\mu\nu}_{ii}$ (=$E(\mu_{ih}\nu_{iht})$) may be of either sign. A negative estimate of this covariance is thus a meaningful result when $\sigma^{\mu\nu}_{ii}$ is a free parameter (which is allowed to be negative), but not when it is restricted to zero *a priori*. We shall recall this generalization of the model when interpreting the estimation results in chapters VI and VII.

---

8) The following line of argument is inspired by Berzeg (1979), who discusses a similar generalization of a single equation error components model. See also Swamy (1974, p.149).

Let us write the covariance structure in matrix notation.[9]
Define the 2 x 1 vector

$$(3.9)\qquad \varepsilon_{ih} = \begin{pmatrix} \varepsilon_{ih1} \\ \varepsilon_{ih2} \end{pmatrix},$$

containing the disturbances in the demand equation for commodity i from household h. From (3.6) we obtain

$$(3.10)\qquad E(\varepsilon_{ih}\,\varepsilon'_{jh}) = \begin{pmatrix} \sigma^{\mu}_{ij}+\sigma^{\nu}_{ij} & \sigma^{\mu}_{ij} \\ \sigma^{\mu}_{ij} & \sigma^{\mu}_{ij}+\sigma^{\nu}_{ij} \end{pmatrix} = \sigma^{\nu}_{ij}\, I_2 + \sigma^{\mu}_{ij}\, E_2 = \Sigma_{ij} \quad (i,j=1,\dots,N),$$

where (in general) I_n is the n x n identity matrix, E_n is the n x n matrix with all elements equal to one, and the last equality defines the 2 x 2 matrix Σ_{ij}. The complete 2N x 1 disturbance vector of household h,

$$(3.11)\qquad \varepsilon_h = \begin{pmatrix} \varepsilon_{1h} \\ \varepsilon_{2h} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_{Nh} \end{pmatrix},$$

thus has the covariance matrix

$$(3.12)\qquad E(\varepsilon_h\varepsilon'_h) = \Omega_* = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1N} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2N} \\ \vdots & \vdots & & \vdots \\ \Sigma_{N1} & \Sigma_{N2} & \cdots & \Sigma_{NN} \end{pmatrix} = \Sigma_{\nu} \otimes I_2 + \Sigma_{\mu} \otimes E_2 ,$$

where $\Sigma_{\nu} = (\sigma^{\nu}_{ij})$, $\Sigma_{\mu} = (\sigma^{\mu}_{ij})$, and $\otimes$ is the Kronecker product operator.[10]
In view of (3.6), the M vectors ε_h are all mutually uncorrelated, i.e.

9) The following is a special case of the situation discussed in Biørn (1981, section 3.2).
10) Confer appendix F for a formal definition and a list of some important properties of Kronecker products which are used in this study.

$E(\varepsilon_h \varepsilon_k') = 0_{2N,2N}$, and thus the covariance matrix of the *complete* 2MN x 1 disturbance vector

$$(3.13) \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_M \end{pmatrix}$$

can be written compactly as

$$(3.14) \quad E(\varepsilon\varepsilon') = \Omega = I_M \otimes \Omega_* = I_M \otimes \Sigma_\nu \otimes I_2 + I_M \otimes \Sigma_\mu \otimes E_2 .$$

3.3. An alternative interpretation of the disturbance component specification[11)]

In the literature on disturbance components models, one may find authors who prefer an autoregressive specification of the form

$$\varepsilon_{iht} = \phi_i \, \varepsilon_{ih,t-1} + \tau_{iht} \qquad (0 < \phi_i < 1),$$

or, more generally

$$\varepsilon_{iht} = \sum_{j=1}^{N} \phi_{ij} \, \varepsilon_{jh,t-1} + \tau_{iht},$$

where the ϕ's are unknown parameters and

$$E(\tau_{iht}) = 0,$$

$$E(\tau_{iht} \, \tau_{jks}) = \delta_{hk} \delta_{ts} \sigma_{ij}^{\tau} ,$$

to the additive two component specification (3.2)-(3.4) considered above.[12)] The reason for this may be the notion that the individual effects are not time invariant constants,[13)] but that the underlying factors - habits, tastes, etc. - fade gradually away as time passes.

11) This section may be skipped without loss of continuity.
12) This specification is considered by e.g. Anderson and Hsiao (1981), in the context of a single equation model. Confer also Parks (1967).
13) Recall that conditional on the h'th individual, μ_{ih} is a constant: for each individual only one 'random drawing' is made from the underlying distribution of the individual components.

At first glance, these two hypotheses may seem essentially different. However, when each individual is under observation only twice, as is the case in our data set, they are in fact two alternative ways of formalizing the same basic hypothesis. In this section, we explore this equivalence in the *single equation* case.

Formally, our problem then is the following: Consider the *autoregressive specification*

$$(3.15)\quad \varepsilon_{ht} = \phi\varepsilon_{ht-1} + \tau_{ht},$$

$$(3.16)\quad E(\tau_{ht}) = 0,$$

$$(3.17)\quad E(\tau_{ht}\,\tau_{ks}) = \begin{cases} \sigma_\tau^2 & \text{for } k=h, s=t \\ 0 & \text{otherwise,} \end{cases} \qquad \begin{pmatrix} h,k=1,\ldots,M \\ t,s=1,2 \end{pmatrix}.$$

How can this be transformed to a *disturbance component specification* of the form

$$(3.18)\quad \varepsilon_{ht} = \mu_h + \nu_{ht},$$

$$(3.19)\quad E(\mu_h) = E(\nu_{ht}) = 0,$$

$$E(\mu_h\mu_k) = \begin{cases} \sigma_\mu^2 = \rho\sigma^2 & \text{for } k=h \\ 0 & \text{for } k\neq h \end{cases}$$

$$(3.20)\quad E(\nu_{ht}\nu_{ks}) = \begin{cases} \sigma_\nu^2 = (1-\rho)\sigma^2 & \text{for } k=h, s=t \\ \text{otherwise} & \end{cases}$$

$$E(\mu_h\nu_{ks}) = 0, \qquad \begin{pmatrix} h,k=1,\ldots,M \\ t,s=1,2 \end{pmatrix},$$

where $\sigma^2 = \text{var}\ \varepsilon_{ht} = \sigma_\mu^2 + \sigma_\nu^2$ and $\rho = \sigma_\mu^2/\sigma^2$?

We first note that for the two specifications to be equivalent we must have

$$(3.21)\quad \phi = \rho = \frac{\sigma_\mu^2}{\sigma^2}.$$

This follows from the fact that (3.15)-(3.17) imply $E(\varepsilon_{h1}\,\varepsilon_{h2}) = \phi\sigma^2$, whereas (3.18)-(3.20) imply $E(\varepsilon_{h1}\,\varepsilon_{h2}) = E(\mu_h^2) = \rho\sigma^2$.

From (3.15) and (3.21) it follows that

$$(3.22)\quad \begin{aligned} \varepsilon_{h1} &= \rho\ \varepsilon_{h0} + \tau_{h1}, \\ \varepsilon_{h2} &= \rho^2 \varepsilon_{h0} + \rho\tau_{h1} + \tau_{h2}. \end{aligned}$$

Assume that the process (3.15)-(3.17) has been effective in all the previous periods, i.e. for t=0,-1,-2,.... . Then,

$$(3.23)\quad \varepsilon_{h0} = \sum_{s=0}^{\infty} \phi^s \tau_{h,-s} = \sum_{s=0}^{\infty} \rho^s \tau_{h,-s},$$

and hence

$$(3.24)\quad \mathrm{var}\ \varepsilon_{h0} = \sigma^2 = \sum_{s=0}^{\infty} \rho^{2s} \sigma_\tau^2 = \frac{\sigma_\tau^2}{1-\rho^2}.$$

Eqs. (3.17) and (3.23) show that ε_{h0}, τ_{h1} and τ_{h2} are uncorrelated. This is important for our problem since it implies that (3.22) expresses ε_{h1} and ε_{h2} as linear combinations of three mutually uncorrelated components, with weights equal to $(\rho,1,0)$ and $(\rho^2,\rho,1)$, respectively.

Define now μ_h, the individual component in the disturbance component specification, as a linear[14)] combination of ε_{h0}, τ_{h1}, and τ_{h2}:

$$(3.25)\quad \mu_h = B\ \varepsilon_{h0} + C_1\ \tau_{h1} + C_2\ \tau_{h2}.$$

Combining this with (3.18) and (3.22), it follows that

$$(3.26)\quad \begin{aligned} \nu_{h1} &= (\rho-B)\varepsilon_{h0} + (1-C_1)\tau_{h1} - C_2\tau_{h2}, \\ \nu_{h2} &= (\rho^2-B)\varepsilon_{h0} + (\rho-C_1)\tau_{h1} + (1-C_2)\tau_{h2}. \end{aligned}$$

We determine the unknown coefficients B, C_1, and C_2 by utilizing the restrictions that μ_h, ν_{h1}, and ν_{h2} are mutually uncorrelated (cf. (3.20)) and paying regard to the relation $\sigma_\tau^2 = \sigma^2(1-\rho^2)$ (cf. (3.24)). This gives

---

14) Since ε_{h0}, τ_{h1}, and τ_{h2} are uncorrelated and have zero expectations and positive variances, μ_h cannot have zero expectation (cf. (3.19)) unless this transformation is linear.

$$(\rho-B)B + \{C_1(1-C_1) - C_2^2\}(1-\rho^2) = 0,$$

$$(\rho^2-B)B + \{C_1(\rho-C_1) + C_2(1-C_2)\}(1-\rho^2) = 0,$$

$$(\rho-B)(\rho^2-B) + \{(1-C_1)(\rho-C_1) - C_2(1-C_2)\}(1-\rho^2) = 0.$$

After some algebraic manipulations, with which we shall not bore the reader, it turns out that this quadratic equation system has the following two solutions:

$$B = \rho^2 \pm (1-\rho)\sqrt{\rho}\ ,$$

$$(3.27)\quad C_1 = \frac{\rho}{1-\rho^2}(1-B) = \rho \mp \frac{\rho\sqrt{\rho}}{1+\rho}\ ,$$

$$C_2 = \frac{\rho}{1+\rho}\ .$$

Finally, substituting these expressions in (3.25)-(3.26), we find

$$\mu_h = \{\rho^2 \pm (1-\rho)\sqrt{\rho}\}\varepsilon_{h0} + \rho\{1 \mp \frac{\sqrt{\rho}}{1+\rho}\}\tau_{h1} + \frac{\rho}{1+\rho}\tau_{h2}\ ,$$

$$(3.28)\quad \nu_{h1} = (1-\rho)\{\rho \mp \sqrt{\rho}\}\varepsilon_{h0} + \{1-\rho \pm \frac{\rho\sqrt{\rho}}{1+\rho}\}\tau_{h1} - \frac{\rho}{1+\rho}\tau_{h2},$$

$$\nu_{h2} = \mp(1-\rho)\sqrt{\rho}\ \varepsilon_{h0} \pm \frac{\rho\sqrt{\rho}}{1+\rho}\tau_{h1} + \frac{1}{1+\rho}\tau_{h2}.$$

These equations define two alternative disturbance component specifications which satisfy the conditions (3.18)-(3.20). The choice is arbitrary. If, as would seem natural, we impose the additional restriction that ε_{h0} should always be assigned a non-negative weight in μ_h (i.e. $B \geq 0$) and τ_{h1} be assigned a non-negative weight in ν_{h1} (i.e. $1 - C_1 \geq 0$), then we should choose the first solution. Consider as an example ρ=0.4. The first solution defined by (3.28) is then

$$\mu_h = 0.5395\ \varepsilon_{h0} + 0.2193\ \tau_{h1} + 0.2857\ \tau_{h2},$$

$$\nu_{h1} = -0.1395\ \varepsilon_{h0} + 0.7807\ \tau_{h1} - 0.2857\ \tau_{h2},$$

$$\nu_{h2} = -0.3795\ \varepsilon_{h0} + 0.1807\ \tau_{h1} + 0.7143\ \tau_{h2}.$$

The alternative solution is

$$\mu_h = -0.2195\ \varepsilon_{h0} + 0.5807\ \tau_{h1} + 0.2857\ \tau_{h2},$$

$$\nu_{h1} = 0.6195\ \varepsilon_{h0} + 0.4193\ \tau_{h1} - 0.2857\ \tau_{h2},$$

$$\nu_{h2} = 0.3795\ \varepsilon_{h0} - 0.1807\ \tau_{h1} + 0.7143\ \tau_{h2}.$$

For $\rho=0$ we get the quite sensible result

$$\mu_h = 0,\ \nu_{h1} = \tau_{h1},\ \nu_{h2} = \tau_{h2}.$$

3.4. The complete model

We now put the pieces from sections 2.3, 3.1 and 3.2 (or more precisely, eqs. (2.23), (3.1), (3.2), (3.5), (3.6), and (3.8)) together and write the complete econometric model as follows:

$$(3.29)\quad w_{iht} = s_{io} + s_{in}\, n_{ht} + s_{iA}\, A_{ht} + (t_i\, v_{iht}^{\beta} - s_{io} - s_{in}\, n_{ht} - s_{iA}\, A_{ht}) \cdot$$

$$\{(a_o + a_n\, n_{ht} + a_A\, A_{ht}) \log u_{ht} + b_o + b_n\, n_{ht} + b_A\, A_{ht}\}/u_{ht} + \varepsilon_{iht}$$

$$(i=1,\ldots,N;\ h=1,\ldots,M,\ t=1,2),$$

where

$$v_{iht} = \frac{p_{iht}}{P_{ht}},$$

$$u_{ht} = \frac{y_{ht}}{P_{ht}},$$

$$P_{ht} = \left(\sum_{j=1}^{N} t_j\, p_{jht}^{\beta}\right)^{\frac{1}{\beta}},$$

$$\varepsilon_{iht} = \mu_{ih} + \nu_{iht},$$

$$E(\varepsilon_{iht}) = 0,$$

$$E(\varepsilon_{iht}\varepsilon_{jks}) = \begin{cases} \sigma^{\mu}_{ij} + \sigma^{\nu}_{ij} & \text{for } k=h,\ s=t \\ \sigma^{\mu}_{ij} & \text{for } k=h,\ s\neq t \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{i=1}^{N} \sigma^{\mu}_{ij} = \sum_{i=1}^{N} \sigma^{\nu}_{ij} = 0 \qquad \begin{array}{l}(j=1,\ldots,N)\\ (i=1,\ldots,N;\ h=1,\ldots,M;\ t=1,2),\end{array}$$

letting, as before, i be the commodity subscript, h be the household subscript, and t indicate the number of the report. Since t represents the number of the report *from each household* and not the historic time, we have to furnish the price variables p_i, P and v_i with a household subscript: p_{iht} represents, for example, the price of commodity i in the period where the h'th household gives its t'th report, and P_{ht} is the value of the total price index in this period.[15)] From (2.5), (2.6), and (3.1) it follows that the t and s coefficients will be subject to the adding-up constraints

$$(3.30) \qquad \begin{aligned} \sum_{i=1}^{N} t_i &= \sum_{i=1}^{N} s_{i0} = 1, \\ \sum_{i=1}^{N} s_{iA} &= \sum_{i=1}^{N} s_{in} = 0. \end{aligned}$$

We assume that the vector of structural explanatory variables $(n_{ht}, A_{ht}, v_{iht}, u_{ht})$ is uncorrelated with both disturbance components (μ_{ih}, ν_{iht}). This may be a somewhat unrealistic assumption as far as the individual disturbance component is concerned. It will be violated if, for instance, tastes, habits, experiences, and other unobserved individual effects are correlated with the structural variables income or age.[16)] Nevertheless, we shall, for simplicity, disregard such effects in the following.

---

15) Of course, p_{iht} and P_{ht} will take the same values for several constellations of h and t, since all households which report in the same historic time period face the same price vector. Moreover, we also have $n_{h2} = n_{h1}$ for households which do not change in size between the two reporting periods, and $A_{h2} = A_{h1} + 1$ for all households in which the head person is the same in the two periods.

16) Correlation between observable and unobservable individual variables in models for CS/TS data has received some attention in recent years. See Mundlak (1978), Taylor (1980), and Hausman and Taylor (1981) for discussions of the nature of this problem in a single equation context, and possible remedies.

IV. ESTIMATION

4.1. The log-likelihood function

Let w denote the 2MN x 1 vector of budget shares w_{iht} ordered in the same way as the disturbance vector ε (see eqs. (3.9), (3.11), and (3.13)), and let Z symbolize the matrix of observations on the exogenous variables (total expenditure, prices and background variables), and α the vector of demand coefficients. The model (3.29) can then be written compactly as

$$(4.1) \qquad w = F(Z;\alpha) + \varepsilon,$$

where F is the vector representation of the budget share demand functions.[1)] Provided that all disturbances are *normally distributed*, the log-likelihood function of w is

$$(4.2) \qquad \log \Lambda = l(w,Z;\alpha,\Omega) = -\frac{2MN}{2}\log(2\pi) - \frac{1}{2}\log|\Omega| - \frac{1}{2}\varepsilon'\Omega^{-1}\varepsilon,$$

where ε is a shorthand expression for $w-F(Z;\alpha)$. We assume tacitly that one commodity group has been deleted from the model to prevent singularity of Ω[2)], and reinterpret N as the number of commodities *minus one*.

Since, according to (3.14), $\Omega = I_M \otimes \Omega_*$, we have

$$(4.3) \qquad |\Omega| = |\Omega_*|^M,$$

and

$$(4.4) \qquad Q = Q(\alpha,\Omega_*) = \varepsilon'\Omega^{-1}\varepsilon = \sum_{h=1}^{M} \varepsilon_h'\Omega_*^{-1}\varepsilon_h.$$

Inserting (4.3) and (4.4) into (4.2), we find that Full Information Maximum Likelihood (FIML) estimation of α and Ω is equivalent to minimization of

$$(4.5) \qquad g = g(\alpha,\Omega_*) = \log|\Omega| + \varepsilon'\Omega^{-1}\varepsilon = M\log|\Omega_*| + \sum_{h=1}^{M}\varepsilon_h'\Omega_*^{-1}\varepsilon_h$$

with respect to the unknown parameters in α and Ω_*.

1) Since the only endogenous variable in each equation is its left hand variable, and all the right hand variables are assumed to be exogenous, the model is, formally, in the (non-linear) seemingly unrelated regressions format. See Zellner (1962), for a discussion of linear seemingly unrelated regression models in the context of Generalized Least Squares estimation.
2) From eqs. (3.12) and (3.14) it follows that Ω will be singular when (3.8) is satisfied. The Full Information Maximum Likelihood estimate of the coefficient vector α will be the same irrespective of which commodity is deleted. See Barten (1969).

4.2. The nature of the estimation problem

It is extremely difficult to minimize the function (4.5) with respect to the unknown parameters in α and Ω_* directly. We shall therefore adopt a procedure which solves the problem *iteratively* - from appropriately chosen starting values - by switching between the following two subproblems:

(i) Minimize Q, as given by (4.4), with respect to α, conditionally on Ω_*. (Non-linear GLS (Generalized Least Squares).)

(ii) Minimize g, as given by (4.5), with respect to Ω_*, conditionally on α.

Oberhofer and Kmenta (1974) have established the conditions for such a zig-zag procedure to be convergent in the context of maximum likelihood estimation of a generalized *linear* regression model. In appendix C, we explore how their results can be generalized to be valid for our non-linear model as well. A set of sufficient (although not necessary) conditions for this is given.

This simple two-stage algorithm is particularly suited to the case where Ω_* is a symmetric and positive definite, but otherwise *unrestricted* 2N x 2N matrix. However, eq. (3.12) shows that in our case it has the following particular structure:

$$\Omega_* = \Sigma_\nu \otimes I_2 + \Sigma_\mu \otimes E_2,$$

where Σ_ν and Σ_μ are symmetric and positive definite, but otherwise unrestricted N x N matrices. Thus, the $2N(2N+1)/2 = N(2N+1)$ elements in Ω_* are uniquely determined by the $2N(N+1)/2 = N(N+1)$ elements in Σ_ν and Σ_μ, i.e. Ω_* is subject to N^2 *linear restrictions* - the symmetry restrictions apart.

The above algorithm can, however, easily be tailor-made so as to take such additional constraints explicitly into account. We have chosen to do this by dividing the estimation problem into *two separate stages*, to be denoted as stage A and stage B. This serves the further purpose of facilitating the choice of *initial values*. Specifically, we use the end result from stage A as starting values for the parameters in stage B. It is a common experience that a judicious choice of starting values for non-linear estimation problems may be crucial for obtaining a reasonably rapid convergence and thus for economization in terms of computer costs.[3)]

---

3) An additional argument for this subdivision of the estimation procedure is that it, as a by-product, provides the information required for testing the restrictions implied by the various specifications. Confer section 6.1 and appendix E below.

In *stage A*, we neglect all individual disturbance components, i.e. we set $\Sigma_\mu = 0_{N,N}$ *a priori*. The two subproblems then become

(Ai) Minimize Q with respect to α, *conditionally on* Σ_ν, *with* Σ_μ *restricted to zero.*

(Aii) Minimize g with respect to Σ_ν, *conditionally on* α, *with* Σ_μ *restricted to zero.*

When convergence is obtained, we drop the binding on the Σ_μ matrix and enter *stage B*, the subproblems of which are

(Bi) Minimize Q with respect to α, *conditionally on* Σ_ν *and* Σ_μ.

(Bii) Minimize g with respect to Σ_ν *and* Σ_μ, *conditionally on* α.

This proceeds until convergence is obtained.

To start this algorithm, initial values have to be assigned to the vector α and the matrix Σ_ν. Since we have estimated several variants of the same basic model (cf. chapters VI and VII), we had to solve this problem several times. To save computer time we found it convenient always to proceed from the most restricted versions of the model to the gradually more general ones, by using the final parameter estimates from a restrictive version as starting values for the estimation of the next, more general one.[4)]

Below we shall give a step-by-step description of this algorithm for iterative FIML estimation. The complete model with N equations is treated in sections 4.3 and 4.4. Section 4.5 is devoted to the special case N=1.

4.3. The algorithm for iterative FIML estimation of the complete model[5)]

Stage A: Iterative minimization of g, given that $\Sigma_\mu = 0$, *i.e.* $\Sigma = \Sigma_\nu$.

A.1. Let $\hat{\alpha}$ denote the vector of the values initially assigned to the vector of demand coefficients α. Calculate the corresponding residuals $\hat{\varepsilon}_{iht} = \hat{\varepsilon}_{iht}(\hat{\alpha})$, form the vector $\hat{\varepsilon}_h = \hat{\varepsilon}_h(\hat{\alpha})$, estimate the covariance matrix $\Sigma = \Sigma_\nu$ by

---

4) For more details, see section H.3 of appendix H. It must be stressed that this strategy is followed for the sole purpose of making computer work more efficient. It does not affect the statistical testing procedures. The logic of multiple testing schemes is, of course, always to proceed from the general to the gradually more restrictive hypotheses, cf. sections 6.1, 7.4, and 7.5.

5) This algorithm is tailor-made for the situation with two replications of all individuals (households). For a generalization to situations with varying number of replications, see Biørn (1981b, section 4.2). Confer also the brief remarks on "unbalanced data" in Fuller and Battese (1974, pp. 77-78) in the context of Generalized Least Squares estimation.

$$(4.6)\qquad \hat{\sigma}_{ij} = \hat{\sigma}_{ij}(\hat{\alpha}) = \frac{1}{2M}\sum_{h=1}^{M}(\hat{\varepsilon}_{ih1}\hat{\varepsilon}_{jh1} + \hat{\varepsilon}_{ih2}\hat{\varepsilon}_{jh2}) \qquad (i,j=1,\ldots,N),$$

and let $\hat{\Sigma} = \hat{\Sigma}(\hat{\alpha}) = (\hat{\sigma}_{ij}(\hat{\alpha}))$. Eq. (4.6) is in accordance with the first order conditions for FIML estimation of the 'total' covariance matrix Σ.[6]

Remark 1: More generally, (4.6) agrees with the formulae for FIML estimation of the covariance matrix of the reduced form of a linear simultaneous model, when there are no restrictions on the elements of this matrix, except the symmetry condition.[7]

A.2. Form the matrix

$$(4.7)\qquad \hat{\Omega}_* = \hat{\Omega}_*(\hat{\alpha}) = \hat{\Sigma} \otimes I_2,$$

and its inverse

$$\hat{\Omega}_*^{-1} = \hat{\Sigma}^{-1} \otimes I_2.$$

Since $\hat{\Omega}_*$ is real and symmetric, so is $\hat{\Omega}_*^{-1}$. Symmetry alone ensures that we can factorize $\hat{\Omega}_*^{-1}$ as follows:

$$(4.8)\qquad \hat{\Omega}_*^{-1} = U'U,$$

where $U = U(\hat{\alpha})$ is a non-singular 2N x 2N matrix. The value of the quadratic form (4.4) which corresponds to the coefficient vector $\alpha = \hat{\alpha}$ can be written as

$$\hat{Q} = \hat{Q}(\hat{\alpha}) = \sum_{h=1}^{M} \hat{\varepsilon}_h(\hat{\alpha})'\hat{\Omega}_*(\hat{\alpha})^{-1}\hat{\varepsilon}_h(\hat{\alpha}) = \sum_{h=1}^{M} \hat{\eta}_h(\hat{\alpha})'\hat{\eta}_h(\hat{\alpha}),$$

where

$$(4.9)\qquad \hat{\eta}_h = \hat{\eta}_h(\hat{\alpha}) = U(\hat{\alpha})\ \hat{\varepsilon}_h(\hat{\alpha}) \qquad (h=1,\ldots,M).$$

The 2N x 1 vector $\hat{\eta}_h$ is the vector of transformed residuals relating to household h which corresponds to the factorization (4.8).

---

6) See appendix D, especially eq. (D.24).
7) Confer Malinvaud (1966, p. 568-570).

Remark 2: There exists an infinite number of factorizations which satisfy (4.8). We have chosen a *Cholesky factorization*, i.e.,

$$\hat{\Omega}_*^{-1} = LDL',$$

where L is a lower triangular 2N x 2N matrix with diagonal elements equal to 1, and D is a diagonal matrix of the same order. The fact that Ω_*^{-1} is real, symmetric, and positive definite guarantees that this factorization exists and is unique.[8)] This implies that our transformation matrix can be written as

$$U = \sqrt{D}\, L'.$$

A.3. Subproblem (Ai), minimization of Q with respect to α, conditionally on Ω_*, is now equivalent to minimization of $\sum_{h=1}^{M} \hat{\eta}_h' \hat{\eta}_h$ with respect to α, which - under certain conditions - can be interpreted as a non-linear least squares estimation problem based on the transformed disturbances.[9)]

A.4. We then return to step A.1, by using the minimizing value of α from step A.3 as new starting value. In this way, we solve subproblem (Aii).

The iterative procedure A.1 - A.4 proceeds until convergence is obtained with respect to Σ. Let $\hat{\Sigma}^{(k)}$ denote the estimated covariance matrix from the k'th iteration. The chosen convergence criterion is

$$(4.10) \quad ||\hat{\Sigma}^{(k)} - \hat{\Sigma}^{(k-1)}|| < \xi,$$

i.e. the norm of the change in $\hat{\Sigma}$ shall not exceed ξ, which is a scalar with a preassigned small value.

Stage B: Iterative minimization of g with respect to all unknown parameters, including Σ_μ.

B.1. Compute the residuals that correspond to the estimated coefficient vector α in the final iteration in stage A, estimate the covariances of the error components, ν_{iht} and μ_{ih}, as follows:

---

8) For a statement of this and other properties of Cholesky factorizations, see Lau (1978).
9) See appendices G and H.

$$(4.11)\quad \hat{\sigma}^{\nu}_{ij} = \frac{1}{2M}\sum_{h=1}^{M}(\hat{\varepsilon}_{ih2}-\hat{\varepsilon}_{ih1})(\hat{\varepsilon}_{jh2}-\hat{\varepsilon}_{jh1})$$

$$(4.12)\quad \hat{\sigma}^{\mu}_{ij} = \frac{1}{2M}\sum_{h=1}^{M}(\hat{\varepsilon}_{ih1}\hat{\varepsilon}_{jh2}+\hat{\varepsilon}_{ih2}\hat{\varepsilon}_{jh1}) \qquad (i,j=1,\ldots,N)$$

and form the matrices $\hat{\Sigma}_{\nu} = (\hat{\sigma}^{\nu}_{ij})$ and $\hat{\Sigma}_{\mu} = (\hat{\sigma}^{\mu}_{ij})$. We show in appendix D (see eqs. (D.22) and (D.23)) that these formulae agree with the first order conditions for FIML estimation of σ^{ν}_{ij} and σ^{μ}_{ij}.

B.2. Form the matrix

$$(4.13)\quad \hat{\Omega}_{*} = \hat{\Sigma}_{\nu} \otimes I_2 + \hat{\Sigma}_{\mu} \otimes E_2.$$

We calculate its inverse, $\hat{\Omega}_{*}^{-1}$, apply the following factorization

$$(4.14)\quad \hat{\Omega}_{*}^{-1} = U'U,$$

and proceed as in step A.2 above.

Remark 3: It is straightforward to establish that (4.13) can be written as

$$(4.15)\quad \hat{\Omega}_{*} = \hat{\Sigma}_{\nu} \otimes (I_2 - \frac{E_2}{2}) + (2\hat{\Sigma}_{\mu} + \hat{\Sigma}_{\nu}) \otimes \frac{E_2}{2}.$$

By utilizing the fact that the 2 x 2 matrices $(I_2 - \frac{E_2}{2})$ and $\frac{E_2}{2}$ are both idempotent and orthogonal, we directly find that the inverse of $\hat{\Omega}_{*}$ has the following structure:[10)]

$$(4.16)\quad \Omega_{*}^{-1} = \hat{\Sigma}_{\nu}^{-1} \otimes (I_2 - \frac{E_2}{2}) + (2\hat{\Sigma}_{\mu} + \hat{\Sigma}_{\nu})^{-1} \otimes \frac{E_2}{2}.$$

B.3 and B.4. These steps are formally equivalent to steps A.3 and A.4 above.

Let $\hat{\Sigma}^{(k)}_{\nu}$ and $\hat{\Sigma}^{(k)}_{\mu}$ denote the estimated covariance matrices from the k'th iteration in stage B. Our convergence criterion is then defined by

---

10) We do not, however, exploit this particular structure of Ω_{*}^{-1} when performing the factorization (4.14). Confer section 4.4 and appendix H (especially, section H.3 and footnote 8) below.

$$(4.17)\qquad \begin{aligned} &||\hat{\Sigma}_\nu^{(k)} - \hat{\Sigma}_\nu^{(k-1)}|| < \xi \text{ and} \\ &||\hat{\Sigma}_\mu^{(k)} - \hat{\Sigma}_\mu^{(k-1)}|| < \xi , \end{aligned}$$

i.e. the norms of the changes in $\hat{\Sigma}_\nu$ and $\hat{\Sigma}_\mu$ shall both be less than a small scalar ξ.

4.4. An alternative factorization of Ω_*^{-1}

The algorithm outlined in step B.2 above proceeds by factorizing the 2N x 2N matrix $\hat{\Omega}_*^{-1}$ as indicated in eq. (4.14). In this section, we briefly describe an alternative procedure based on a factorization of the "elementary" N x N matrices $\hat{\Sigma}_\nu^{-1}$ and $(2\hat{\Sigma}_\mu + \hat{\Sigma}_\nu)^{-1}$, i.e. a procedure which exploits the specific structure of Ω_*^{-1} pointed out in remark 3 above (eq. (4.16)).

Since $\hat{\Sigma}_\nu^{-1}$ and $(2\hat{\Sigma}_\mu + \hat{\Sigma}_\nu)^{-1}$ are both symmetric, there exist non-singular N x N matrices T_1 and T_2 such that

$$(4.18)\qquad T_1'T_1 = \hat{\Sigma}_\nu^{-1},$$

$$(4.19)\qquad T_2'T_2 = (2\hat{\Sigma}_\mu + \hat{\Sigma}_\nu)^{-1}.$$

From (4.16), (4.18), and (4.19), while again utilizing the fact that $(I_2 - \frac{E_2}{2})$ and $\frac{E_2}{2}$ are idempotent and orthogonal, it follows that Q, as defined in (4.4), can be transformed as follows:

$$(4.20)\qquad Q = Q(\alpha,\hat{\Sigma}_\nu,\hat{\Sigma}_\mu) = \sum_{h=1}^{M} \varepsilon_h' [T_1 \otimes (I_2 - \frac{E_2}{2})]'[T_1 \otimes (I_2 - \frac{E_2}{2})]\varepsilon_h$$

$$+ \sum_{h=1}^{M} \varepsilon_h' [T_2 \otimes \frac{E_2}{2}]'[T_2 \otimes \frac{E_2}{2}]\varepsilon_h,$$

or simply as

$$(4.21)\qquad Q = \sum_{h=1}^{M} \eta_h'\eta_h,$$

where

$$(4.22)\qquad \eta_h = [T_1 \otimes (I_2 - \frac{E_2}{2}) + T_2 \otimes \frac{E_2}{2}]\varepsilon_h \qquad (h=1,\ldots,M).$$

It is quite clear that this is just another way of factorizing Ω_*^{-1} and transforming Q into a sum of squares.[11] The possible advantage of this alternative is *only* a computational one: We reduce the problem of inverting one 2N x 2N matrix ($\hat{\Omega}_*$) to a problem of inverting two N x N matrices ($\hat{\Sigma}_\nu$ and $(2\hat{\Sigma}_\mu + \hat{\Sigma}_\nu)$) by exploiting the particular structure of $\hat{\Omega}_*^{-1}$ given by (4.16).

4.5. Single equation FIML estimation[12]

When $N=1$, the iterative estimation procedure can be simplified substantially. In this case, Σ_ν and Σ_μ are scalars, which we denote as σ_ν^2 and σ_μ^2, respectively. The matrix Ω_*, defined in (3.12), then simply becomes

$$(4.23)\quad \Omega_* = \sigma_\nu^2 I_2 + \sigma_\mu^2 E_2 = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

where we for convenience have introduced the symbols $\sigma^2 = \sigma_\nu^2 + \sigma_\mu^2$ for the total disturbance variance and $\rho = \sigma_\mu^2/\sigma^2$ for the part of this total which is due to individual variations. It is easy to show that ρ has the alternative interpretation as the coefficient of correlation between the disturbance of the first and the second observation of any household.[13]

In the single equation case, stage A degenerates to non-linear least squares estimation of the coefficient vector α (subproblem (Ai)) and the disturbance variance $\sigma^2 = \sigma_\nu^2$ (subproblem (Aii)), since $\rho=0$ implies $\Omega_* = \sigma^2 I_2$. Below we consider stage B and its subproblems (Bi) and (Bii) in turn and sketch the complete algorithm in a concluding paragraph.

Subproblem (Bi): Minimization of Q with respect to α, conditionally on σ^2 and ρ.

Considering the factorization of Ω_*^{-1} given in section 4.4 above, we find from equations (4.18) and (4.19)

---

11) Since $\hat{\Sigma}_\nu^{-1}$ and $(2\hat{\Sigma}_\mu+\hat{\Sigma}_\nu)^{-1}$ are both real, symmetric, and positive definite, we may, of course, choose the Cholesky factorization in this case as well.
12) This case is given an extensive and generalized treatment in Biørn (1981a, pp. 226-231). See also the last part of section 4.2 in Biørn (1981b) for a further generalization.
13) This is based on the assumption that the disturbance components μ_{ih} and ν_{iht} are uncorrelated. For a qualification of this assumption, see section 3.2.

$$(4.24)\quad T_1 = \frac{1}{\sigma_\nu} = \frac{1}{\sigma\sqrt{1-\rho}} ,$$

$$(4.25)\quad T_2 = \frac{1}{\sqrt{2\sigma_\mu^2+\sigma_\nu^2}} = \frac{1}{\sigma\sqrt{1+\rho}} .$$

The transformed disturbance vector, defined in (4.22), then becomes

$$\eta_h = \frac{1}{\sigma}[(1-\rho)^{-\frac{1}{2}}(I_2 - \frac{E_2}{2}) + (1+\rho)^{-\frac{1}{2}}\frac{E_2}{2}]\varepsilon_h$$

$$= \sigma^{-1}(1-\rho)^{-\frac{1}{2}}[I_2-\{1-(\frac{1-\rho}{1+\rho})^{\frac{1}{2}}\}\frac{E_2}{2}]\,\varepsilon_h.$$

Writing $\varepsilon_h = \begin{pmatrix}\varepsilon_{h1}\\ \varepsilon_{h2}\end{pmatrix}$ (cf. eq. (3.9)) and $\eta_h = \begin{pmatrix}\eta_{h1}\\ \eta_{h2}\end{pmatrix}$, this implies that η_{ht} is proportional to

$$\varepsilon_{ht} - \{1 - (\frac{1-\rho}{1+\rho})^{\frac{1}{2}}\}\frac{\varepsilon_{h1}+\varepsilon_{h2}}{2} \qquad \begin{matrix}(h=1,\ldots,M)\\ (t=1,2),\end{matrix}$$

when we disregard the irrelevant constant $\sigma^{-1}(1-\rho)^{-\frac{1}{2}}$.

Thus, minimization of Q can be obtained by means of the following simple algorithm:

> *Subtract $1-((1-\rho)/(1+\rho))^{\frac{1}{2}}$ times the corresponding individual average from the original disturbances, and minimize the resulting sum of squares with respect to the coefficient vector α.*

The fraction to be subtracted, $1-((1-\rho)/(1+\rho))^{\frac{1}{2}}$, is an increasing function of ρ, the individual share in the total variance, or the coefficient of correlation between the disturbance from the first and the second observation. Thus, the transformation which solves problem (Bi) has a considerable intuitive appeal.

Subproblem (Bii): Minimization of g with respect to σ^2 and ρ, conditionally on α.

When N=1, it follows from (4.11) and (4.12) that

$$\hat{\sigma}_\nu^2 = \frac{1}{2M}\sum_{h=1}^{M}(\hat{\varepsilon}_{h2}-\hat{\varepsilon}_{h1})^2,$$

$$\hat{\sigma}_\mu^2 = \frac{1}{M}\sum_{h=1}^{M}\hat{\varepsilon}_{h1}\hat{\varepsilon}_{h2},$$

where $\hat{\varepsilon}_{ht}$ denotes the residuals which emerge from the solution to subproblem (Bi). Hence, the conditional estimators of σ and ρ become[14)]

$$(4.26)\quad \hat{\sigma}^2 = \hat{\sigma}_\nu^2 + \hat{\sigma}_\mu^2 = \frac{1}{2M}\sum_{h=1}^{M}(\hat{\varepsilon}_{h1}^2+\hat{\varepsilon}_{h2}^2),$$

$$(4.27)\quad \hat{\rho} = \frac{\hat{\sigma}_\mu^2}{\hat{\sigma}^2} = \frac{2\sum_{h=1}^{M}\hat{\varepsilon}_{h1}\hat{\varepsilon}_{h2}}{\sum_{h=1}^{M}(\hat{\varepsilon}_{h1}^2+\hat{\varepsilon}_{h2}^2)}\ .$$

It is easy to verify that (4.26) - (4.27) are in fact the solution to subproblem (Bii) in the single equation case. From (4.23) we find

$$|\Omega_*| = \sigma^4(1-\rho^2)\ ,$$

$$\Omega_*^{-1} = \frac{1}{\sigma^2(1-\rho^2)}\begin{pmatrix}1 & -\rho\\ -\rho & 1\end{pmatrix},$$

which gives the following expression for the g function (4.5):

$$(4.28)\quad g = g(\sigma^2,\rho,\alpha) = M\{2\log\sigma^2 + \log(1-\rho^2)\} + \frac{1}{\sigma^2(1-\rho^2)}\,Q_1(\rho),$$

where

$$(4.29)\quad Q_1(\rho) = \sum_{h=1}^{M}\varepsilon_h'\begin{pmatrix}1 & -\rho\\ -\rho & 1\end{pmatrix}\varepsilon_h = \sum_{h=1}^{M}\{\varepsilon_{h1}^2 - 2\rho\varepsilon_{h1}\varepsilon_{h2} + \varepsilon_{h2}^2\}.$$

---

14) As noted in section 3.2, we are not ensured that $\hat{\rho}$ is positive. These estimates are identical with the "analysis of variance estimates" of the disturbance components. This formal equivalence between maximum likelihood estimation and analysis of variance estimation is, however, only confined to the two components specification considered here. In three component models, such equivalence holds only asymptotically. See Amemiya (1971, sections 3-5).

Minimizing g partially with respect to σ^2 we find the conditional estimator:

$$(4.30)\quad \hat{\sigma}^2(\rho) = \frac{Q_1(\rho)}{2M(1-\rho^2)} .$$

Inserting this into (4.28), we obtain the concentrated g function

$$g^* = g^*(\rho,\alpha) = \text{constant} + M\{2 \log Q_1(\rho) - \log (1-\rho^2)\}.$$

Minimization of g^* with respect to ρ, gives (4.27). From (4.29) and (4.27), we find

$$Q_1(\hat{\rho}) = (1-\hat{\rho}^2) \sum_{h=1}^{M} (\hat{\varepsilon}_{h1}^2+\hat{\varepsilon}_{h2}^2).$$

Combining this expression with (4.30), we directly obtain the variance estimator (4.26).

The algorithm for iterative single equation FIML estimation

Summing up, the procedure we have established for finding the iterative solution to the FIML estimation problem in the single equation case, can be described as a step-by-step algorithm, which is analogous to the one stated for the simultaneous estimation of the complete N equation model. The only difference is that whereas the algorithm for the complete model requires an initial guess at the coefficient vector $\hat{\alpha}$, the present algorithm is initiated from the value $\rho=0$ in stage A. Then subproblem (Ai) degenerates to ordinary non-linear least squares estimation of α and subproblem (Aii) to ordinary non-linear least squares estimation of the disturbance variance σ^2. Stage B can then be condensed into the following four step algorithm:

1) Find the residuals $\hat{\varepsilon}_{ht}$ (for all h and t) that correspond to the estimated equation in stage A.

2) Calculate the estimate of ρ, $\hat{\rho}$, from (4.27).

3) Calculate the transformed residuals $\hat{\eta}_{ht}$, and minimize the resulting sum of squares with respect to α.

4) Go to step 1), while using the solution value of α from step 3) as new starting value.

This iterative procedure proceeds until convergence is obtained with respect to ρ. Let $\hat{\rho}^{(k)}$ denote the estimated ρ value in the k'th iteration. The convergence criterion in stage B is then defined by

$$|\hat{\rho}^{(k)} - \hat{\rho}^{(k-1)}| < \xi ,$$

i.e. the absolute value of the change in $\hat{\rho}$ shall be smaller than some prescribed scalar ξ.

V. THE DATA

The present study is part of a larger project which has as its data base Norwegian Surveys of Consumer Expenditure from five consecutive years, 1973 - 1977. In Biørn and Jansen (1980) we have given an extensive discussion of this data source, which contains a total of 8 110 individual observation points. For reasons discussed in section 2.1 of that report, the sample size and the sampling design have undergone substantial changes during this period. This is visualized in figure 5.1.

In this study, we focus solely on the reports from those households which have been observed twice. The sampling design with partly overlapping samples was introduced in the Norwegian Surveys of Consumer Expenditure from the year 1975 onwards. Our data file contains 209 complete reports from households observed in 1975 and 1976, and 235 reports from those participating in 1976 and 1977. No household participated more than twice. These are the basic data for the present study. However, to simplify matters we have discarded 26 households from the second group, which leaves us with a data set consisting of *M=418 households* and *2M=836 reports* (individual observation points). The data for this study thus consist of only 10 per cent of the original data base. (See figure 5.1.)

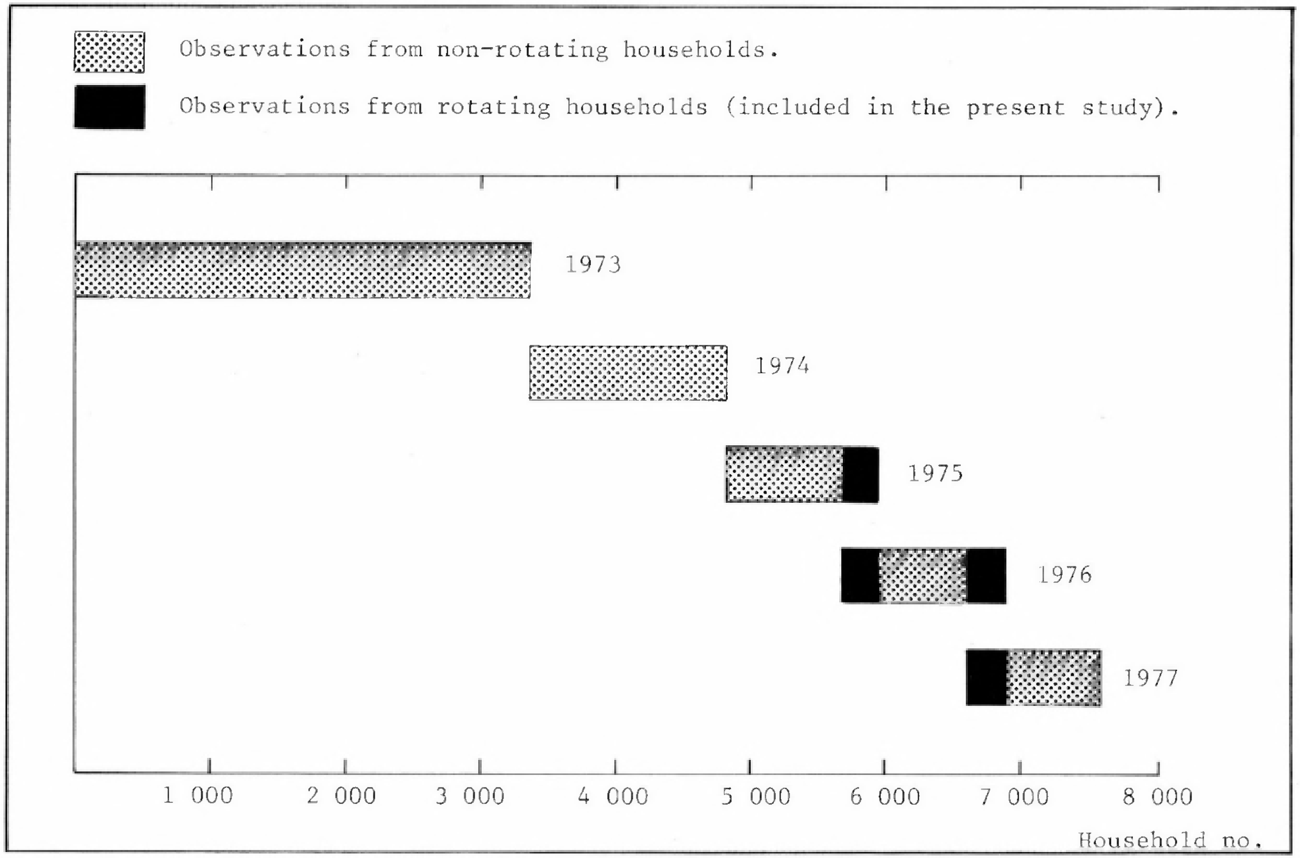
We have divided the consumption expenditures into *N=5* aggregated commodity groups. This is the upper level of aggregation in our study and it is primarily intended for simultaneous equations estimation. (See chapter VI.) These commodity groups are further disaggregated to *N=28* groups, which form the basis for a more detailed examination of the consumption pattern to be presented in chapter VII.[1)]

The *compiling method* in the Norwegian Surveys of Consumer Expenditure is a recording of consumption expenses by a combination of detailed book-keeping and interview. Each household is asked to keep accounts of its expenses during a 14 days' period. In addition, for commodities and services with a low purchase frequency, expenses during the last 12 months are registered in a concluding interview at the end of the accounting period. Expenses on housing are measured by rent (including maintenance and repairs), whereas other durable goods are simply represented by the value of last year's total purchases.[2)]

---

1) The commodity groups are listed in table 5.1. A detailed record of the commodity definitions at both levels of aggregation is given in Biørn and Jansen (1980, section 2.2 and appendix 1).
2) Confer tables 3 and 4 in Biørn and Jansen (1980, p. 12) giving the shares of the expenditure on each commodity group which are recorded by interview and the shares of the expenditure which represent purchases of durable goods.

Figure 5.1. Household reports in the Norwegian Surveys of Consumer Expenditure. 1973 - 1977. Overview of the data structure



In table 5.1, we have recorded summary statistics for the 836 household reports which underlie the present study. The first column gives the unweighted average and standard deviation of the budget shares[3)] (measured at current prices), while the second column contains the same statistics for the expenditure at constant 1974 prices, calculated by deflating the expenditure on each item by a corresponding price index to be explained below. These summary statistics concur with those reported in Biørn and Jansen (1980) for the complete 1973 - 1977 data set. We observe a relatively high standard deviation for some of the commodity groups. This reflects, *inter alia*, the presence of zero-reporting in our data, i.e. the fact that a non-negligible fraction of the respondents report zero expenditure on some commodity groups. We have not, however, taken any specific action to counteract that problem in this study.

Table 5.1. Budget shares (measured at current prices) and expenditures (measured at constant 1974 prices) for the 836 reports from the rotating households, 1975 - 1977.
Average values and standard deviations (in parenthesis)

Commodity group	Budget share. Percentages	Expenditure at constant 1974 prices. Nkr
1. Flour and bread	2.63 (2.37)	1 019 (750)
2. Meat and eggs	6.70 (6.25)	3 195 (4 254)
3. Fish	1.60 (2.24)	684 (980)
4. Canned meat and fish	0.60 (0.91)	263 (373)
5. Dairy products	3.73 (2.83)	1 448 (942)
6. Butter and margarine	0.98 (1.07)	366 (357)
7. Potatoes and vegetables	5.13 (4.13)	2 290 (1 900)
8. Other food	4.29 (3.36)	1 624 (1 235)
9. Beverages	2.35 (3.49)	1 165 (1 874)
10. Tobacco	1.74 (2.69)	752 (1 140)
11. Clothing	7.58 (7.99)	4 071 (4 728)

3) See Biørn and Jansen (1980, subsection 3.2.4) for a discussion of the distinction between weighted and unweighted average budget shares.

Table 5.1 (cont.). Budget shares (measured at current prices) and expenditures (measured at constant 1974 prices) for the 836 reports from the rotating households, 1975 - 1977.
Average values and standard deviations (in parenthesis)

Commodity group	Budget share. Percentages	Expenditure at constant 1974 prices. Nkr
12. Footwear	1.72 (3.98)	981 (2 217)
13. Housing	11.04 (10.17)	5 465 (6 839)
14. Fuel and power	4.78 (4.15)	1 606 (875)
15. Furniture	4.45 (7.41)	2 570 (5 299)
16. Household equipment .	2.93 (5.61)	1 488 (3 889)
17. Misc. household goods	2.60 (3.66)	1 096 (1 919)
18. Medical care	1.70 (4.70	785 (2 212)
19. Motorcars and bicycles	4.95 (12.51)	3 644 (9 870)
20. Running costs of vehicles	6.91 (9.58)	4 484 (9 220)
21. Public transport	2.54 (5.42)	1 343 (4 427)
22. PTT[a] charges	1.47 (5.60	776 (3 739)
23. Recreation	5.89 (8.53)	3 445 (6 927)
24. Public entertainment	3.10 (5.41)	1 514 (2 821)
25. Books and newspapers	2.26 (4.48)	1 059 (3 393)
26. Personal care	2.00 (2.31)	934 (1 100)
27. Misc. goods and services	1.49 (4.85)	1 131 (8 083)
28. Restaurants, hotels, etc.	2.82 (5.49)	1 608 (3 611)

a) I.e. post, telephone, and telegraph.

Table 5.1 (cont.). Budget shares (measured at current prices) and expenditures (measured at constant 1974 prices) for the 836 reports from the rotating households, 1975 - 1977.
Average values and standard deviations (in parenthesis)

Commodity group	Budget share. Percentages	Expenditure at constant 1974 prices. Nkr
I. Food, beverages and tobacco	29.75 (14.07)	12 782 (8 011)
II. Clothing and footwear	9.30 (9.41)	5 049 (5 855)
III. Housing, fuel and furniture	25.81 (14.28)	12 194 (11 373)
IV. Travel and recreation	27.12 (18.21)	16 264 (19 383)
V. Other goods and services	8.03 (8.80)	4 464 (9 616)
Total	100,00 (..)	44 433 (30 770)

The sample averages and standard deviations of the main background variables are given in table 5.2. The main income earner of the 'rotating' households has an average age of 52.6 years. This is about 1.7 years higher that the average age based on all the 3 284 household reports from the years 1975 - 1977; i.e. the main income earners of the rotating households are on the average older than those of the non-rotating households. The average family size, 3.03 persons, is very close to the overall sample mean of the households observed in the years 1975 - 1977 (cf. tables 1 and 2 in Biørn and Jansen (1980, p. 10)).

Table 5.2. Main background variables. Sample averages and standard deviations based on the 836 reports from the rotating households, 1975 - 1977

Variable	Sample average	Standard deviation
Age of the main income earner in the households, years	52.61	15.44
Household size, number of persons	3.03	1.62

The *price indices* are constructed from the basic data used in calculating the official Norwegian Consumer Price Index.[4] First, we established a Laspeyres price index for each commodity group on a monthly basis. We then performed a smoothing of each series to a periodicity of 14 days to make the periodicity of the indices coincide with the length of the accounting period for the consumption expenses. The smoothing algorithm is described in greater detail in section 2.3 of Biørn and Jansen (1980, pp. 11-15). In table 5.3 below, we reproduce the resulting annual averages of the price indices only.

Table 5.3. Price indices for the commodity groups. Annual averages (1974 = 100)[a)]

Commodity group		1975	1976	1977
	All commodities	111.67	121.91	132.97
1.	Flour and bread	112.33	117.29	123.39
2.	Meat and eggs	116.31	132.58	145.30
3.	Fish	101.38	102.56	114.23
4.	Canned meat and fish	106.64	114.24	125.52
5.	Dairy products	120.71	133.96	125.65
6.	Butter and margarine	106.61	105.19	118.04
7.	Potatoes and vegetables	118.09	135.33	139.96
8.	Other food	115.56	123.88	156.30
9.	Beverages	117.32	128.65	141.06
10.	Tobacco	108.39	122.87	126.42
11.	Clothing	107.42	114.55	126.01
12.	Footwear	106.93	119.47	134.85
13.	Housing	108.34	116.05	125.03
14.	Fuel and power	113.62	125.30	140.37
15.	Furniture	111.12	119.24	129.51
16.	Household equipment	111.44	119.29	125.70
17.	Misc. household goods	119.25	132.50	146.11
18.	Medical care	115.00	127.53	138.14
19.	Motorcars, bicycles	113.17	124.02	134.31
20.	Running costs of vehicles	106.60	116.76	130.42
21.	Public transport	111.61	129.55	146.09
22.	PTT charges	102.44	102.93	113.23
23.	Recreation	109.42	117.08	124.28
24.	Public entertainment	105.98	110.86	119.97
25.	Books and newspapers	112.79	128.39	142.34
26.	Personal care	115.99	126.93	137.76
27.	Misc. goods and services	118.41	127.51	135.60
28.	Restaurants, hotels, etc.	114.26	125.30	141.46
I.	Food, beverages and tobacco	114.89	126.72	137.03
II.	Clothing and footwear	107.34	115.37	127.48
III.	Housing, fuel and furniture	111.26	120.29	130.69
IV.	Travel and recreation	109.17	119.29	130.92
V.	Other goods and services	115.62	126.66	138.66

a) Extracted from table 7 in Biørn and Jansen (1980, p. 15).

4) This is a Laspeyres index based on monthly registration of the prices of a selection of 770 goods and services.

VI. EMPIRICAL RESULTS: THE COMPLETE 5 COMMODITY MODEL

The empirical work with the simultaneous estimation of the model, (3.29), has brought a considerable amount of interesting results for the five aggregated commodity groups. In this chapter, we give a survey of the main findings. Several versions of the same basic model will be investigated. Our inference procedure will be a combination of hypothesis testing and estimation.[1)]

First, in section 6.1, we test the general model specification by means of a multiple testing strategy based on the Likelihood Ratio principle. We focus on the following questions: (i) What is the effect of the background variables: which are significant, and which are not? (ii) Do our data indicate the presence of significant individual disturbance components? (iii) Can we impose restrictions on the coefficients of the price index function, i.e. β and t_i, without incurring a loss of goodness of fit? Then, in section 6.2, we comment on the magnitude and sign of the coefficient estimates obtained. In section 6.3, we examine more systematically the effect of incorporating individual disturbance components, *inter alia* by investigating the sensitivity of the estimates of the structural coefficients. The next two sections deal specifically with the demand elasticities implied by the coefficient estimates. Section 6.4 contains an extensive discussion of the Engel, Cournot, and Slutsky elasticities together with the estimated budget shares, while section 6.5 is devoted to the income flexibility. We find that the qualitative implications of Frisch's hypothesis on the variation of the latter parameter with income are supported. In both these sections, we use diagrams to summarize the main findings. Finally, section 6.6 reports some experiences concerning the effect of relaxing constraints on the coefficients of the price index function, i.e. β and t_i, which we have imposed during most of the estimation work. The purpose is to illustrate a problem of collinearity to which the Fourgeaud-Nataf model and related models are vulnerable.

The complete set of tables underlying the work reported in this chapter is reproduced in part A of the table annex. The tables included in the text section are, in most cases, extracts from the table annex and give the main results only.[2)]

---

1) For details on the estimation strategy and the computer work, see chapter IV and appendix H.
2) In this chapter, and in table annex A, we number the aggregate commodities by Arabic numerals (1,2,..,5). In chapters V and VII, we represent these commodities by Roman numerals, to prevent confusion with the disaggregated groups. The reason for this change of notation is only typographic.

6.1. Test of model specification

In principle, our model specification could be tested by testing all the different hypotheses and subhypotheses involved in the three questions raised above simultaneously within the framework of one multiple testing scheme. Although this is a recommendable approach from a theoretical point of view, it would hardly be practicable in the present case. Not only would such a scheme contain a substantial number of hypotheses, a lot of computer problems would also be likely to arise owing to the large number of parameters involved, the high degree of non-linearity of the model, and the complexity of the estimation algorithm. In these circumstances, we should desist from asking too many questions simultaneously.

Our strategy will be first to concentrate on hypotheses about the form of the price index function, i.e. question (iii). Conditional on the outcome of these tests, we proceed to questions (i) and (ii) and perform a simultaneous test of the hypotheses involved by means of one single testing scheme. This implies that we focus simultaneously on the background variables and the structure of disturbances, but consider these hypotheses as separated from those concerning the form of the price index function.

6.1.1. Constraints on the price index function

Unconstrained FIML estimation of the demand model with no background variables included gave the following estimates:[3)]

	i: 1	2	3	4	5
$\hat{s}_i$	0.1368	0.1175	0.1882	0.4486	0.1089
$\hat{t}_i$	0.5687	0.0553	0.4103	-0.0602	0.0259

$\hat{a} = 0.3954 \quad \hat{b} = 0.6597 \quad \hat{\beta} = 2.2666$

These estimates satisfy the adding-up and non-negativity constraints on the s_i and t_i coefficients (cf. (2.5), (2.6), (2,17), and (2.19)), except that $\hat{t}_4$ is negative. If eq. (2.4) is interpreted as a price index function, this is obviously an unacceptable result since it implies that commodity 4, Travel and recreation, occurs with a

3) Extracted from the first column of table A.3 in the table annex. Attempts were also made to include background variables. In this we did not succeed, however, as the estimation process did not converge.

negative weight in the price index P (cf. eq. (2.18)). The standard error of $\hat{t}_4$ is, however, 0.37, so we cannot reject the hypothesis $t_4 \geqq 0$, i.e. it would not be in conflict with our data to restrict this coefficient to be non-negative. Imposing $t_4 = 0$ as an a priori constraint, we find that all estimates fall within the admissible region:

	i				
	1	2	3	4	5
$\hat{s}_i$	0.1367	0.1176	0.1882	0.4486	0.1089
$\hat{t}_i$	0.5198	0.0628	0.3830	0	0.0344

$\hat{a} = 0.4459$ $\hat{b} = 0,7421$ $\hat{\beta} = 2.0556$

We then turn to β, the coefficient which characterizes the curvature of the "indifference surfaces" of the price index function.[4] The point estimate, $\hat{\beta} = 2.0556$, indicates that these surfaces are *concave*, but it has a standard error as large as 1.3241, so an approximate 5 per cent confidence interval for β extends from -0.55 to 4.66. Thus, we cannot reject neither that the price index function is linear ($\beta = 1 : P = \Sigma_j t_j p_j$) nor that it is log-linear ($\beta = 0 : \log P = \Sigma_j t_j \log p_j$). In his analysis of the Fourgeaud-Nataf model based on French data, Nasse found an estimated β value slightly less than one ($\hat{\beta} = 0.9862$) (Nasse (1973, p. 1152)), i.e. also his analysis supports the hypothesis that the implicit price index function is linear.[5]

If we impose the linearity restriction $\beta=1$ from the outset, we find that all our coefficient estimates satisfy the a priori constraints:

	i				
	1	2	3	4	5
$\hat{s}_i$	0.1366	0.1177	0.1880	0.4488	0.1089
$\hat{t}_i$	0.3609	0.0859	0.2948	0.1936	0.0648

$\hat{a} = 0.7786$ $\hat{b} = 1.2872$

The evidence from our data is, in other words, that *neither the hypothesis that all t_i's are non-negative nor the hypothesis $\beta=1$ can be rejected.* This conclusion holds not only when the two hypotheses are tested separately (by means of t tests), but also when they are tested

4) Cf. footnote 10 in chapter II.
5) Nasse reports an associated standard error estimate equal to 0.1568, but does not, however, comment on its interpretation.

jointly (by means of a Likelihood Ratio test; see the bottom line of table A.3). In the following, we shall therefore impose $\beta=1$, $t_i\in[0,1]$ as *a priori* constraints on the price index function when drawing inferences on the other demand parameters. This is obviously a simplification. We shall discuss its implications more closely in section 6.6.

6.1.2. A testing scheme for background variables and unobserved individual effects

Our starting point for testing the significance of the background variables and the individual disturbance components - subject to the constraints $\beta=1, t_i\in[0,1]$, $i=1,\dots,N(=5)$ - is the following *basic hypothesis:*

> H_*: Both age (A), number of persons (n), and individual disturbance components are significant factors in the demand functions.

Against this we set the following three more restrictive hypotheses:

H_A: Age (A) is insignificant: $s_{iA} = a_A = b_A = 0 \qquad (i=1,\dots,N).$

H_n: Number of persons (n) is insignificant:
$s_{in} = a_n = b_n = 0 \qquad (i=1,\dots,N).$

H_μ: Individual disturbance components are insignificant:
$\sigma^{\mu}_{ij} = 0 \qquad (i=1,\dots,N;\ j=1,\dots,N).$

These are the hypotheses on the *first level* of our testing scheme. By combining these hypotheses in all possible pairs, we get the three *second level* hypotheses, to be confronted with the first level hypotheses:

$H_A \cap H_n$: Neither A nor n are significant:
$s_{iA} = s_{in} = a_A = a_n = b_A = b_n = 0 \qquad (i=1,\dots,N).$

$H_A \cap H_\mu$: Neither A nor individual components are significant:
$s_{iA} = a_A = b_A = \sigma^{\mu}_{ij} = 0 \qquad (i=1,\dots,N;\ j=1,\dots,N).$

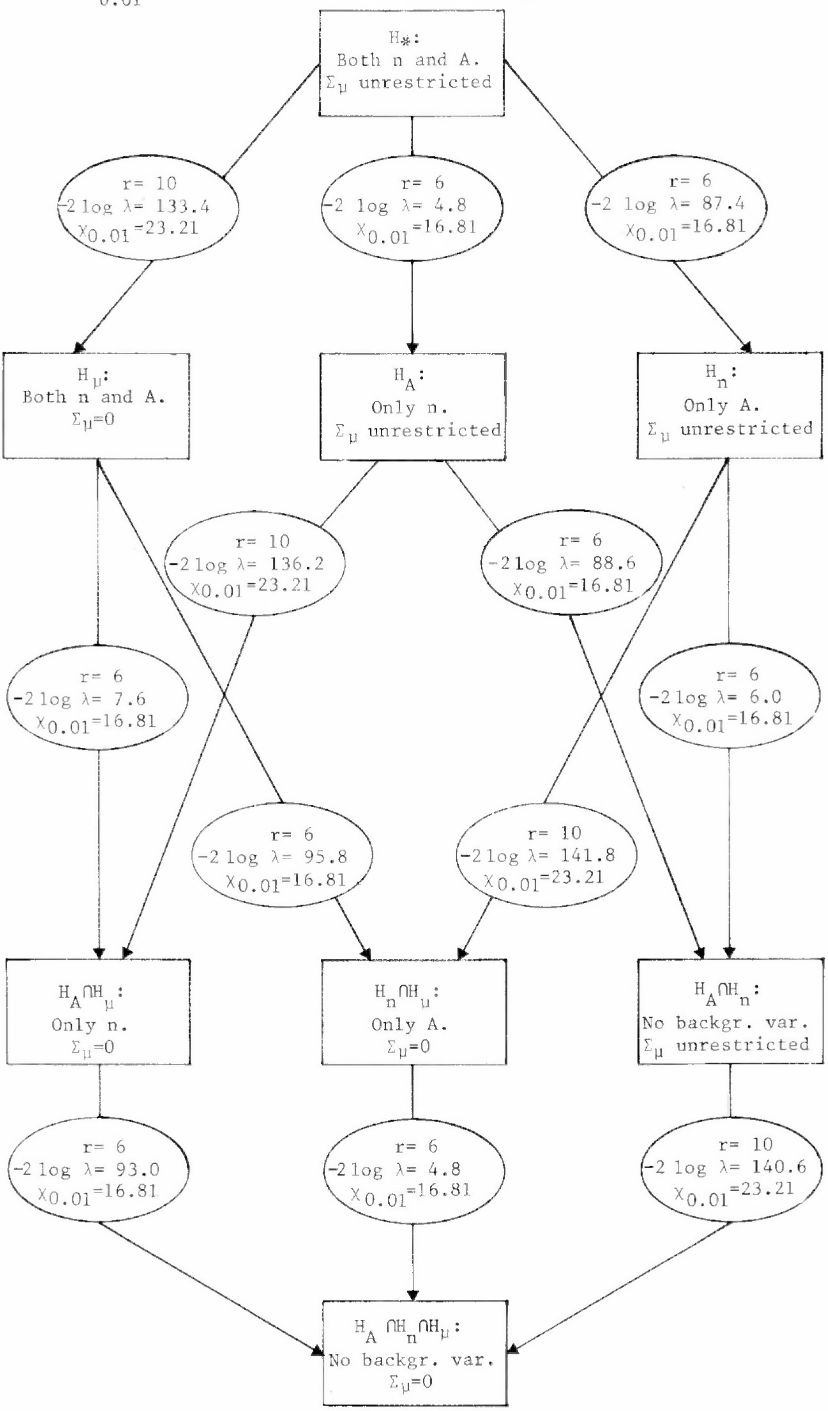
$H_n \cap H_\mu$: Neither n nor individual components are significant:
$s_{in} = a_n = b_n = \sigma^{\mu}_{ij} = 0 \qquad (i=1,\dots,N;\ j=1,\dots,N).$

On the *bottom level*, we find the most restrictive hypothesis:

$H_A \cap H_n \cap H_\mu$: Neither A, nor n, nor individual components are significant:

$$s_{iA} = s_{in} = a_A = a_n = b_A = b_n = \sigma^{\mu}_{ij} = 0 \qquad (i=1,\dots,N;\ j=1,\dots,N).$$

Figure 6.1. Test of model specification in the simultaneous equation model. The testing tree.
A priori restrictions: $\beta = 1$, $t_i \in [0,1]$ $(i = 1,2,\ldots,5)$.
r= no. of restrictions.
$-2 \log \lambda$= Likelihood Ratio test statistic. (λ=Likelihood Ratio).
$\chi_{0.01}$=critical 1 per cent value in the χ^2 (r) distribution



The test tree is illustrated in figure 6.1, with the arrows symbolizing the subtests. We proceed by testing each of the more restrictive hypotheses against the one, or the ones, of which it is a special case, i.e. testing H_A, H_n, and H_μ against H_*, $H_A \cap H_n$ against H_A, etc. This gives a total of 12 subtests. Our test criterion is to *control the probability of making at least one false rejection*, i.e. to reject a restrictive hypothesis against a more general one erroneously at least once. Assigning a 1 per cent probability level to each subtest, the overall level of significance - i.e. the probability of making at least one false rejection - will have an upper bound of 12 per cent.[6)]

Each test is carried out by means of the *Likelihood Ratio* principle from the FIML estimates for the 8 variants of the model reported in table A.2 in the table annex. The method derived from this principle is: Let L_{H0} and L_{H1} be the value of the log-likelihood function corresponding to the FIML estimates of a model under the null hypothesis H_0 and the less restrictive alternative hypothesis H_1, respectively. Then H_0 should be rejected in favour of H_1 if the statistic

$$-2 \log \lambda = -2[L_{H0} - L_{H1}]$$

exceeds a pre-selected fractile in the $\chi^2(r)$ distribution, where r is the number of restrictions imposed in H_0.[7)] In non-linear models, as the one considered here, this test is only asymptotically valid.[8)]

We find that 8 of the 12 subtests lead to rejection. Of the three first level hypotheses, both H_μ and H_n are rejected against H_*. Only H_A passes this test, i.e. the age of the head of the household drops out as explanatory variable in the demand system on the first level of the testing scheme. On the other hand, the more restrictive hypotheses, $H_A \cap H_\mu$ and $H_A \cap H_n$, are both rejected against H_A. Thus, we may say that H_A is the 'optimal', or 'preferred', specification of the model, i.e. our data indicate that *both the number of household members and the unobservable individual factors represented by the* μ_{ih}*'s are significant explanatory variables in consumer demand with a five commodity classification, while the age of the head person is insignificant.* This conclusion

---

6) This follows straightly from basic rules in probability calculus.
7) When determining r we, of course, exclude the adding-up conditions (3.8) and (3.30) and the symmetry conditions $\sigma^\mu_{ij} = \sigma^\mu_{ji}$, which are a priori restrictions in all model variants.
8) For details, see appendix E and Kendall and Stuart (1973).

is strengthened by the fact that in all the four subtests in which the presence of age specific effects is tested in our testing scheme, these effects turn out to be insignificant, whereas all the eight subtests concerning the number of persons or the unobserved individual effects indicate that these effects are highly significant.

6.2. Estimates of the structural coefficients

We now turn to the estimates of the structural coefficients. Our attention will be confined to those specifications which satisfy the constraints $\beta=1$ and $t_i \in [0,1]$, $i=1,\ldots,5$; as we noted in section 6.1, these are restrictions on the price index function which cannot be rejected from our data. Needless to say, the implications of the results emerge only when we take the economic interpretation of the coefficients into account.

We shall be particularly concerned with the sensitivity of the estimates with respect to the specification of background variables in our model. For this reason, we shall consider not only the specification H_A (A excluded), which the multiple test in section 6.1 pointed out as the 'preferred' variant, but also the specifications H_* (both n and A included) and H_n (n excluded). We first focus on the coefficients a and b which characterize the C(u) function, and which are common parameters in all the demand equations (subsection 6.2.1). Then, we discuss the commodity specific parameters s_i and t_i, which - as we saw in chapter 2 - have a 'budget share dimension' (subsection 6.2.2).

6.2.1. The function $C(u) = a \log u + b$

Before interpreting the empirical estimates of a and b, i.e. the coefficients of the C(u) function - in the following frequently to be referred to as the 'income coefficients' - let us consider two important restrictions on this function which follow from the underlying hypothesis of maximizing behaviour. The first restriction,

$$(6.1) \qquad u > C(u) = a \log u + b,$$

is a direct implication of positivity of the marginal utility of income, confer (2.16). The second restriction is that the income flexibility ω is negative.[9] From the expression for ω, given in (2.31), it is easily seen that this is equivalent to imposing the following additional constraint:

9) This follows from the fact that for maximal utility to be attained, the Hessian of the utility function must be negative definite. See e.g. Theil (1975, p. 29).

(6.2) $\quad u > a.$

Together, (6.1) and (6.2) are equivalent to

(6.3) $\quad u > u_o = \max(u_*, a),$

where u_o is the *effective* lower bound on u, for which the model is valid, and $u_* = u_*(a,b)$ is determined by solving the equation

(6.4) $\quad u_* = a \log u_* + b.$

This equation will have two solutions, one solution, or no solution at all, depending on the numerical values of a and b. Assume that $a>0$. If two solutions exist, u_*' and u_*'' ($>u_*'$) say, i.e. if the function $C(u) = a \log u + b$ has two points of intersection with the function u, then the first inequality constraint, (6.1), will not be satisfied for $u \in \{u_*', u_*''\}$. This follows from the fact that C(u) is concave. Figure 6.2 illustrates this situation. Then we have $u_*'' > a > u_*'$, and u_*'' will represent the effective lower bound on u, i.e. $u_o = u_*''$. The converse situation, with no intersection between the functions C(u) and u, is illustrated in figure 6.3. Here, the first inequality restriction, (6.1), will always be satisfied, i.e. only the second, (6.2), is binding: $u_o = a$.

Which constellations of a and b will give a situation with two, one, and no point(s) of intersection, respectively? Consider the difference between the functions C(u) and u. By using simple calculus, we find that this difference attains its maximum, $a \log a + b - a = b - a(1-\log a)$, for $u = a$. Obviously, a situation with two points of intersection arises if this maximum is positive (confer figure 6.2). If the maximum is zero, the function u will be tangent to C(u) for $u = a$, i.e. $u_o = u_* = a$. Finally, if the maximal difference is negative, no point of intersection will exist (confer figure 6.3), i.e. no real root u_* exists. Thus, the joint implication of (6.1) and (6.2) can be summarized as follows:

i) If $b > a(1 - \log a)$, the effective lower bound on u is $u_o = u_*$, where u_* is the largest root of equation (6.4).

ii) If $b \leqq a(1 - \log a)$, the effective lower bound on u is $u_o = a$.

Figure 6.2. The situation with two points of intersection between u and C(u):
a log a + b > a.
$u_o = u''_* > a$

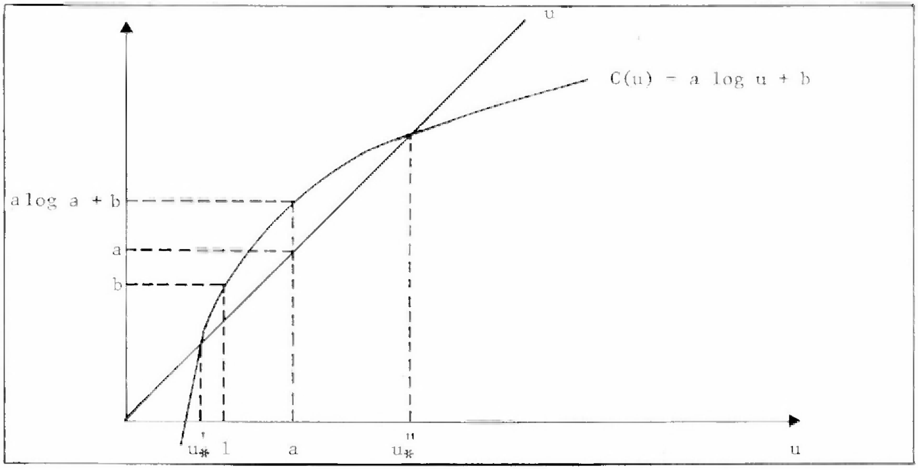


Figure 6.3. The situation with no point of intersection between u and C(u):
a log a + b < a.
$u_o = a$

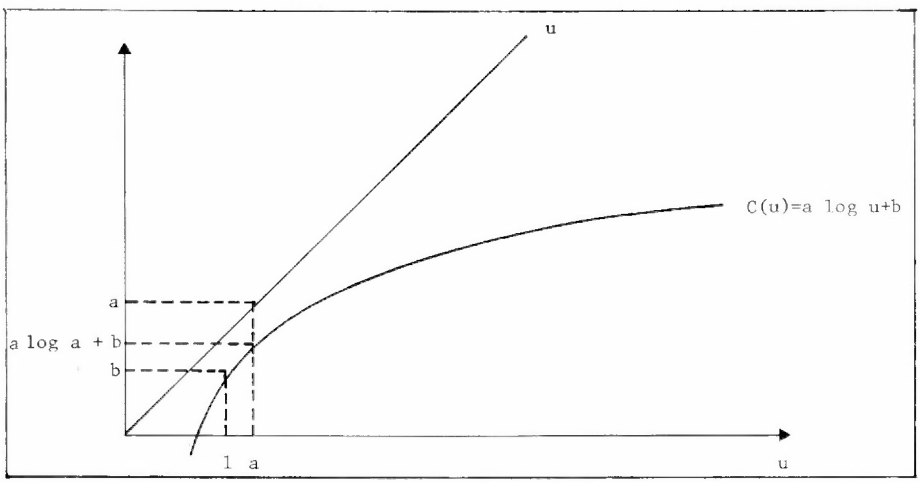


Table 6.1 reports the estimated value of a and b, as well as a(1 - log a), for different specifications of background variables, calculated at the approximate sample averages of n and A, i.e. n=3 and A=50. Observe that the unit of measurement of u=y/P is 10 000 Nkr, at constant 1974 prices.[10] This transformation affects the numerical estimate of the C(u) function but, of course, not its economic implications. The sample mean estimates satisfy b > a(1 - log a) in all alternatives, i.e. the situation is as in case i) above, and the effective lower bound on u, u_o, is the maximal root of (6.4). These estimates of u_o (= u_*) are (after multiplication by 10 000):

Background variables included in the model	Sample mean estimate of $u_o = u_*$, Nkr, measured at constant 1974 prices
n and A	7 828
n	8 064
A	17 537
None	17 006

The results are fairly sensitive with respect to the underlying specification of background variables.

The effects on the income coefficients and the estimated lower bound u_o of including background variables into the model are further analyzed in table 6.2. We find that both a and b increase with the household size, whereas a decreases and b is nearly constant (but increasing) with increasing age of the household head. The estimated value of b - a(1 - log a) is negative for a one person household (n=1) - i.e. we have a situation like the one depicted in figure 6.3 - whereas b - a(1 - log a) is positive for n=3 and 5 - i.e. for these households, the situation is like the one depicted in figure 6.2.[11] The resulting estimate of the lower bound on u, u_o, is sensitive with respect to household size - it is roughly three times as large for a five person household as for a one person household - but does not show much variation with the age of the household head.

---

10) Note also that the value of the price index P is not the same in these four alternatives, owing to the differences in the estimates of the underlying weights, cf. table 6.4 below. Constancy of u across alternatives does not imply constancy of y, and *vice versa*.

11) Simple interpolation indicates that the critical value of n, i.e. the value at which the effective restriction on u switches from (6.2) to (6.1), is about 2.7 persons for A=30 years, decreasing to approximately 1.6 persons for A=70 years.

Table 6.1. Sample mean estimates of the income coefficients, a and b,[a)] derived from the FIML estimates of the 5 commodity model with different specifications of background variables. A priori restrictions: $\beta=1$ and $t_i \in [0,1]$, $i=1,2,\dots,5$.[b)] (Σ_μ unrestricted)

Coefficients, etc.	Background variables included			
	n and A	n	A	None
a	0.4403	0.4196	0.8518	0.7786
b	0.8906	0.8967	1.2752	1.2872
a(1 - log a)	0.8015	0.7840	0.9884	0.9735

a) $a = a_o + a_n n + a_A A$ and $b = b_o + b_n n + b_A A$. a and b are evaluated at the *approximate* sample mean values of the background variables, n = 3 and A = 50. The unit of measurement of u is 10 000 Nkr.
b) Confer table A.2 of the table annex.

Table 6.2. Estimates of a and b and corresponding estimates of the lower bound on real income, u_o, derived from FIML estimates of the model with both background variables included.[a)] A priori restrictions: $\beta = 1$ and $t_i \in [0,1]$, $i=1,\dots,5$.[b)] (Σ_μ unrestricted)

Value of background variables		a	b	b - a(1 - log a)	Effective lower bound on u, u_o, Nkr[c)]
A = 30	n=1	0.4532	0.6183	-0.1936	4 532
	n=3	0.5024	0.8859	0.0377	7 228
	n=5	0.5516	1.1536	0.2738	12 971
A = 50	n=1	0.3911	0.6230	-0.1353	3 911
	n=3	0.4403	0.8906	0.0891	7 828
	n=5	0.4895	1.1583	0.3191	12 786
A = 70	n=1	0.3290	0.6277	-0.0670	3 290
	n=3	0.3782	0.8953	0.1494	8 205
	n=5	0.4274	1.1630	0.3723	12 627

a) $a = a_o + a_n n + a_A A$ and $b = b_o + b_n n + b_A A$. The unit of measurement of u is 10 000 Nkr.
b) Confer table A.2 of the table annex.
c) The figure in this column is equal to 10 000 times the maximal root of (6.4) if b - a(1 - log a) is non-negative, and equal to 10 000 a if b - a(1 - log a) is negative.

Approximate standard errors of the estimated structural coefficients are given in table A.2 of the table annex. From the first column of this table we find that the 'constant terms' in a and b, i.e. a_o and b_o, are both significantly different from zero (at the five per cent level), whereas a_n, a_A and b_A are insignificant. The only significant effect of background variables on the C(u) function goes through the coefficient b_n. Since at least one of the coefficients of $\underline{a} = a_o + a_n n + a_A A$ is significantly different from zero, we conclude that the hypothesis a = 0 (or rather $a_o = a_n = a_A = 0$) is rejected. *Our data thus lead to rejection of the linear expenditure system (LES)* as an appropriate parametrization, since that model is characterized by a = 0 (and $\beta = 1$); cf. section 2.3.

6.2.2. The commodity specific coefficients, s_i and t_i

If we compare the estimates obtained for the remaining structural coefficients – i.e. s_i and t_i – we find notable differences in their stability with respect to the specification of background variables in the model.

In table 6.3, we have recorded the sample mean estimates of s_i, i.e. the estimates for n=3 and A=50. Conditional on these values of the background variables, the estimates are fairly stable across the different model variants. From chapter 2 we know that s_i may be interpreted as the *asymptotic* budget share of commodity i:

$$\lim_{u\to\infty} w_i = s_i \qquad (i=1,\ldots,5). \tag{6.5}$$

This motivates putting the s_i estimates along with the sample means of the budget shares, i.e. the *average* budget shares $\bar{w}_i$, in table 6.3. The most striking observation is that the asymptotic budget share of Food, beverages and tobacco (commodity 1) is only about one third of the sample mean of its observed average budget share, whereas s_4 for Travel and recreation (commodity 4) exceeds the corresponding sample average, $\bar{w}_4$, by 65-75 per cent.

Table 6.3. Sample mean estimates of the demand coefficients s_i [a)] derived from the FIML estimates of the 5 commodity model, with different specifications of background variables. A priori restrictions: $\beta = 1$ and $t_i \in [0,1]$, $i=1,2,\ldots,5$. [b)] (Σ_μ unrestricted)

Coefficient	Background variables included				Sample mean of budget share, $\bar{w}_i$
	n and A	n	A	None	
s_1	0.0848	0.0896	0.1217	0.1366	0.2975
s_2	0.1182	0.1173	0.1201	0.1177	0.0930
s_3	0.2063	0.2070	0.1839	0.1880	0.2581
s_4	0.4706	0.4675	0.4633	0.4488	0.2712
s_5	0.1200	0.1186	0.1100	0.1089	0.0803

a) $s_i = s_{io} + s_{in}n + s_{iA}A$ is evaluated at the *approximate* sample mean values of the background variables, n=3 and A=50.
b) Confer table A.2 of the table annex.

Table 6.4 displays the estimated t_i coefficients, which in the model specifications under scrutiny represent the weights assigned to p_j (j=1,...,5) in the price index function.[12)] Again, the average budget shares in the sample may serve as reference points since such shares are often used to define the weights in simple price index functions. The annual averages of the price indices P in the years 1975, 1976, and 1977, based on these alternative sets of weights are recorded in the bottom section of table 6.4. The maximal difference between the indices varies from 1.0 to 1.5 per cent.

Comparing tables 6.3 and 6.4, we find that the ratio between t_i and $\bar{w}_i$ is negatively correlated with the ratio between s_i and $\bar{w}_i$ (i=1,2,..,5). The largest discrepancy between t_i and $\bar{w}_i$ is found for commodities 1 and 4, and the sign of $(t_i - \bar{w}_i)$ is in all model variants the opposite of that of $(s_i - \bar{w}_i)$ for all i. If we compare the model variants in which the estimated value of t_4 is positive with those where t_4 attains its lower bound, we observe that the weight of commodity 1 increases with more than the "lost" weight of commodity 4.

12) Or to be precise, the weights in the *basic year* of the price index function, i.e. when $p_1 = p_2 = \ldots = p_5 = P = 1$.

Table 6.4. FIML estimates of the price index coefficients t_i, average budget shares, and corresponding annual averages of the price index P.
A priori restrictions: $\beta=1$, $t_i \in [0,1]$, $i=1,\dots,5$.
(Σ_μ unrestricted)[a)b)]

Coefficient	Background variables included				Sample mean of budget share, $\bar{w}_i$
	n and A	n	A	None	
t_1	0.5727	0.5715	0.3644	0.3609	0.2975
t_2	0.0607	0.0619	0.0851	0.0859	0.0930
t_3	0.3415	0.3397	0.2944	0.2948	0.2581
t_4	0	0	0.1909	0.1936	0.2712
t_5	0.0250	0.0269	0.0652	0.0648	0.0803
Annual average of price index P (1974=100)[c)]					
1975	113.20	113.21	112.13	112.11	111.77
1976	123.82	123.83	122.44	122.41	122.00
1977	134.31	134.33	133.29	133.26	132.99

a) Since $\beta=1$, the price index function is simply $P = \Sigma_j t_j p_j$.
b) Confer table A.2 of the table annex.
c) The price index P is calculated as $P = \Sigma_j \bar{w}_j p_j$.

The main results concerning the impact of the background variables on the coefficients s_i are summarized in table 6.5. An increase in household size leads to an increase in the asymptotic budget share of commodity groups 1 Food, beverages, and tobacco, 2 Clothing and footwear, and 4 Travel and recreation and a decrease in the asymptotic budget share of commodity groups 3 Housing, fuel, and furniture and 5 Other goods and services. Similarly, the asymptotic budget share increases with the age of the household head for commodity groups 1 Food, beverages, and tobacco, 2 Clothing and footwear, and 5 Other goods and services and decreases for commodity groups 3 Housing, fuel, and furniture and 4 Travel and recreation.

Tabell 6.5. Sign of the estimated effect of the background variables on the asymptotic budget shares s_i.
A priori restrictions: $\beta=1$ and $t_i \in [0,1]$.
(Σ_μ unrestricted)[a]

Structural coefficient	Sign of the coefficient of background variables	
	n	A
$s_1 = s_{1o} + s_{1n}n + s_{1A}A$	+	+
$s_2 = s_{2o} + s_{2n}n + s_{2A}A$	+	+
$s_3 = s_{3o} + s_{3n}n + s_{3A}A$	-	-
$s_4 = s_{4o} + s_{4n}n + s_{4A}A$	+	-
$s_5 = s_{5o} + s_{5n}n + s_{5A}A$	-	+

a) This table is based on the FIML estimates of the model variant which includes both background variables simultaneously, i.e. the first column of table A.2 of the table annex.

The approximate standard errors of the s_i and t_i coefficients - recorded in table A.2 of the table annex - reveal that the t_i coefficients and the 'constant terms' of s_i (s_{io}) are in general sharply determined in the model specifications which include background variables. The only exceptions are the estimates of s_{1o} and t_4, which are not significantly different from zero at the 5 per cent level. Among the coefficients which represent the effects of the background variables, only one, s_{3n}, is significantly different from zero (and negative).

6.3. Empirical impact of the individual disturbance components

In section 6.1, we concluded that the unobservable individual factors represented by the individual disturbance components μ_{ih} are significant factors in explaining observed differences in consumption (cf. figure 6.1). We shall now examine the empirical impact of these unobservable individual factors further. First, we present the FIML estimates of the 'total' disturbance covariance matrix Σ and the 'individual' matrix Σ_μ (subsection 6.3.1), and then discuss briefly the effect on the estimated structural coefficients of neglecting individual disturbance components, i.e. imposing $\Sigma_\mu = 0$ *a priori* (subsection 6.3.2).

6.3.1. FIML estimates of the disturbance covariance matrices

The complete set of FIML estimates of the disturbance covariances for the model specifications with $\beta=1$ and $t_i\in[0,1]$ imposed *a priori*, is reported in tables A.4–A.5 of the table annex. The results for the model variant with n as the only background variable – i.e. the 'preferred' specification according to our multiple test in section 6.1 – are collected in table 6.6. We find that a substantial part of the total disturbance covariances may be attributed to the unobserved individual factors, μ_{ih}.

There is nothing in our method of estimation that ensures a reduction in the *total* disturbance covariances when we take account of individual specific error components. Indeed, despite our clear rejection of the hypothesis $\Sigma_\mu = 0$, the estimates of the total disturbance variances and covariances are left virtually unchanged when we introduce this restriction in the model, see table A.5. The importance of the error component specification thus mainly lies in the increased knowledge it gives about the structure of the second order moments of the disturbances.

Table 6.6. FIML estimates of the 'total' disturbance covariance matrix $\Sigma = (\sigma_{ij})$ and the 'individual' matrix $\Sigma_\mu = (\sigma^\mu_{ij})$ for the preferred model variant, i.e. the model with n as the only background variable[a].
A priori restrictions: $\beta=1$, $t_i\in[0,1]$

$\hat{\sigma}_{ij}\cdot 10^2$

i	j				
	1	2	3	4	5
1	1.397				
2	-0.112	0.867			
3	-0.376	-0.196	1.940		
4	-0.795	-0.486	-1.151	2.771	
5	-0.114	-0.077	-0.217	-0.343	0.751

$\sigma^\mu_{ij}\cdot 10^2$

i	j				
	1	2	3	4	5
1	0.562				
2	-0.007	0.212			
3	-0.143	-0.015	0.351		
4	-0.358	-0.146	-0.228	0.816	
5	-0.054	-0.044	0.035	-0.084	0.147

a) Confer table A.4 of the table annex.

Estimates of the part of the disturbance variance which is due the individual component (ρ_i), are shown in table 6.7 for different model specifications. For all commodity groups, we observe that this share is not affected very much by the inclusion of background variables in the model, i.e. the numerator (σ^{μ}_{ii}) and the denominator (σ_{ii}) change by roughly the same proportion. The total disturbance variance decreases for all commodity groups (see table A.4), in particular when family size is included. If we rank the commodities according to the size of ρ_i, we find that commodity group 1 Food, beverages, and tobacco comes on top, with the individual disturbance component accounting for about 40 per cent of the total disturbance variance. The lowest ρ_i (18 per cent) is found for commodity group 3 Housing, fuel, and furniture.

Table 6.7. Part of the disturbance variance which is due to the individual component: $\rho_i = \sigma^{\mu}_{ii}/\sigma_{ii}$.a)
A priori restrictions: $\beta=1$, $t_i \in [0,1]$

	Background variables included			
	Both n and A	Only n	Only A	None
ρ_1	0.397	0.402	0.414	0.412
ρ_2	0.245	0.244	0.245	0.246
ρ_3	0.180	0.181	0.181	0.180
ρ_4	0.297	0.294	0.294	0.286
ρ_5	0.194	0.196	0.202	0.201

a) Calculated from table A.4 of the table annex.

6.3.2. Effects of the restriction $\Sigma_{\mu} = 0$ on the structural coefficient estimates

We are also in a position to compare the estimates of the structural coefficients of the model in the case where individual disturbance components are accounted for ($\Sigma_{\mu} \neq 0$), with those obtained when individual differences are ruled out *a priori* ($\Sigma_{\mu} = 0$). Table 6.8 contains the results for the 'preferred' model specification, i.e. the one with $\beta=1$, $t_i \in [0,1]$, for all i, and n as the only background variable included. Among those coefficients which have a comparable scaling (i.e. the t_i and s_i coefficients), it is the elements of s_1 and s_4 (i.e. s_{10}, s_{1n} and s_{40}, s_{4n}) which are most sensitive. This is not surprising in view of the fact that the individual component in the covariance between the disturbances of commodity 1 Food, beverages, and tobacco, and commodity 4 Travel and recreation, $\hat{\sigma}^{\mu}_{41}$ = −0.358, is by far the largest in absolute value among the off-diagonal element of $\hat{\Sigma}_{\mu}$ (table 6.6). We observe, however, that the changes in s_{io} (i = 1 and 4) are approximately compensated by the changes in s_{in} (i = 1 and 4) for values of n close to sample average.

Table 6.8. Effect of the restriction $\Sigma_{\mu} = 0$ on the FIML coefficient estimates in the model variant with n included as the only background variable.
A priori restrictions: $\beta=1$, $t_i \in [0,1]$[a)]

Coefficient	Σ_{μ} unrestricted	$\Sigma_{\mu} = 0$
s_{1o}	0.0577	0.0823
$s_{1n} \cdot 10^2$	1.0647	0.8014
s_{2o}	0.0939	0.0943
$s_{2n} \cdot 10^2$	0.7798	0.7738
s_{3o}	0.2749	0.2749
$s_{3n} \cdot 10^2$	−2.2637	−2.2202
s_{4o}	0.4452	0.4190
$s_{4n} \cdot 10^2$	0.7428	0.9541
s_{5o}	0.1283	0.1295
$s_{5n} \cdot 10^2$	−0.3236	−0.3091
a_o	0.3029	0.2569
$a_n \cdot 10^2$	3.8914	4.6449
b_o	0.5102	0.4929
$b_n \cdot 10^2$	12.883	11.351
t_1	0.5715	0.5779
t_2	0.0619	0.0570
t_3	0.3397	0.3458
t_4	0	0
t_5	0.0269	0.0193

a) Extracted from table A.1 of the table annex.

On the whole, the sensitivity of the estimated structural coefficients with respect to the specification of the covariance structure is smaller than might be expected. This may be taken as a support to our maintained hypothesis of zero correlation between the structural explanatory variables - income, age, and household size - and the unobservable individual factors represented by the individual component of the disturbances (cf. section 3.4). The implications of these changes for the demand elasticities are further discussed in subsection 6.4.3 below.

6.4. The demand elasticities and the estimated budget shares

Like the budget shares, the demand elasticities are functions of the explanatory variables, real income and real prices, as well as the background variables, family size and age of the household head. We shall in this section discuss main properties of these functions, confining, as before, attention to those specifications where $\beta=1$ and $t_i \in [0,1]$ are imposed as a priori restrictions on the price index function P.

When $\beta=1$, the expressions for the demand elasticities, given in eqs. (2.27)-(2.30), can be simplified substantially. The Engel elasticity of commodity i becomes[13)]

$$(6.6)\qquad E_i = \frac{s_i u + (t_i v_i - s_i)\, a}{s_i u + (t_i v_i - s_i)\,(a \log u + b)} \qquad (i=1,\dots,N)$$

and the Cournot and Slutsky elasticities of commodity i with respect to the price of commodity j are, respectively,

$$(6.7)\qquad e_{ij} = S_{ii}^{*}(\delta_{ij} - \pi_j) - E_i \pi_j ,$$

$$(6.8)\qquad S_{ij} = S_{ii}^{*}(\delta_{ij} - \pi_j) + E_i (w_j - \pi_j) \qquad (i=1,\dots,N;\ j=1,\dots,N),$$

where π_j, the elasticity of the price index function P with respect to p_j, is now simply equal to

$$(6.9)\qquad \pi_j = \frac{\partial P}{\partial p_j}\frac{p_j}{P} = t_j v_j \qquad (j=1,\dots,N),$$

$\delta_{ij} = 1$ for i=j, and 0 otherwise, and

13) Recall that s_i, a, and b are functions of the background variables.

$$(6.10)\quad S_{ii}^{*} = \frac{-s_i(u - a\log u - b)}{t_i v_i(a\log u + b) + s_i(u - a\log u - b)} \qquad (i=1,\ldots,N).$$

We note that

$$(6.11)\quad -1 \leq S_{ii}^{*} < 0 \qquad (i=1,\ldots,N)$$

when the inequalities $u > a \log u + b$ and $s_i > 0$ $(i=1,\ldots,N)$ are satisfied (cf. (2.16) and (2.17)). S_{ii}^{*} attains its lower bound, -1, if and only if t_i attains its lower bound, zero. Thus, our constraints on the price index function influence the possible range of variation of the price elasticities e_{ij} and S_{ij}. This should be kept in mind when considering the following estimates.

6.4.1. Demand elasticity estimates

In table 6.9, we have summarized the results for the Engel elasticities, the direct Cournot, and the direct Slutsky elasticities. We have also included the estimated budget shares for the five aggregated commodity groups, along with their corresponding sample means. All the estimates are calculated at the sample mean value of the (relative) prices and at the approximate sample mean values of the other explanatory variables: u=50 000 Nkr (at constant 1974 prices), n=3 persons, and A=50 years.

Table 6.9. Estimates of budget shares, Engel, direct Cournot, and direct Slutsky elasticities based on simultaneous estimation of the five commodity model.
n=3 persons, A=50 years, u=50 000 Nkr, and all prices equal to their sample means[a].
A priori restrictions: $\beta=1$, $t_i \in [0,1]$.
(Σ_μ unrestricted)

Commodity group (i)	Engel elasticity E_i	Direct Cournot elasticity e_{ii}	Direct Slutsky elasticity S_{ii}	Estimated budget share, per cent w_i	Sample mean of budget share, per cent[b]
1 Food, beverages and tobacco ...	0.527	-0.406	-0.277	24.46	29.75
2 Clothing and footwear	1.144	-0.834	-0.721	9.86	9.30
3 Housing, fuel and furniture .	0.881	-0.673	-0.455	24.68	25.81
4 Travel and recreation	1.341	-1.000	-0.571	32.01	27.12
5 Other goods and services ..	1.243	-0.917	-0.805	8.98	8.02

a) The estimates refer to the model variant with both n and A included, confer tables A.7, A.9, and A.16.
b) Confer table 5.1.

We find that the *estimated budget shares* of the commodity groups 1 Food, beverages and tobacco and 4 Travel and recreation show large deviations - about 5 percentage units - from their respective sample means. It is no accident that it is for the same commodities that we found the largest discrepancies between the observed sample means of budget shares and the corresponding asymptotic ones, s_i (cf. table 6.3). For the other three commodities, the differences between the observed sample means of the budget shares and their estimates based on the demand system are much smaller.

Among the five commodity groups, Food, beverages and tobacco comes out with the lowest estimate for the *Engel elasticity*, 0.527. This is in accordance with the majority of previous estimates for food alone. It is of particular interest to compare our estimate with the ones reported for food by Nasse (1973), since he uses the same parametric specification of the Fourgeaud-Nataf demand model as the one considered here.[14] Using aggregate national accounts data from France 1949 - 1969, Nasse obtained an estimate of the Engel elasticity of food equal to 0.75 (calculated at sample mean values of real income and prices). Nasse also reports an alternative estimate of 0.35, based on data in the form of cell means from a family expenditure survey from 1965.

Other studies - based on cross-sectional data and single equation estimation methods, e.g. Prais and Houthakker (1955), Bojer (1977), and Biørn (1978), as well as studies based on aggregate time series data and simultaneous equation estimation techniques, e.g. Deaton (1974, 1975a) - all yield higher estimates of Engel elasticities for tobacco and beverages than for food alone. We may therefore conclude that our estimate for the composite commodity Food, beverages and tobacco are relatively low.

Commodity group 3 Housing, fuel and furniture is also characterized as a necessity by an estimated Engel elasticity of 0.881. The remaining three commodity groups are all "luxuries" with estimated Engel elasticities above unity. Our estimated Engel elasticities are in remarkable accordance with those reported in Salvas-Bronsard (1978). She estimated a double-logarithmic demand system comprising six commodity groups from two vintages of French family expenditure surveys (1965 and 1972), but, like Nasse, she used group means instead of the primary individual observations. Her estimates of the Engel elasticities, calculated at the approximate sample mean values of the explanatory variables, were:

14) Unfortunately, food is the only commodity group in Nasse (1973) which is comparable to our aggregate commodity groups. Nasse considered four commodity groups: food, durables, semi-durables, and non-durables other than food.

Food	0.63
Clothing	1.14
Housing	0.60
Personal and medical care ..	1.33
Transport and communications	1.65
Recreation and others	1.64

The estimated *own price elasticities* are all negative and less than (or equal to) one in absolute value, i.e. the demand is inelastic for all commodity groups. The ranking of the commodity groups, by increasing absolute value of the Cournot elasticities is identical to the ranking by increasing value of the Engel elasticity. From eq. (6.7) we observe that the direct Cournot elasticity of commodity i, e_{ii}, is a weighted average of S_{ii}^* and $-E_i$, with non-negative weights equal to $(1-t_i v_i)$ and $t_i v_i$, respectively. Even though our restrictions on the structural coefficients, $\beta=1$ and $t_i \in [0,1]$, imply limits to the range of S_{ii}^*, cf. (6.11), *none of the above conclusions are implied by the a priori restrictions.*

Considering the specific estimates, it may be argued that the demand for the composite commodity group 1 is surprisingly price-inelastic ($e_{11} = -0.41$). Again, this estimate is close to the one obtained by Nasse (1973) for food alone, -0.37,[15)] whereas other studies, based on aggregate time-series data (e.g. Deaton (1974), using the LES and the Rotterdam model) indicate that the demand for tobacco and - in particular - beverages is more price sensitive. As far as the other commodity groups are concerned, it should be noted that our estimate of -1 for the price elasticity of Travel and recreation follows trivially from the estimate of t_4, which is equal to its lower bound, zero.

The direct Slutsky elasticities are also all negative - as they should be to concur with utility maximizing behaviour. Since no commodity group is inferior ($E_i < 0$), the direct Slutsky elasticity is always smaller (in absolute value) than the corresponding direct Cournot elasticity.

---

15) Based on time-series data from the French national accounts, 1949 - 1969.

The complete set of cross-price (indirect) Cournot elasticities is given in table 6.10. All uncompensated cross-price elasticities turn out to be non-positive. That is, all commodity groups are substitutes in demand, except that the demand for commodity groups 1, 2, 3, and 5 is independent of the price of commodity group 4 Travel and recreation. This result, which is not plausible, follows immediately from the fact that the estimated t_4 is zero, which implies that the price of Travel and recreation does not enter the budget share functions of the other commodity groups.

Table 6.10. Estimates of the cross-price Cournot elasticities e_{ij} (i.e. uncompensated cross-price elasticities), based on simultaneous estimation of the five commodity model. n=3 persons, A=50 years, u=50 000 Nkr, and all prices set equal to their sample means.a)
A priori restrictions: $\beta=1$, $t_i \in [0,1]$.
(Σ_μ unrestricted)

Commodity group (i)	Commodity group (j)				
	1	2	3	4	5
1 Food, beverages and tobacco		-0.017	-0.097	0.000	-0.007
2 Clothing and footwear	-0.192		-0.109	0.000	-0.008
3 Housing, fuel and furniture	-0.182	-0.018		0.000	-0.008
4 Travel and recreation	-0.199	-0.019	-0.113		-0.009
5 Other goods and services	-0.196	-0.019	-0.111	0.000	

a) The estimates refer to the model variant with n and A included, confer table A.16, part ii).

From eqs. (6.7) and (6.9) it follows that the expression for the cross-price elasticities can be written

$$(6.12) \quad e_{ij} = -t_j v_j (S_{ii}^* + E_i), \qquad i \neq j, \quad (i=1,\ldots N;\ j=1,\ldots,N).$$

Since $t_j v_j$ is always non-negative, the sign of the non-zero cross-price elasticities follows from the fact that the absolute value of the estimate of S_{ii}^* is less than that of E_i for all commodities at the sample point we are considering. Moreover, the *difference* between E_i and $|S_{ii}^*|$ is nearly constant across commodity groups:

Commodity group	$E_i - \lvert S_{ii}^* \rvert$
1 Food, beverages and tobacco	0.291
2 Clothing and footwear	0.328
3 Housing, fuel and furniture	0.312
4 Travel and recreation	0.341
5 Other goods and services	0.335

This explains our finding in table 6.10 that the cross-price Cournot elasticities e_{ij} are fairly insensitive with respect to the subscript i for a given subscript j.[16] Again, these results *are not implied* by the *a priori* restrictions imposed on the demand system.

The discussion so far has been confined to the model variant with both family size and age of household head included as explanatory variables. If we compare the results in table 6.9 with those for other model variants - i.e. with none or only one background variable included, see table A.7 of the table annex - we find that the Engel elasticity estimates are fairly stable across the different specifications, when evaluated at the sample means of the explanatory variables. The estimated Cournot elasticities, however, vary. This reflects primarily the differences between the estimated t_i coefficients from one model variant to another - the crucial point being whether the t_i coefficients are all non-zero or not.[17]

6.4.2. The demand elasticities and the estimated budget shares as functions of the explanatory variables

In this subsection, we shall describe the variation of the estimated budget shares and demand elasticities with real income and the background variables, household size and age of the main income earner. In other words, we shall consider these functions for given values of the relative prices, and we set these values equal to their sample means. When $\beta=1$, the relative price of commodity group i, v_i, enters the various demand elasticity functions in complete symmetry with the coefficient t_i. Tables A.7 and A.8 of the table annex reveal that a reduction in the own price by 15 per cent from its sample mean, all other prices kept constant, causes positive shifts in the Engel elasticity functions for all commodities which have a strictly positive t_i coefficient. The same experiment

16) At other sample points, the differences between $\lvert S_{ii}^* \rvert$ and E_i vary more between the commodity groups.
17) Confer tables A.11-A.16 of the table annex.

gives more ambiguous results for the estimated price responses. The sign of the shift in the Slutsky and the Cournot elasticities, expressed as functions of real income, is different for different combinations of the background variables.

We have chosen to describe the main results concerning the estimated budget share and demand elasticity functions *graphically*. Figures 6.5 - 6.23 contain such functions for real incomes u between 20 000 Nkr and 100 000 Nkr, measured at constant 1974 prices. The household sizes considered are n=1, 3, and 5 persons, and the age of the main income earner is alternatively set equal to A=30, 50, and 70 years.[18] In figure 6.4, we have assembled the budget share functions for all commodities as estimated at the sample mean values of the background variables. We observe that the estimated budget shares for 1 Food, beverages and tobacco and 3 Housing, fuel and furniture decrease with increasing real income; for the other commodities it increases.

The budget share functions approach the estimated value of s_i asymptotically when real income goes to infinity, cf. eq. (6.5) and table 6.3 above. In appendix A, we have derived the corresponding asymptotes of the demand elasticity functions:

$$(6.13)\quad \lim_{u\to\infty} E_i = 1 \qquad (i=1,\dots,N),$$

$$(6.14)\quad \lim_{u\to\infty} e_{ii} = -1 \qquad (i=1,\dots,N),$$

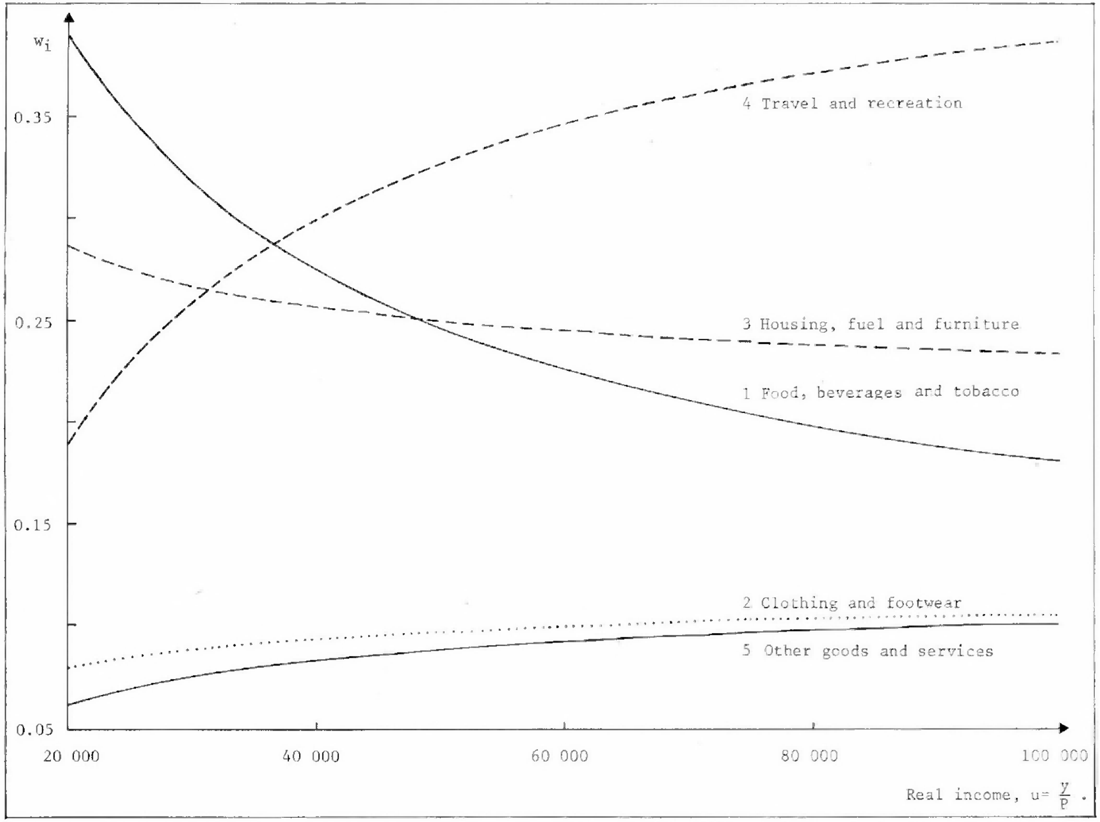
$$(6.15)\quad \lim_{u\to\infty} S_{ii} = s_i-1 \qquad (i=1,\dots,N).$$

The reader should keep these asymptotic values in mind when interpreting the figures drawn for each commodity group.

---

18) For the sake of clarity the combinations (n=1, A=30), (n=1, A=70), (n=5, A=30) and (n=5, A=70) are left out in most figures, i.e. when varying A, we confine attention to household size n=3 only. Figures 6.6 and 6.7, however, are exceptions to this rule.

Figure 6.4. Estimated budget shares for the five aggregated commodity groups, as functions of real income.
n = 3 persons and A = 50 years.
A priori restrictions: $\beta = 1$, $t_i \in [0,1]$



The budget share function for commodity group 1 *Food, beverages and tobacco* (figure 6.5) exhibits positive shifts with increasing age of the main income earner, the magnitude of the shifts being increasing with increasing real income. An increase in household size causes a large shift in the budget share function. All budget share functions are declining in income, which, of course, reflect the fact that the commodity is a necessity ($E_i < 1$). The Engel elasticity (figure 6.6) is first a decreasing and then an increasing function of real income, and its turning point occurs for lower real income the larger is the household size and the higher is the age of the main income earner. Increasing age causes a positive shift in the Engel elasticity function for all real incomes above 30 000 Nkr, whereas increasing household size yields a positive shift in the Engel elasticity for real incomes above 35 000 Nkr.

Similar complex pictures are revealed for the direct Cournot and Slutsky elasticity functions in figures 6.7 and 6.8. The direct Cournot elasticity function is decreasing with increasing real income, one person households (n=1) with real incomes below a certain level being the only exception. Increasing household size causes negative shifts in the Cournot elasticities for real incomes above approximately 40 000 Nkr, whereas increasing age of the main income earner is accompanied by negative shifts for all real incomes considered (figure 6.7). The direct Slutsky elasticity is a monotonically decreasing function of real incomes, whereas the shifts induced in the function by increasing n and A are analogous to the ones observed for the Cournot elasticity function (figure 6.8).

Within the range of the real incomes we consider, the estimated budget share function for commodity group 2 *Clothing and footwear* in figure 6.9 shows uniformly positive shifts for increasing age of the main income earner, whereas increasing household size shifts the function upwards for real incomes above 30 000 Nkr. The Engel elasticity function for this commodity group is monotonically decreasing with increasing real incomes, and it shifts markedly upwards with increasing household size, while being nearly independent of the age of the main income earner, see figure 6.10. The price elasticity functions in figures 6.11 and 6.12 are both monotonically decreasing with real income. An increase in A or a decrease in n give both rise to negative shifts in this function, i.e. they increase the price sensitivity of this commodity group.

Figure 6.5. Estimated budget share for commodity group 1 Food, beverages and tobacco, as a function of real income.
n = 1,3,5 persons. A = 30,50,70 years.
A priori restrictions: $\beta = 1$, $t_i \in [0,1]$

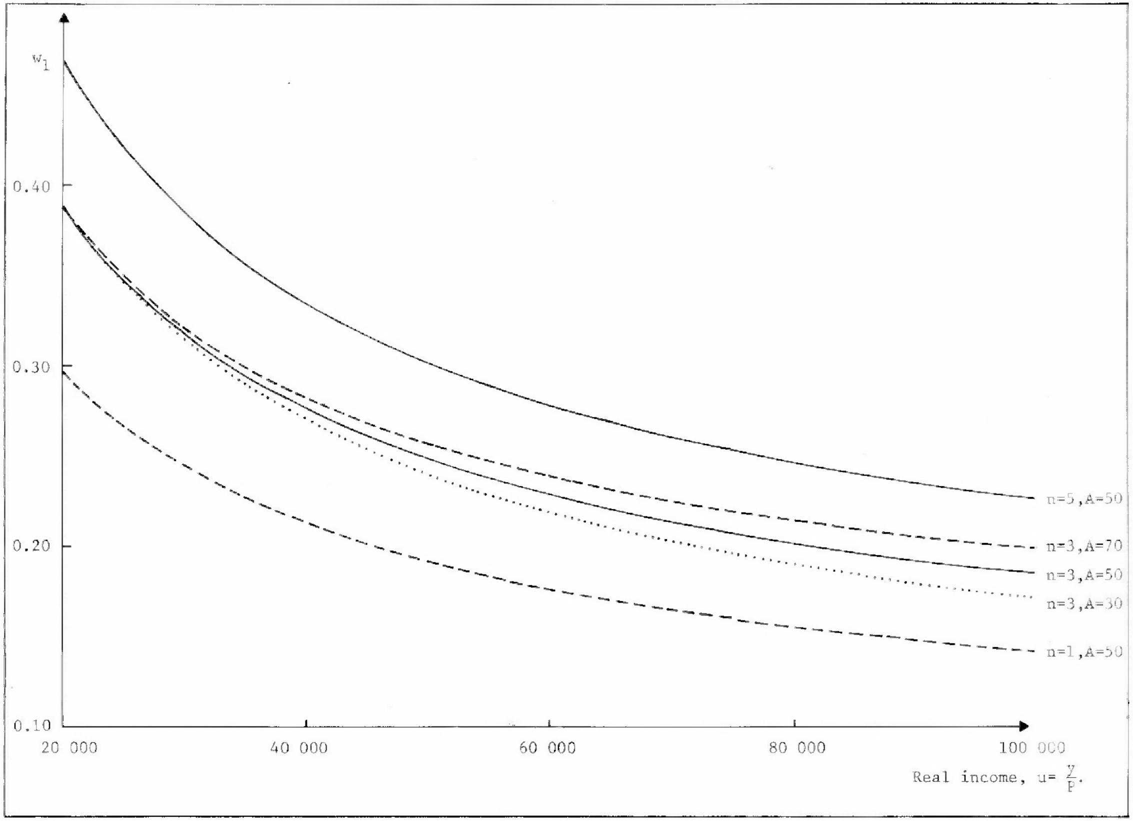


Figure 6.6. Estimated Engel elasticity for commodity group 1 Food, beverages and tobacco, as a function of real income.
n = 1,3,5 persons. A = 30,50,70 years.
A priori restrictions: $\beta = 1$, $t_i \in [0,1]$

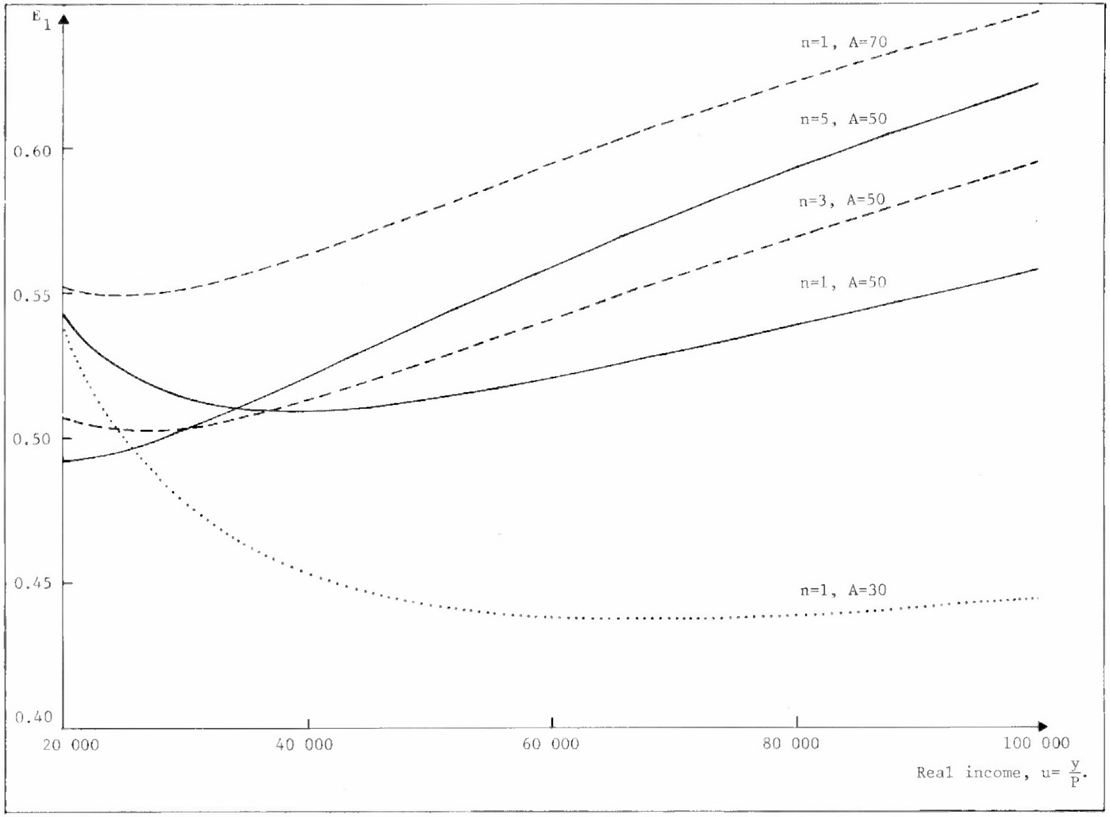


Figure 6.7. Estimated direct Cournot elasticity for commodity group 1 Food, beverages and tobacco, as a function of real income.
n = 1,3,5 persons. A = 30,50,70 years.
A priori restrictions: $\beta = 1$, $t_i \in [0,1]$

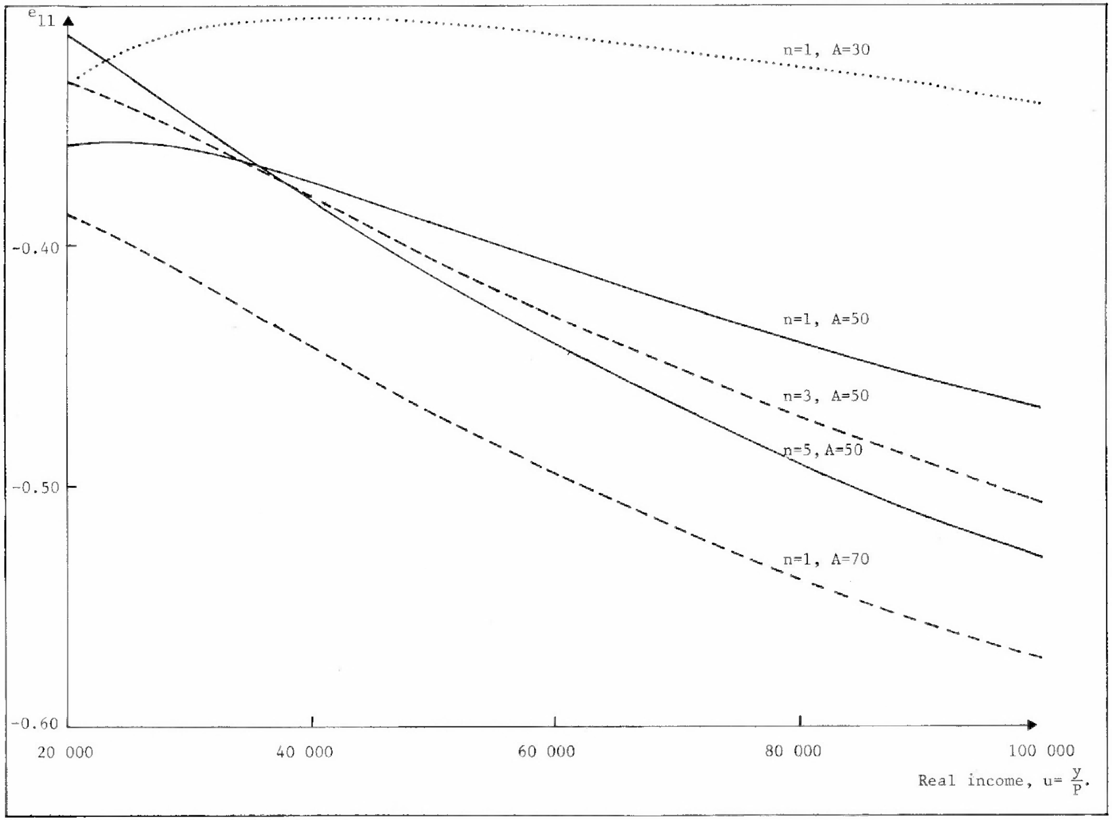


Figure 6.8. Estimated direct Slutsky elasticity for commodity group 1 Food, beverages and tobacco, as a function of real income.
$n = 1, 3, 5$ persons. $A = 30, 50, 70$ years.
A priori restrictions: $\beta = 1$, $t_i \in [0,1]$

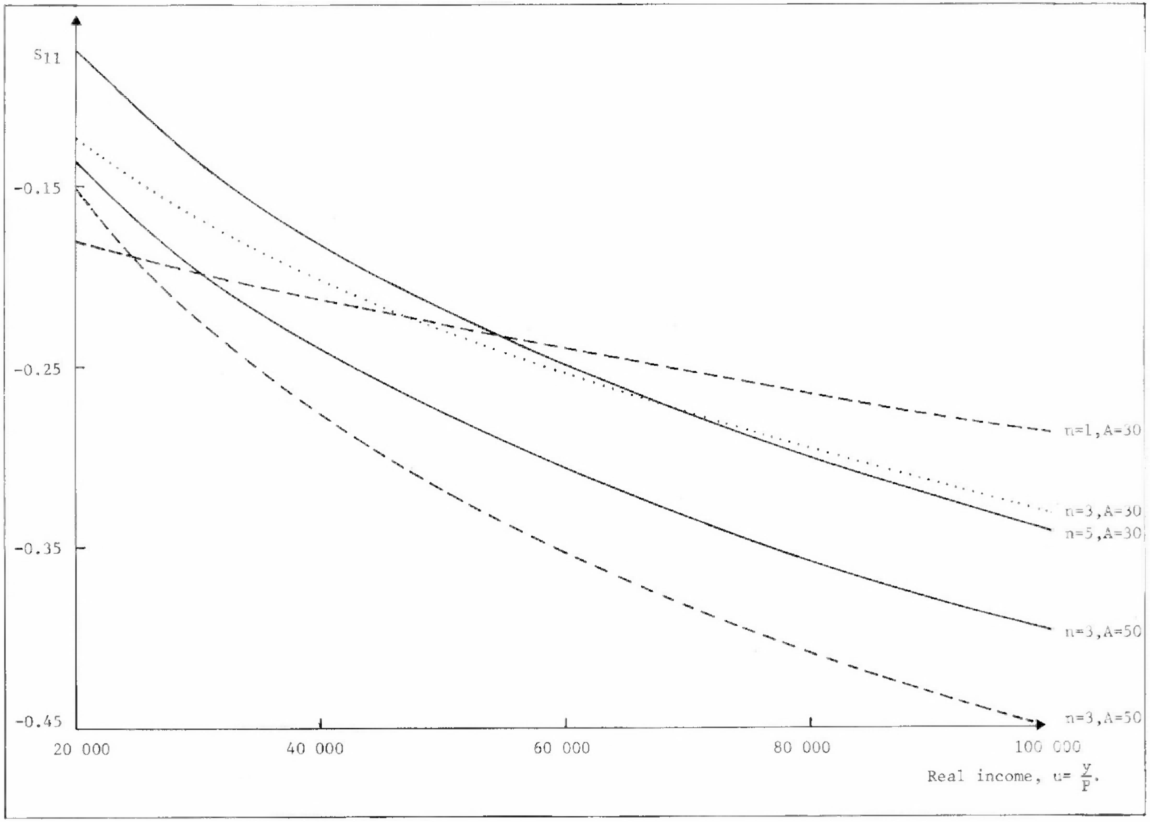


Figure 6.9. Estimated budget share for commodity group 2 Clothing and footwear, as a function of real income.
n= 1, 3, 5 persons. A= 30, 50, 70 years
A priori restrictions: β= 1, $t_i \in [0,1]$

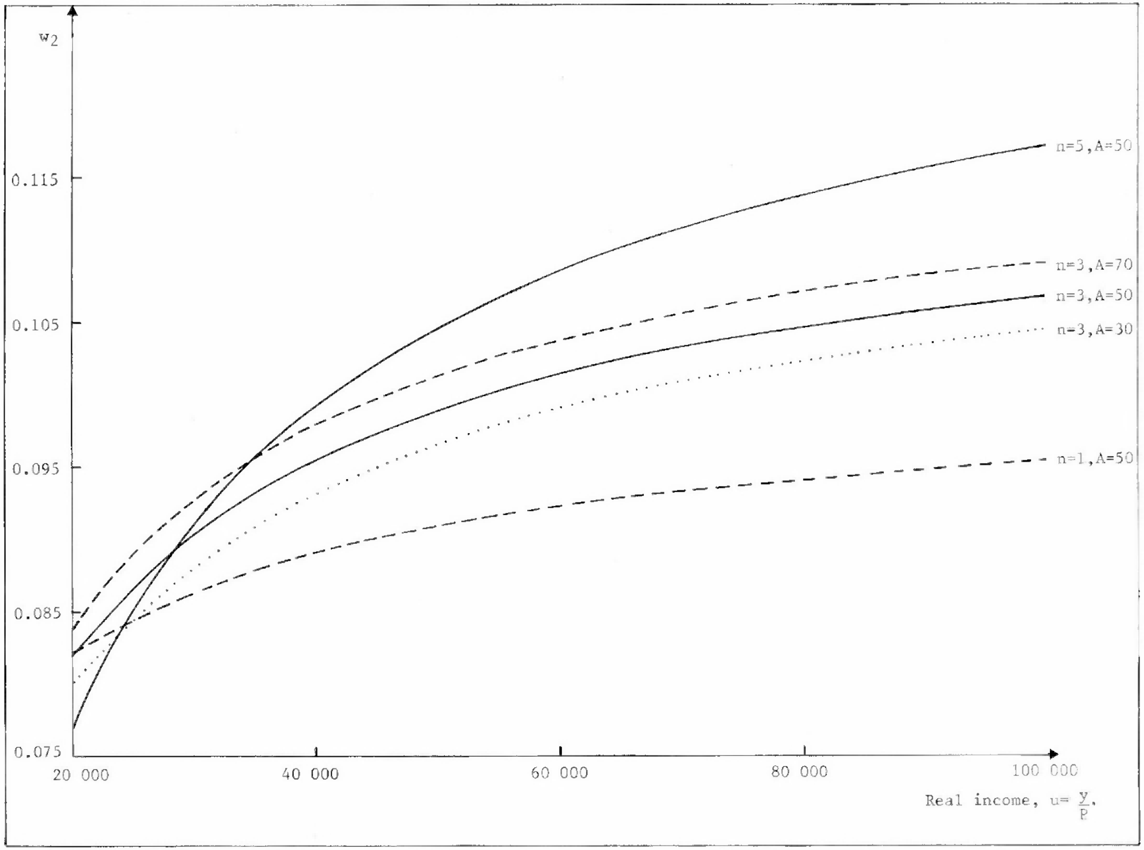


Figure 6.10. Estimated Engel elasticity for commodity group 2 Clothing and footwear, as a function of real income.
$n= 1, 3, 5$ persons. $A= 30, 50, 70$ years.
A priori restrictions: $\beta= 1$, $t_i \in [0,1]$

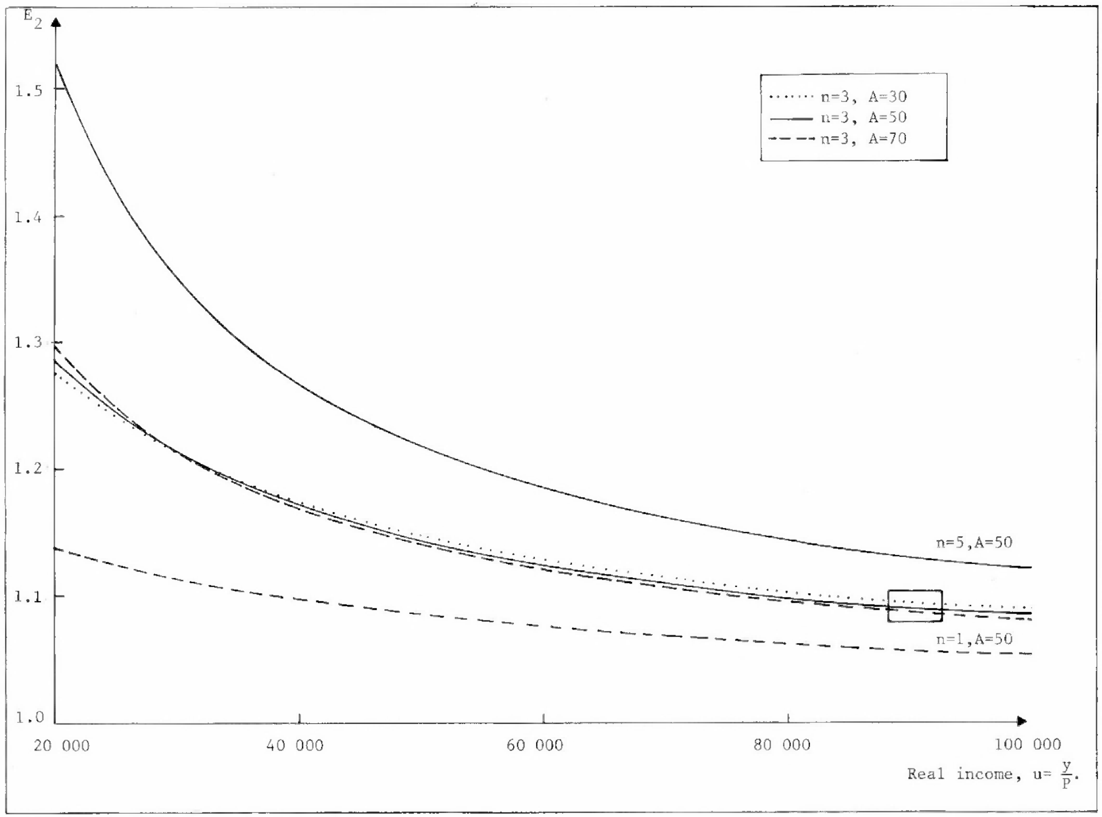


Figure 6.11. Estimated direct Cournot elasticity for commodity group 2 Clothing and footwear, as a function of real income.
n= 1, 3, 5 persons. A= 30, 50, 70 years.
A priori restrictions: β= 1, $t_i \in [0,1]$

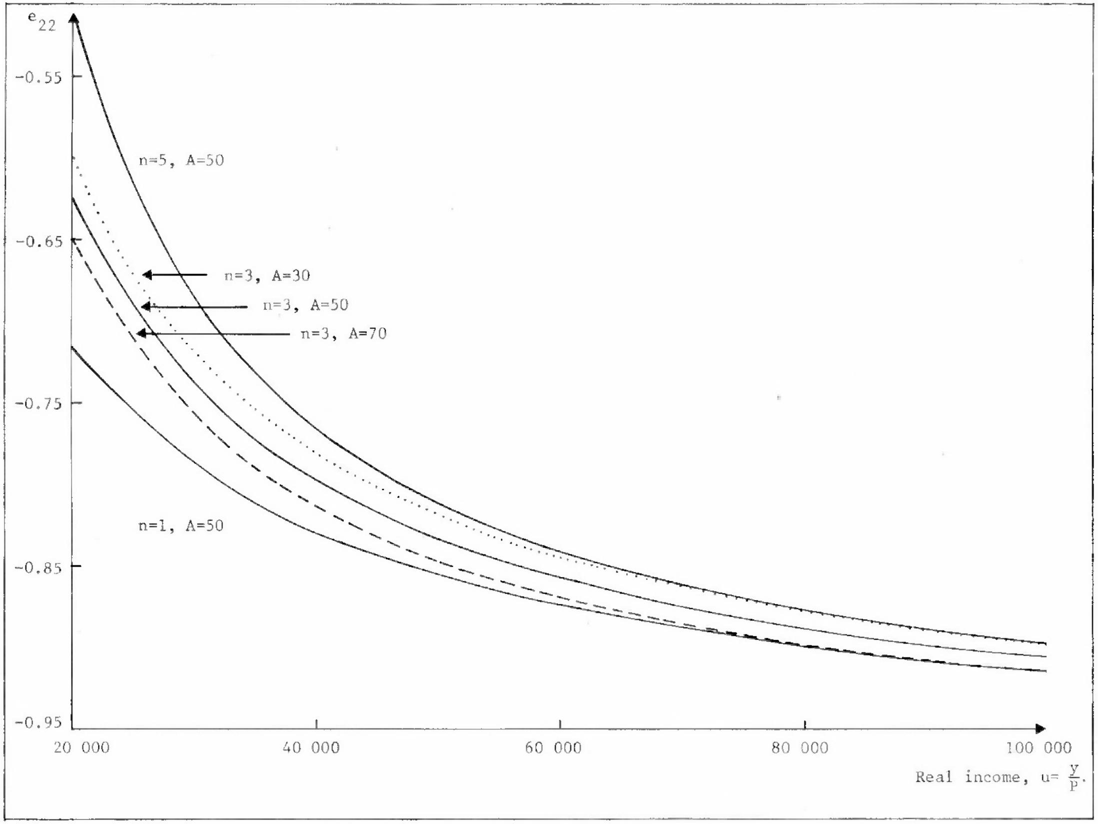
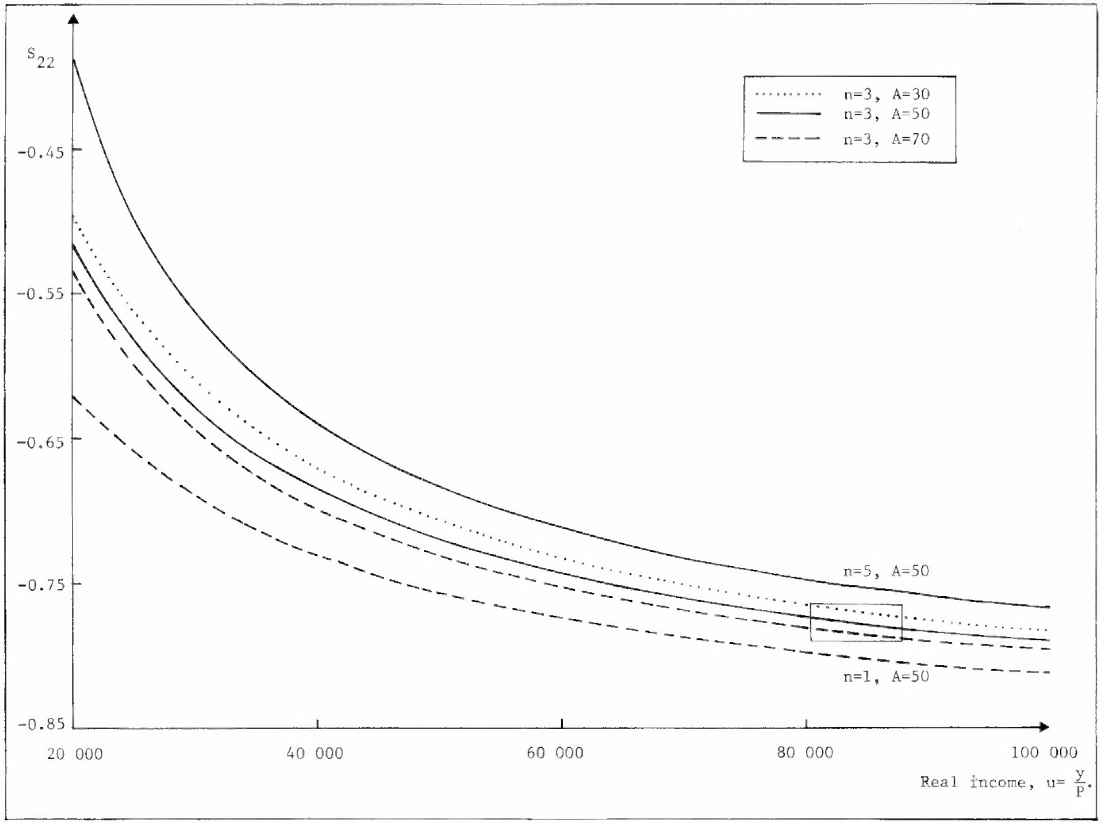


Figure 6.12. Estimated direct Slutsky elasticity for commodity group 2 Clothing and footwear, as a function of real income.
$n = 1, 3, 5$ persons. $A = 30, 50, 70$ years.
A priori restrictions: $\beta = 1$, $t_i \in [0,1]$



The results for commodity group 3 *Housing, fuel and furniture* disclose that the budget share function is decreasing with increasing age of the main income earner at any income level, and decreasing with household size for all real incomes above 25 000 Nkr (cf. figure 6.13). The Engel elasticity for this group is a monotonically increasing function of real income and it is subject to negative shifts both with increasing household size and increasing age of the main income earner (figure 6.14). The Cournot and Slutsky elasticities are uniformly decreasing functions of real income (figures 6.15 and 6.16). For any given level of the real income, the compensated as well as the uncompensated own-price elasticities are reduced (in absolute value) by an increase in the household size.

The budget share function for commodity group 4 *Travel and recreation* is shifted negatively by an increase in household size at any real income level, whereas the function shifts downwards for increasing age of the main income earner for real incomes above 30 000 Nkr (figure 6.17). The estimated Engel elasticity function is monotonically decreasing with increasing real income (figure 6.18), and shifts upwards with increasing household size and downwards with an increase of the age of the household head.[19] Owing to the zero estimate of t_4, the price response functions for this commodity group degenerate: The Cournot elasticity is constant and equal to -1, whereas the Slutsky elasticity function in figure 6.19 is $(1 - w_4 E_4)$, according to eqs. (6.7) - (6.10).

The final commodity group, 5 *Other goods and services*, has a budget share function that shifts downwards with a partial increase in household size and upwards with in increase of the age of the household head, see figure 6.20. The Engel elasticity function for this commodity has the same main features as that for Travel and recreation: It is monotonically decreasing in real income, the Engel elasticity function shifts upwards with increasing household size for all levels of real income and downwards with increasing age of the household head, except for real incomes below 25 000 Nkr (figure 6.21). The own price response functions - figures 6.22 and 6.23 - are monotonically decreasing with increasing real income. Both of them shift downwards with an increase in the household size and only slightly downwards with an increase in the age of the main income earner in the household.

---

19) Since the estimated t_4 is zero, this Engel elasticity function coincides with the negative of the income flexibility, see section 6.5 below.

Figure 6.13. Estimated budget share for commodity group 3 Housing, fuel and furniture, as a function of real income.
n= 1, 3, 5 persons. A= 30, 50, 70 years.
A priori restrictions: β= 1, $t_i \in [0,1]$

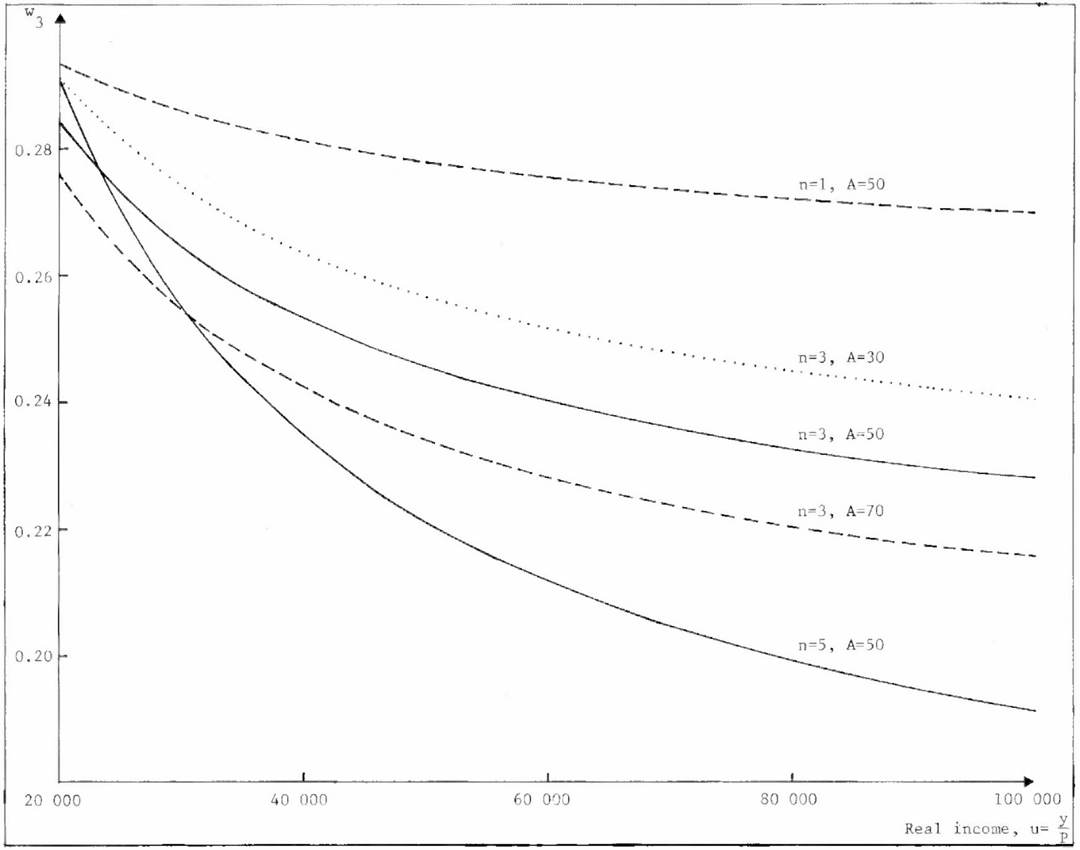


Figure 6.14. Estimated Engel elasticity for commodity group 3 Housing, fuel and furniture, as a function of real income.
n= 1,3,5 persons. A= 30,50,70 years.
A priori restrictions: β= 1, and $t_i \in [0,1]$

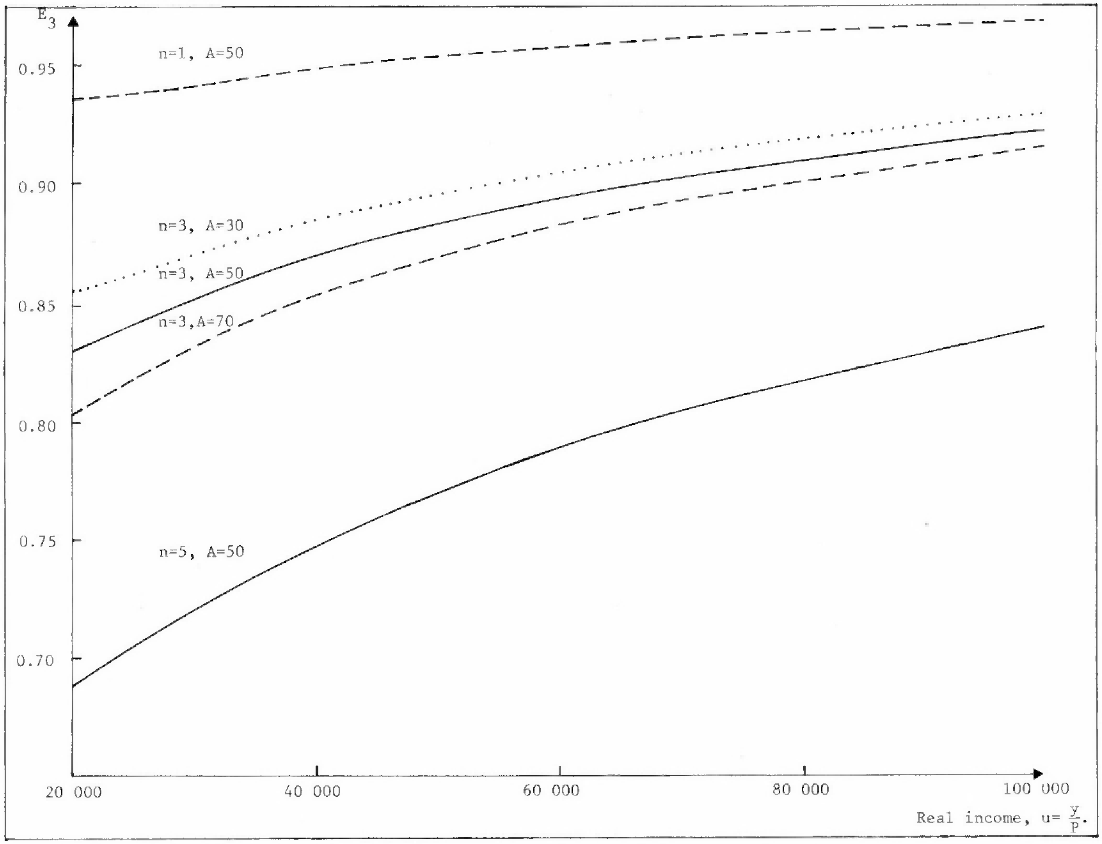


Figure 6.15. Estimated direct Cournot elasticity for commodity group 3 Housing, fuel and furniture, as a function of real income.
n= 1,3,5 persons. A= 30,50,70 years.
A priori restrictions: β= 1 and $t_i \in [0,1]$

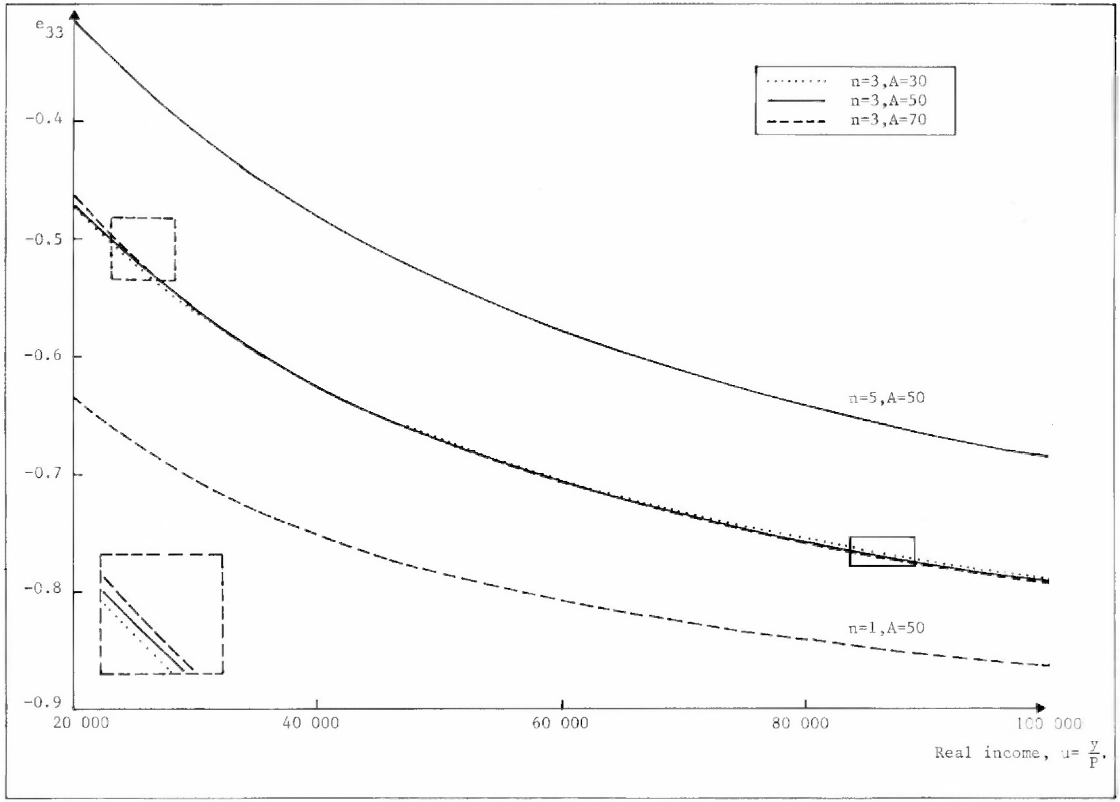


Figure 6.16. Estimated direct Slutsky elasticity for commodity group 3 Housing, fuel and furniture, as a function of real income.
n= 1,3,5 persons. A= 30,50,70 years.
A priori restrictions: β= 1 and $t_i \in [0,1]$

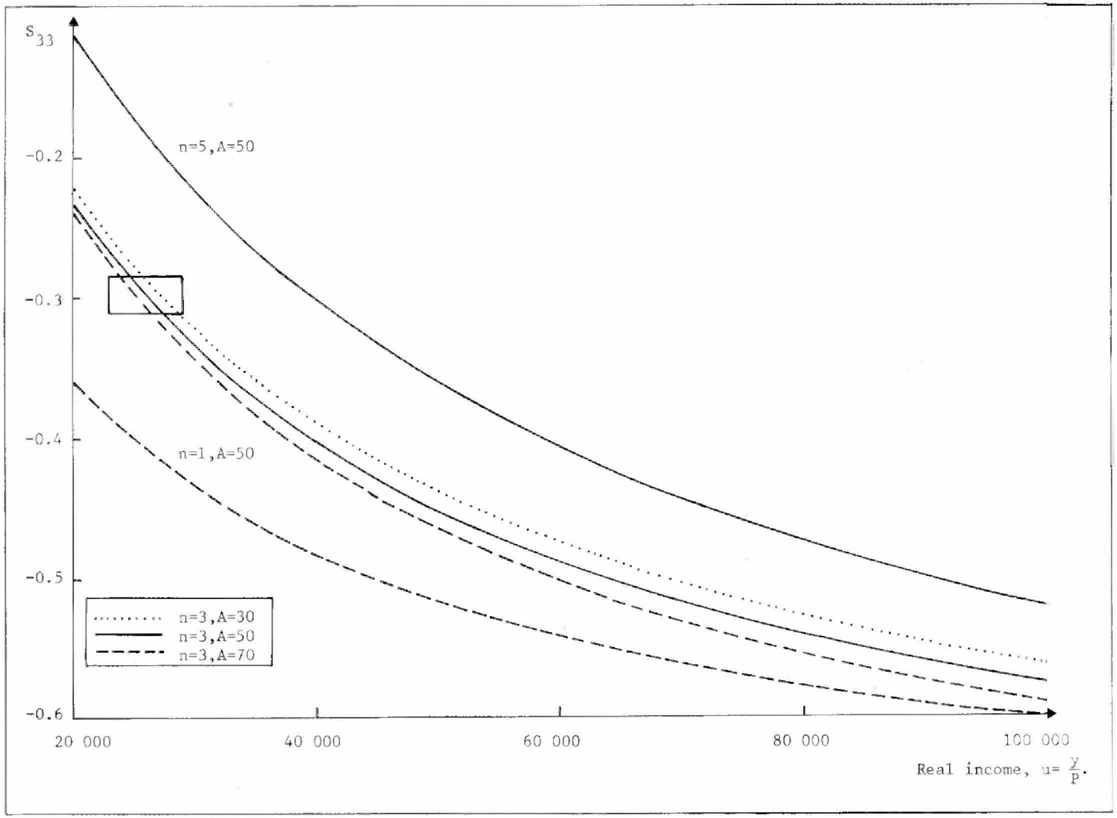


Figure 6.17. Estimated budget share for commodity group 4 Travel and recreation, as a function of real income.
n= 1,3,5 persons. A= 30, 50, 70 years.
A priori restrictions: β= 1 and $t_i \in [0,1]$

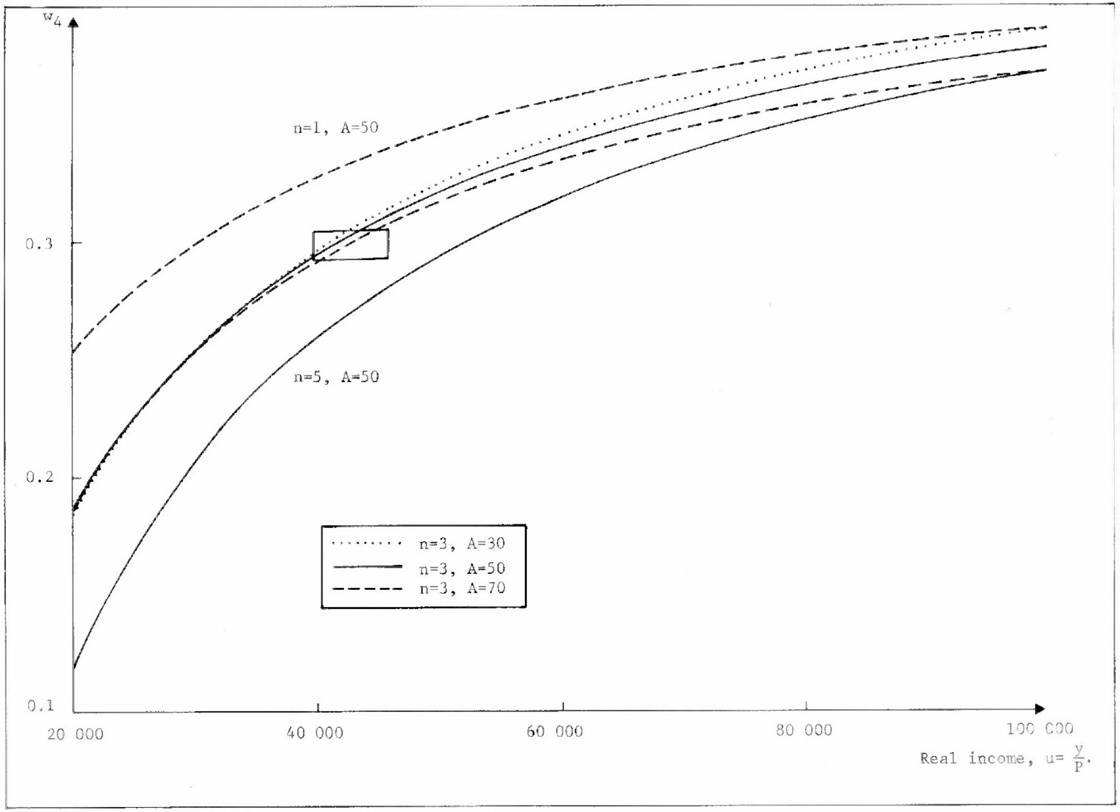


Figure 6.18. Estimated Engel elasticity for commodity group 4 Travel and recreation, as a function of real income.
n= 1,3,5 persons. A= 30,50,70 years.
A priori restrictions: β= 1 and $t_i \in [0,1]$

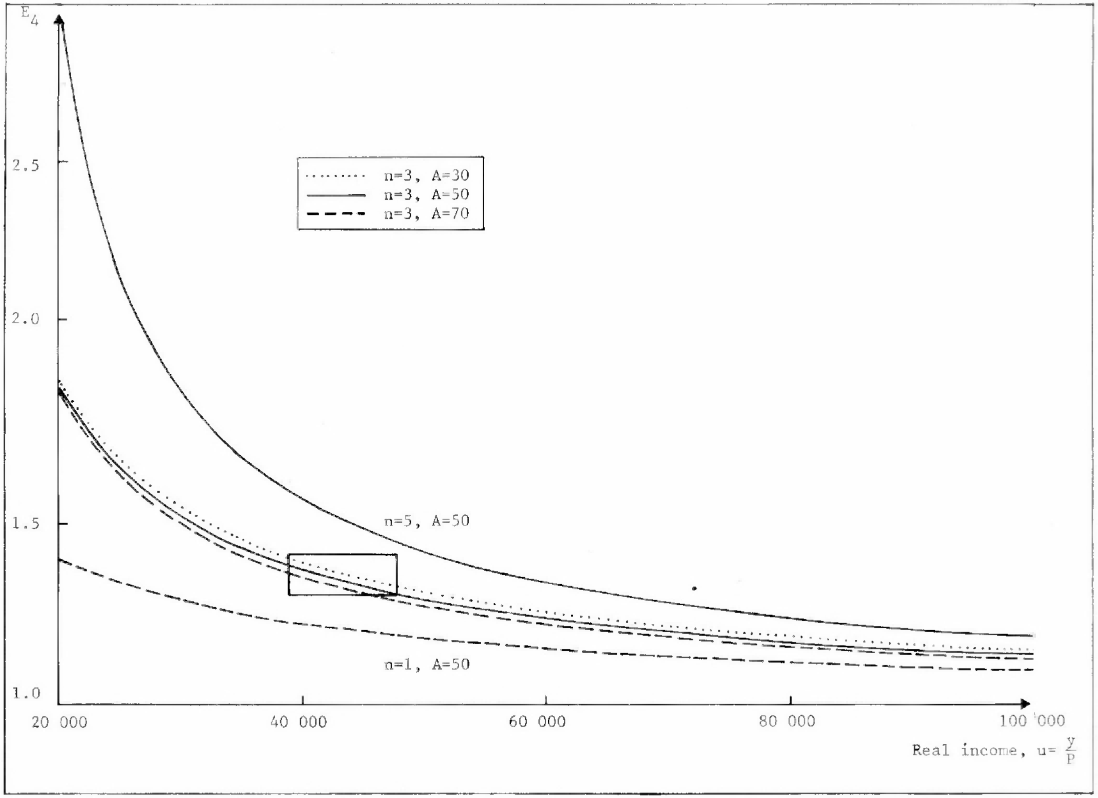


Figure 6.19. Estimated direct Slutsky elasticity for commodity group 4 Travel and recreation, as a function of real income.
n= 1, 3, 5 persons. A= 30, 50, 70 years.
A priori restrictions: β= 1 and $t_i \in [0,1]$

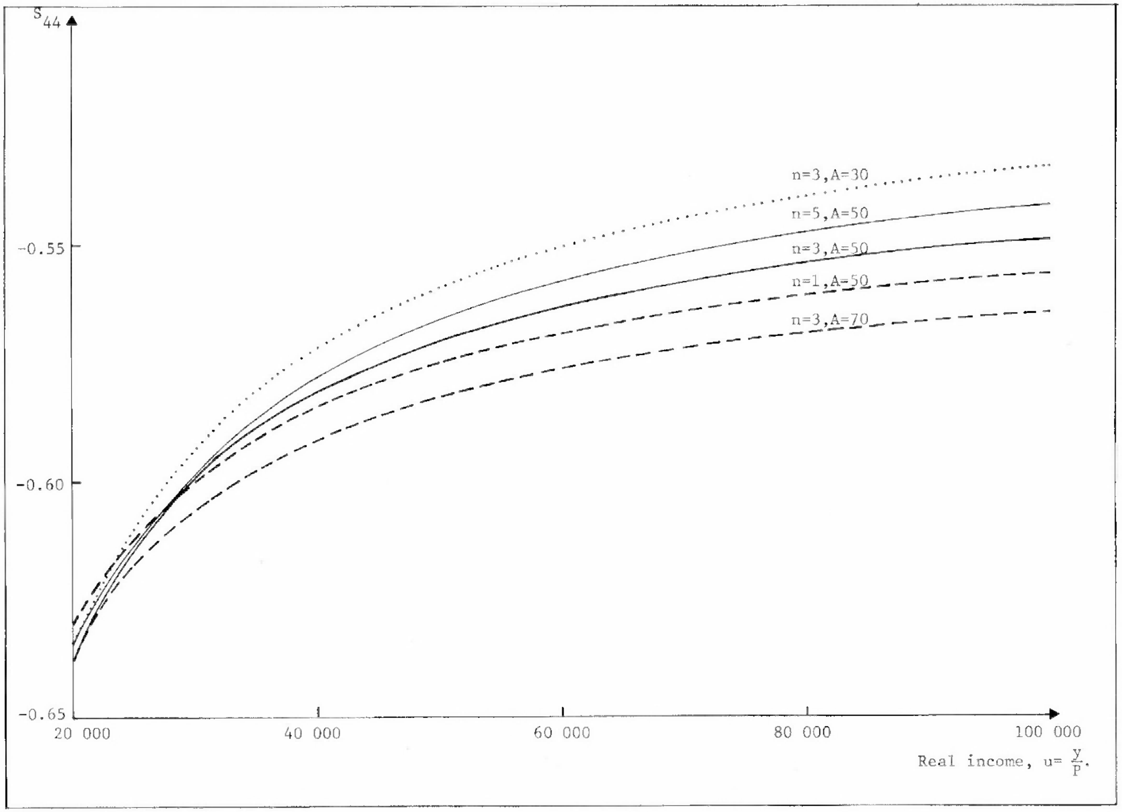


Figure 6.20. Estimated budget share for commodity group 5 Other goods and services, as a function of real income.
n= 1, 3, 5 persons. A= 30, 50, 70 years.
A priori restrictions: β= 1 and $t_i \in [0,1]$

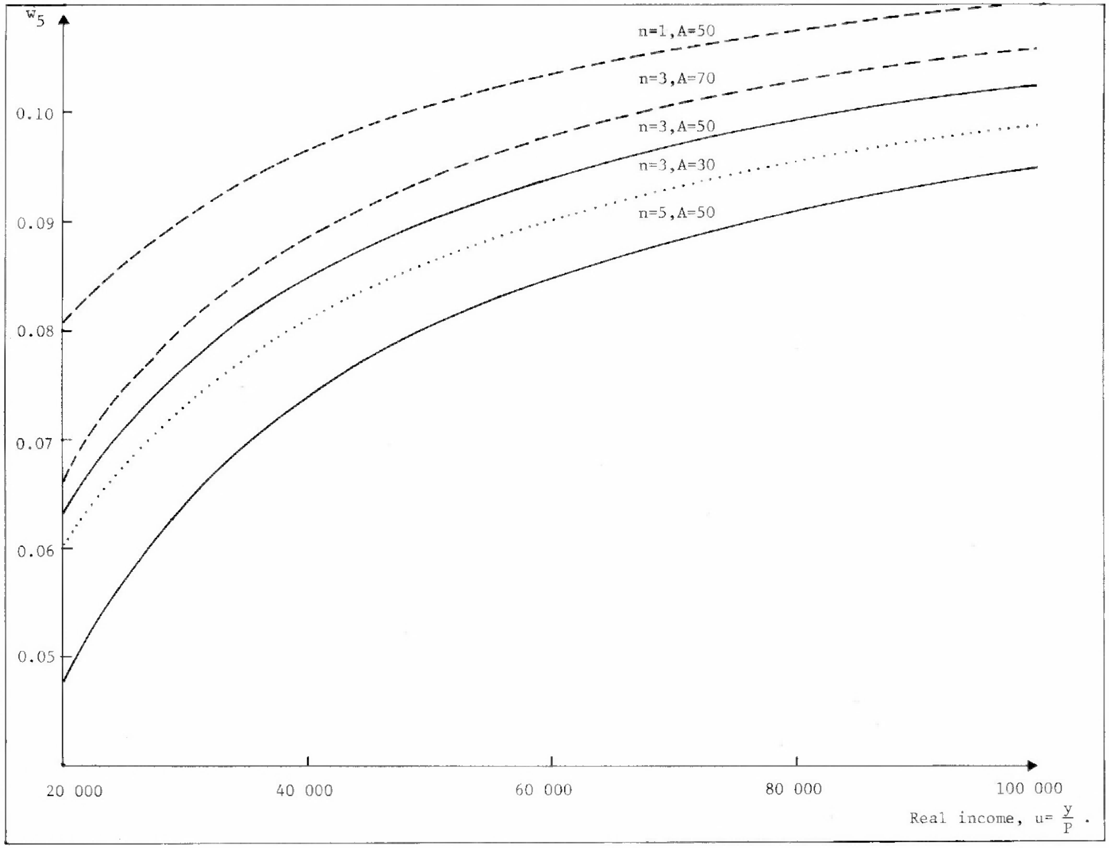


Figure 6.21. Estimated Engel elasticity for commodity group 5 Other goods and services, as a function of real income.
n= 1, 3, 5 persons. A= 30, 50, 70 years.
A priori restrictions: β= 1 and $t_i \in [0,1]$

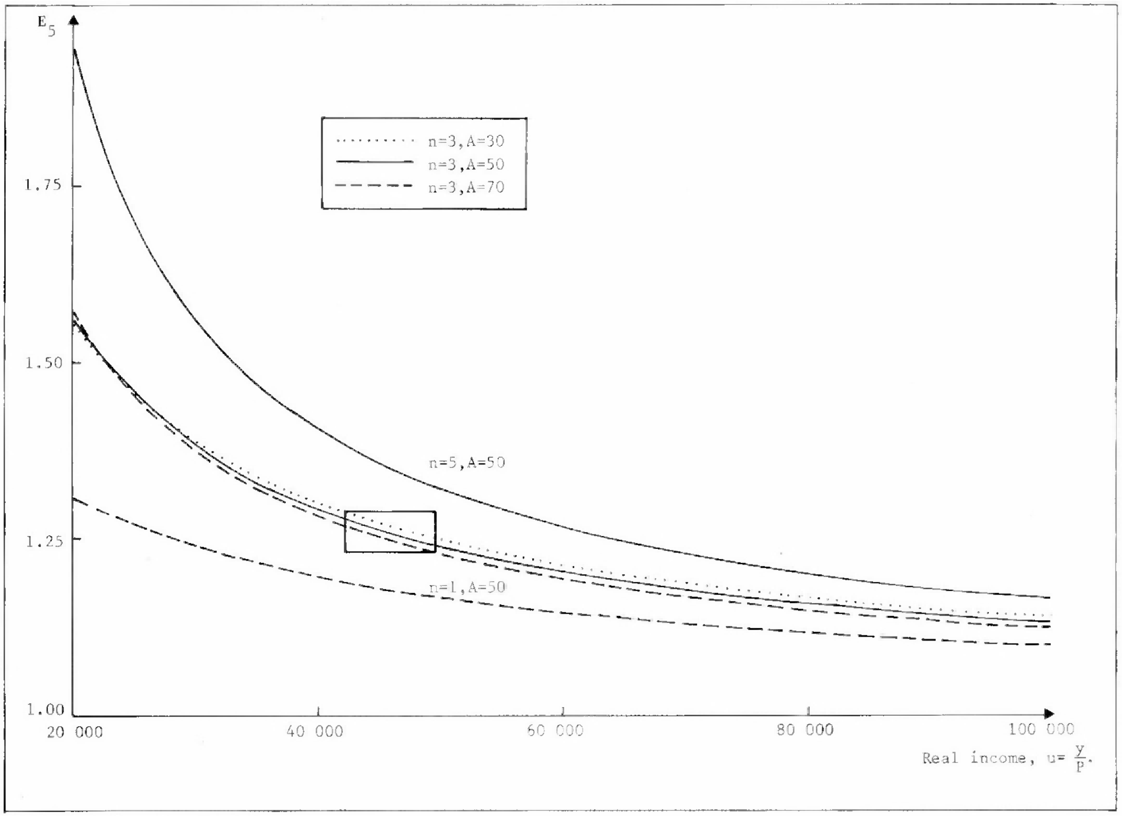


Figure 6.22. Estimated direct Cournot elasticity for commodity group 5 Other goods and services, as a function of real income.
n= 1,3,5 persons. A= 30,50,70 years.
A priori restrictions: β= 1 and $t_i \in [0,1]$

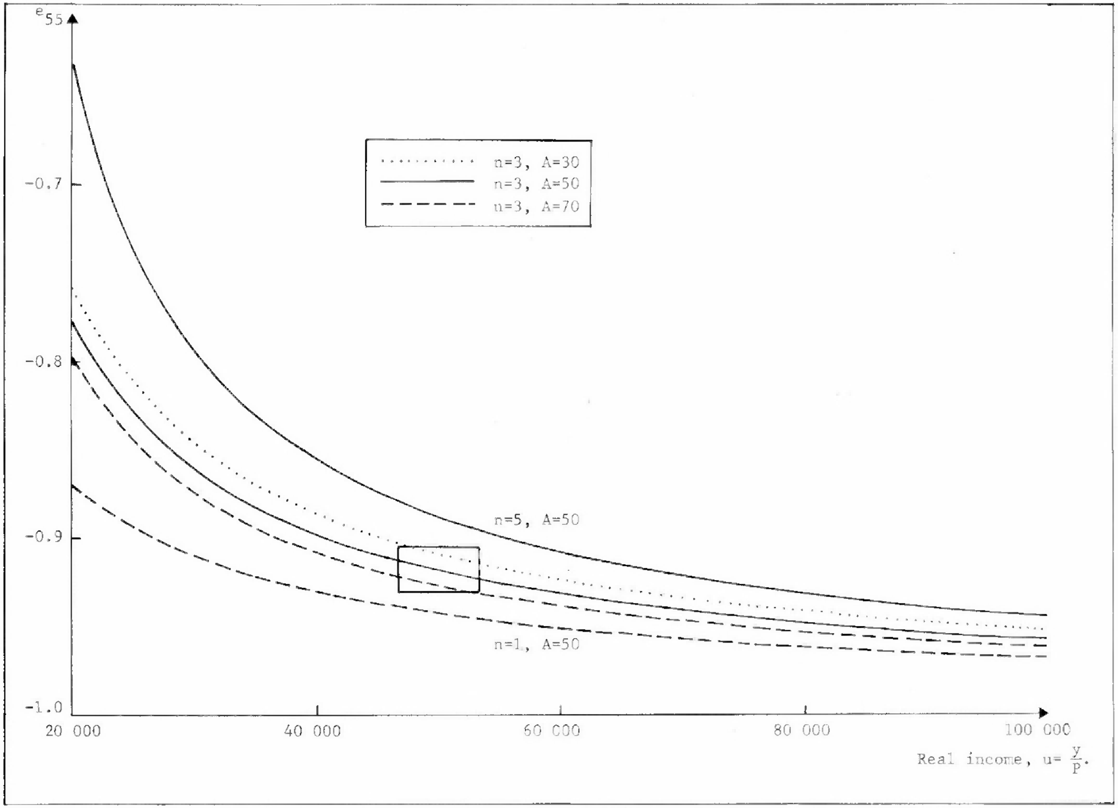
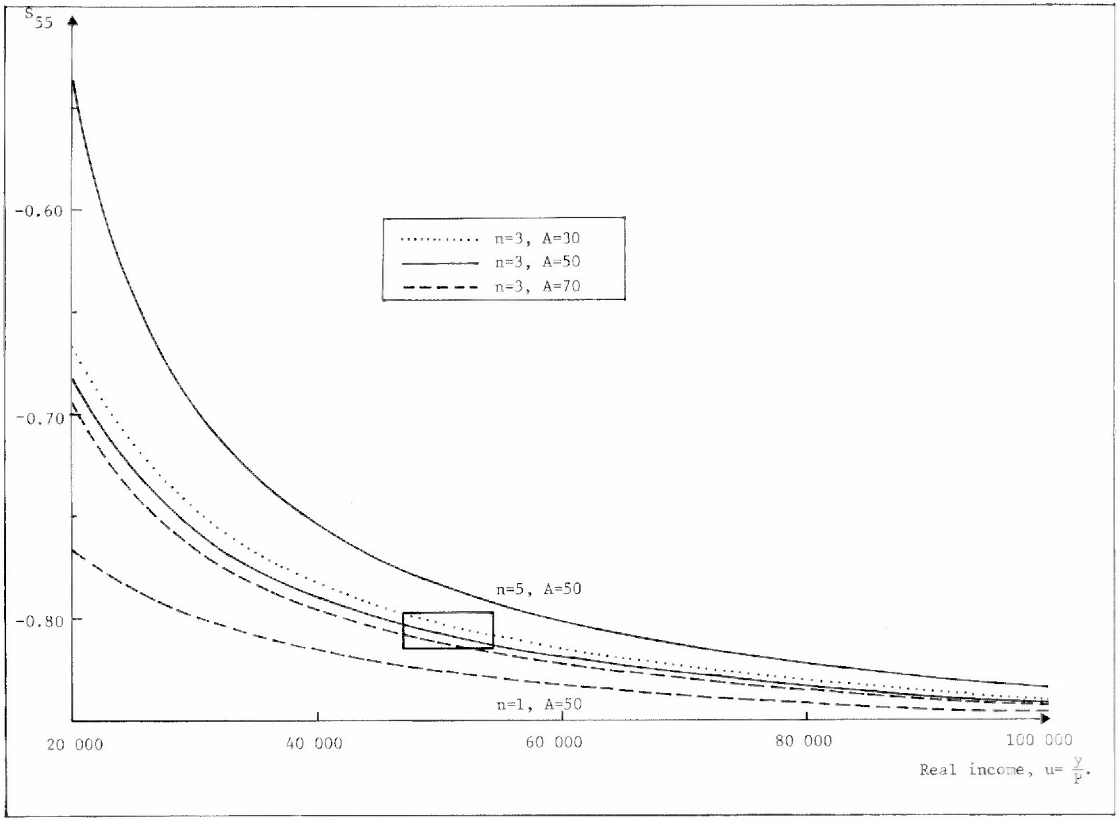


Figure 6.23. Estimated direct Slutsky elasticity for commodity group 5 Other goods and services, as a function of real income.
$n = 1,3,5$ persons. $A = 30,50,70$ years.
A priori restrictions: $\beta = 1$ and $t_i \in [0,1]$



Comparison with empirical evidence from other studies

Until recently, background variables have not, to our knowledge, been included into empirical analyses of complete demand systems based on aggregate time series or combined cross-section/time-series data. The most common approach for cross-sectional studies has been to apply models which are linear or log-linear in income, and introduce family size and other demographic factors as additional explanatory variables in such specifications. Houthakker (1957) reports partial elasticities for four expenditure groups (food, clothing, housing, and miscellaneous) with respect to total expenditure and family size, for 17 different countries, largely based on cell means from national family expenditure surveys. His results, which rely on single equation estimation methods, show a significantly positive partial elasticity of demand for food with respect to family size for all countries considered. This concurs with our findings for commodity group 1 Food, beverages and tobacco, see figure 6.5. For clothing and for housing, however, the partial elasticities with respect to family size differed in signs between the different countries, both signs being equally well represented.

The only previous attempts in the literature to introduce demographic background variables into *complete* demand systems estimated by simultaneous equation methods are made by Pollak and Wales. In their recent series of articles (Pollak and Wales (1978, 1980, 1981)), they consider a three commodity grouping, which comprises nearly fifty per cent of the total expenditure - food, clothing,[20] and miscellaneous goods - and suggest several functional relationships between the coefficients of well-known demand systems (LES, QES, generalized Translog and CES functions) and family characteristics. (The way in which we have introduced background variables in our model corresponds to 'linear demographic translating' in the terminology of Pollak and Wales - confer footnote 2 of chapter III above.) Their data base is British family expenditure surveys from 1966 to 1972, and they perform the usual data reduction by employing *cell means* as basic data instead of the primary individual observations.[21] Another basic difference between our study

20) According to the authors, this commodity group also comprises footwear; it is thus equivalent to our commodity group 2.

21) Pollak and Wales (1978) is based on two vintages of these surveys (1966 and 1972), whereas their Econometrica papers (from 1980 and 1981) cover all the seven years from 1966 to 1972. Pollak and Wales furnish their coefficient estimates with standard errors, but do not, unfortunately, comment on neither the principles of their calculation nor their interpretation. Since their basic data are in the form of cell means, such information would have been most interesting and facilitated comparison with corresponding estimates based on genuine micro data.

and that of Pollak and Wales is that the latter define family size as the number of *children* in the family, while their age variable represents the average age of the children only.[22)]

In table 6.11, we have summarized the findings of Pollak and Wales, as calculated at the approximate sample mean of prices and total expenditure. For food, we find that the sign of the effect on the estimated marginal budget shares of an increasing family size and an increasing average age of the children are both in accordance with our findings recorded in figure 6.5 for the composite commodity group Food, beverages and tobacco.[23)] Their results also agree with our estimates for clothing, calculated at the sample mean of prices and total expenditure. As regards the impact of the demographic variables on the own price (Cournot) elasticities, our sample mean estimates for Food, beverages and tobacco support those of Pollak and Wales for the GTL (Generalized Translog) model, but contradict those based on the QES (Quadratic Expenditure System) and GCES (Generalized CES) models. For Clothing and footwear, our estimates support their results for the QES model for the family size variable and the GTL model for the age variable, but contradict those based on the other functional forms.

---

22) Households consisting of adults only are excluded.

23) The marginal budget share is equal to the product of the Engel elasticity and the estimated (average) budget share. In our case it turns out that the sign of the effect on the estimated (average) budget share of an increasing family size and an increasing age of the main income earner coincide with the sign of the effect on the marginal budget share for Food, beverages and tobacco as well as for Clothing and footwear.

Table 6.11. Summary of the results from the studies of Pollak and Wales (1978, 1980, and 1981)[a)]

Functional form of the demand functions	Food: Effect on marginal budget share	Food: Effect on direct price elasticity	Clothing: Effect on marginal budget share	Clothing: Effect on direct price elasticity
	A. Partial increase in the family size, i.e. the number of children			
QES[b)]	+	..	+	..
LES[b)]	+	..	+	..
QES[c)]	+	+	+	+
GTL[c)]	+	−	+	−
GCES[d)]	Constant	+	Constant	−
	B. Partial increase in the average age of the children			
QES[c)]	+	+	+	+
GTL[c)]	+	−	+	−

a) The effects on the marginal budget shares and on the value of the estimated direct price elasticities are calculated at the approximate sample means.
b) Pollak and Wales (1978), based on the UK Family expenditure surveys 1966 and 1972, cell means. QES = Quadratic Expenditure System. LES = Linear Expenditure System.
c) Pollak and Wales (1980), based on the UK Family expenditure surveys 1968 - 1972, cell means. QES = Quadratic Expenditure System. GTL = Generalized Translog System.
d) Pollak and Wales (1981), based on the UK Family expenditure surveys 1966 - 1972, cell means. GCES = Generalized CES functions.

6.4.3. The effect of the restriction $\Sigma_\mu = 0$ on the estimated demand elasticities

In subsection 6.3.2, we discussed the impact on the estimated structural coefficients of neglecting individual disturbance components, i.e. imposing $\Sigma_\mu = 0$ *a priori*. Let us look briefly at the effect of this restriction on the implied budget shares and income and price elasticities, summarized in table 6.12. We observe that the estimated budget shares and Engel elasticities are affected perceptibly by this restriction only for commodities 1 Food, beverages and tobacco and 4 Travel and recreation, but the changes are by no means dramatic. As regards the direct Cournot elasticities, the restriction $\Sigma_\mu = 0$ is accompanied by a moderate increase in the absolute value of the estimates for all commodities except 4 Travel and recreation for which the elasticity remains fixed at -1.0.

Table 6.12. Effect of the restriction $\Sigma_\mu = 0$ on the estimates of budget shares, Engel, and direct Cournot elasticities, based on simultaneous estimation of the model variant with n as the only background variable.
n = 1, 3, 5 persons, u = 50 000 Nkr, and all (absolute) prices equal to their sample means.[a)]
A priori restrictions: $\beta=1$, $t_i \in [0,1]$

Commodity group	Σ_μ unrestricted[b)]			$\Sigma_\mu = 0$		
	n=1	n=3	n=5	n=1	n=3	n=5
	Estimated budget shares (percentages)					
1 Food, beverages and tobacco	19.08	24.48	29.55	19.96	24.85	29.50
2 Clothing and footwear	9.13	9.87	10.37	9.17	9.89	10.39
3 Housing, fuel and furniture	27.10	24.60	22.78	27.11	24.62	22.79
4 Travel and recreation	34.50	32.05	29.37	33.47	31.59	29.42
5 Other goods and services	10.19	9.00	7.91	10.28	9.04	7.90
	Engel elasticities					
1 Food, beverages and tobacco	0.543	0.535	0.534	0.604	0.582	0.570
2 Clothing and footwear	1.081	1.138	1.210	1.084	1.139	1.208
3 Housing, fuel and furniture	0.951	0.884	0.783	0.951	0.888	0.794
4 Travel and recreation	1.222	1.336	1.479	1.203	1.304	1.431
5 Other goods and services	1.162	1.233	1.312	1.164	1.238	1.323
	Direct Cournot elasticities					
1 Food, beverages and tobacco	-0.430	-0.416	-0.407	-0.501	-0.467	-0.443
2 Clothing and footwear	-0.862	-0.833	-0.805	-0.883	-0.857	-0.831
3 Housing, fuel and furniture	-0.789	-0.678	-0.548	-0.803	-0.695	-0.568
4 Travel and recreation	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000
5 Other goods and services	-0.942	-0.913	-0.876	-0.962	-0.942	-0.916

a) Note that the price index, and thus the relative prices, contain the coefficients t_i as parameters, and will therefore take different values in the two cases considered here (Σ_μ unrestricted, $\Sigma_\mu = 0$) even though all the absolute prices are equal.
b) Confer tables A.7, A.9, and A.14 of the table annex.

6.5. The income flexibility

The estimates of the coefficient vector have also interesting implications for the income flexibility. In appendix A, we establish the following expression for this parameter (cf. eq. (A.18)):

$$(6.16)\quad \omega = -\frac{u - a}{u - a \log u - b}\,.$$

Using this formula as a starting point, we shall first examine the variation of the income flexibility with respect to changes in real income - a recurring issue of dispute in the literature[24] - and second, characterize its dependence on demographic variables.

Before considering the numerical results, let us examine a bit closer which *a priori* restrictions the model imposes on the variation of ω. From (6.16) we find

$$(6.17)\quad -\omega'_u = \frac{\partial(-\omega)}{\partial u} = \frac{2a - \frac{a^2}{u} - a \log u - b}{(u - a \log u - b)^2}$$

$$= \frac{a\{1 - \frac{a}{u} + \log (\frac{a}{u})\} + a(1 - \log a) - b}{(u - a \log u - b)^2}\,.$$

The expression in the curly bracket in the numerator has the form $\Delta + \log(1-\Delta)$, with $\Delta = (u-a)/u$. From (6.17) and the assumption $a > 0$, we have $0 < \Delta < 1$. Then, using Taylor's formula, it follows that

$$\log(1-\Delta) = -\sum_{k=1}^{\infty} \frac{1}{k}\Delta^k,$$

and hence,

$$\Delta + \log(1-\Delta) = -\sum_{k=2}^{\infty} \frac{1}{k}\Delta^k.$$

24) Cf. e.g. Frisch (1959, p. 189), Goldberger and Gamaletsos (1970), van Praag (1971), Theil and Brooks (1970-71), Theil (1975, pp. 29-30, 108-112) and (1976, pp. 419-420).

This expression is obviously negative for $0 < \Delta < 1$, and its absolute value is an increasing function of Δ. Thus, $1-a/u + \log(a/u)$ is always negative, and its absolute value is an increasing function of u (recall that Δ is an increasing function of u). Since the denominator of (6.17) is always positive, the sign of $-\omega_u'$ will be determined by the sign of $a(1 - \log a) - b$. Non-positivity of this expression is a *sufficient* condition for uniform negativity of $-\omega_u'$. However, even if $a(1 - \log a) - b > 0$, the numerator will always be negative for values of u beyond a certain limit, since $1-a/u + \log(a/u)$ is always negative and decreasing and goes to minus infinity as u goes to plus infinity. This implies that $-\omega_u'$ will always be negative for large values of u. Summing up, we have:

1) If $b \geqq a(1 - \log a)$, i.e. if the functions $C(u) = a \log \bar{u} + b$ and u intersect or if the latter is tangent to the former (cf. figure 6.2 above), then the absolute value of the income flexibility, $-\omega$, will be a monotonically decreasing function of real income u for all admissible values of u.

2) A *necessary* condition for the absolute value of the income flexibility to be increasing with income is that $b < a(1 - \log a)$. This condition is identical with the condition for non-intersection of $C(u) = a \log u + b$ and u (cf. figure 6.3 above). In this case, however, $-\omega$ will only be increasing with u up to the value at which $a\{1-a/u + \log(a/u)\} = b - a(1 - \log a)$. After this, it will be monotonically decreasing.

The estimated income flexibility function is illustrated graphically in figures 6.24 A and B for household sizes n = 1, 3, and 5 persons. Figure 6.24 A shows the variation of $-\omega$ with u for a household whose head person is A = 50 years of age (which is close to the average age in the sample). The estimated value for the 'average household' (u = 50 000, n = 3, A = 50) is $\omega = -1.34$. This value is somewhat lower than, but agrees well with the majority of empirical estimates of this parameter from other countries - see e.g. the survey in Brown and Deaton (1972, p. 1206) and Theil (1975, pp. 207-208, 303-305)[25] - and it also concurs with previous estimates based on Norwegian time series data (Biørn (1974)).

---

25) Recall that Theil, following Houthakker (1960), defines the income flexibility as $\Phi = 1/\omega$. Cf. footnote 17 in chapter II above.

Figure 6.24A. Estimated income flexibility, ω, as a function of real income u.
$n = 1,3,5$ persons. $A = 50$ years.
A priori restrictions: $\beta = 1$, $t_i \in [0,1]$

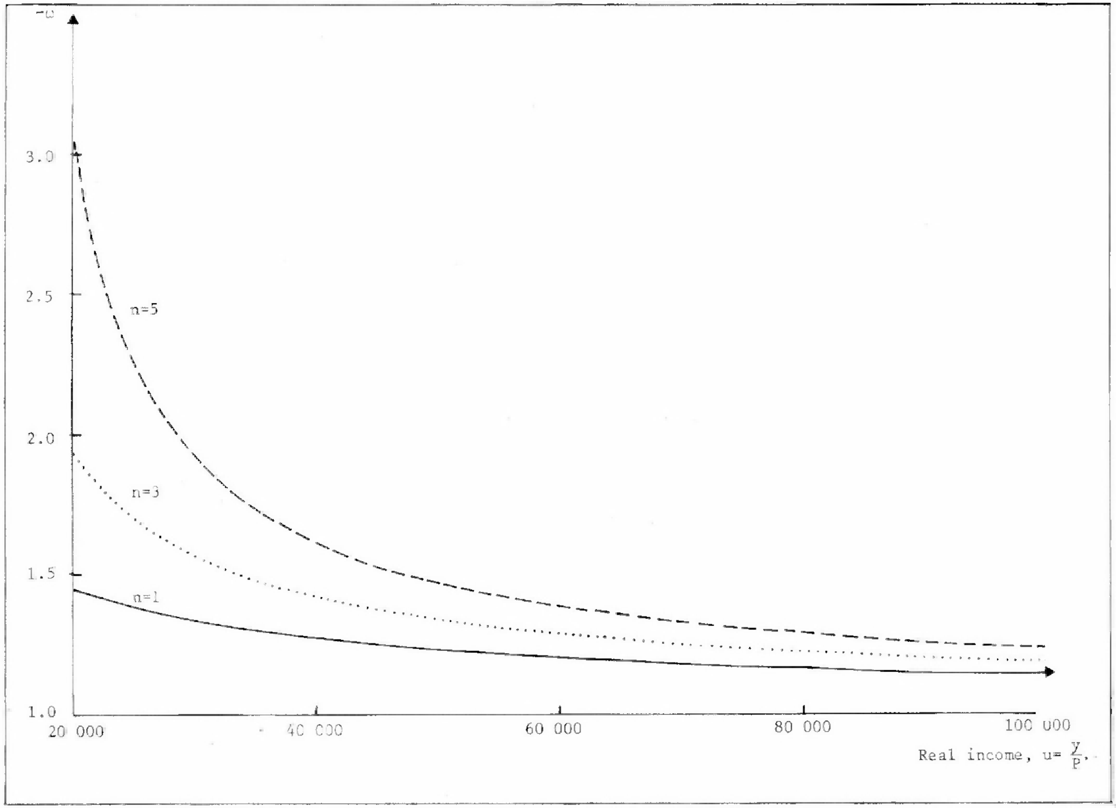
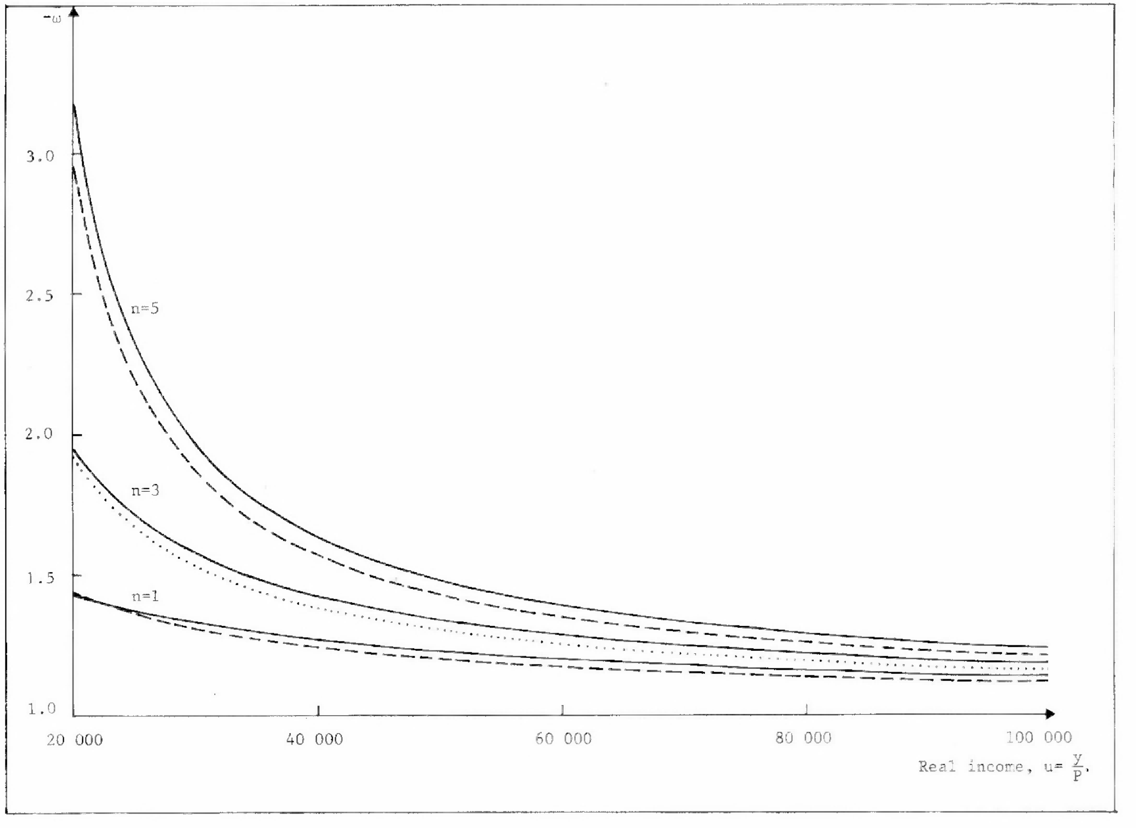


Figure 6.24B. Estimated income flexibility, ω, as a function of real income u.
n = 1,3 and 5 persons.
Solid lines: A = 30 years.
Dotted lines: A = 70 years.
A priori restrictions: $\beta = 1$, $t_i \in [0,1]$



Our results support Frisch's famous conjecture that the absolute value of the money flexibility is a decreasing function of income (Frisch (1959, p. 189)), although not his numerical guesses. This is at variance with, for example, Theil and Brooks (1970-71), who are led to rejection. Of course, our parametrization of the C(u) function places rather strong *a priori* restrictions on the possible pattern of variation of ω - in particular its asymptotic property $\lim_{u\to\infty} \omega = -1$. But as our specification does not a priori *exclude* $-\omega_u' > 0$ over a certain interval, cf. the above discussion - in contrast to, for example, the LES model of which it is a generalization[26] - it is interesting that our estimated $(-\omega)$ function is monotonically declining over the empirically relevant range. For a household of average size and average age (n=3, A=50), the estimated value declines from $-\omega = 1.94$ for u = 20 000 Nkr to $-\omega = 1.18$ for u = 100 000 Nkr (table 6.13).

The estimate is also quite sensitive with respect to changes in the household size, in particular for low incomes. For a household with average income and with a head of average age (u = 50 000, A = 50), the estimate increases from $-\omega = 1.23$ for n=1 person to $-\omega = 1.48$ for n=5 persons. On the other hand, it depends only slightly on age: when u = 50 000 and n=3, the estimate is $-\omega = 1.36$ for A=30 years and $-\omega = 1.32$ for A=70 years (table 6.14). A graphic visualization of the range is given in figure 6.24 B, the solid lines representing households with A=30 years, the dotted lines relating to A=70 years. So, for practical purposes, we may conclude that the income flexibility is age independent.

A final reminder: The income flexibility, interpreted as the elasticity of the marginal utility of income with respect to income, is not an ordinal concept. All the conclusions above are confined to the indirect utility function (2.11) as a representation of the underlying indifference map, cf. section 2.4. If we had chosen another utility indicator function, our numerical estimates of ω would, of course, have become different.

---

26) Recall that LES is equivalent to imposing the restrictions $a=0$, $\beta=1$; cf. section 2.3 above. Then, $-\omega = u/(u-b)$, which is monotonically decreasing for all $u > b > 0$.

Table 6.13. Estimates of the income flexibility ω for selected values of real income u and household size n. A=50 years[a]. A priori restrictions: $\beta=1$, $t_i \in [0,1]$. (Σ_μ unrestricted)

u, Nkr	n=1	n=3	n=5
20 000	-1.45	-1.94	-3.01
50 000	-1.23	-1.34	-1.48
100 000	-1.13	-1.18	-1.23

a) The estimates refer to the model variant with both n and A included.

Table 6.14. Estimates of the income flexibility ω for selected values of real income u and age A. n=3 persons.[a] A priori restrictions: $\beta=1$, $t_i \in [0,1]$. (Σ_μ unrestricted)

u, Nkr	A=30	A=50	A=70
20 000	-1.96	-1.94	-1.92
50 000	-1.36	-1.34	-1.32
100 000	-1.19	-1.18	-1.17

a) The estimates refer to the model variant with both n and A included.

6.6. On the effects of relaxing constraints imposed on the price index function

All inferences obtained in sections 6.2 - 6.5 are conditional on the following restrictions on the coefficients of the price index function: $\beta = 1$ and $t_i \geqq 0$ for all i. These restrictions cannot be rejected from our data (cf. subsection 6.1.1), and when they are imposed, we obtain a significant reduction in the computer costs involved in estimating and testing the other coefficients of the model.

A main explanation of this state of affairs seems to be that the relative prices exhibit a rather limited variation, since our data cover three successive years only. And although by no means negligible[27], the relative variation of the prices is far less than that of the other structural variables, u, n, and A. So it comes as no surprise that it is the coefficients characterizing the price response which cause the most serious estimation problems. In this section, we take a closer look at these problems.

---

27) Cf. tables 7 and A2.1 in Biørn and Jansen (1980).

Consider table 6.15, which contains *coefficient estimates* and standard errors for three different model variants: no restrictions imposed (column 1), non-negativity of the weights t_i of the price index function imposed (column 2), and linearity of the price index function, i.e. $\beta=1$, imposed (column 3). First, we note that the estimates of the s_i's - as well as their standard errors - are virtually the same in the three model variants; i.e. the estimated asymptotic budget shares do not change perceptibly when the coefficients of the price index function are restricted.[28) On the other hand, the estimates of a and b, i.e. the coefficients of the income response function C(u) change markedly. Imposition of the linearity constraint $\beta=1$ almost doubles the estimates of a and b - from 0.40 and 0.66 to 0.78 and 1.29, respectively - and also induces notable changes in the t_i's. The estimate of t_1, for instance, i.e. the weight of Food, beverages and tobacco in the price index function in its basic year, decreases from 0.57 to 0.36 while the estimate of t_4, i.e. the weight of Travel and recreation, increases from -0.06 to 0.19.

The third, and perhaps most striking conclusion which can be drawn from table 6.15, is that fixing t_4 to zero is accompanied by a drastic reduction of the standard errors of the other t_i coefficients and those of a and b, whereas it induces only a moderate reduction of the standard error of the estimate of β (compare columns 1 and 2). On the other hand, restricting the price index function to be linear ($\beta=1$), leads to a moderate decrease in the standard errors of its estimated weights t_i, and, indeed, a substantial *increase* of the standard errors of a and b (compare columns 1 and 3).

---

28) This conclusion is, however, confined to the model variant with no background variables specified. When the number of household members n is included, imposition of the non-negativity constraint on the t_i's change the estimates of both s_{io} and s_{in} markedly - compare the third column of table A.1 with the third column of table A.2 in the table annex.

Table 6.15. Estimates of the demand coefficients with and without constraints imposed on the coefficients of the price index function $P = (\Sigma_i t_i p_i^{\beta})^{\frac{1}{\beta}}$.

No background variables included.

Σ_{μ} unrestricted

Coefficient	A priori restrictions		
	None	$t_i \in [0,1]$	$\beta = 1$
s_1	0.1368 (0.0148)	0.1367 (0.0148)	0.1366 (0.0148)
s_2	0.1175 (0.0087)	0.1176 (0.0087)	0.1177 (0.0087)
s_3	0.1882 (0.0132)	0.1882 (0.0132)	0.1880 (0.0132)
s_4	0.4486 (0.0184)	0.4486 (0.0184)	0.4488 (0.0184)
s_5	0.1089 (..)	0.1089 (..)	0.1089 (..)
t_1	0.5687 (0.2997)	0.5198 (0.0248)	0.3609 (0.2751)
t_2	0.0553 (0.0503)	0.0628 (0.0160)	0.0859 (0.0441)
t_3	0.4103 (0.1668)	0.3830 (0.0203)	0.2948 (0.1272)
t_4	-0.0602 (0.3661)	0 (..)	0.1936 (0.3117)
t_5	0.0259 (..)	0.0344 (..)	0.0648 (..)
a	0.3954 (0.2736)	0.4459 (0.0482)	0.7786 (0.9373)
b	0.6597 (0.4415)	0.7421 (0.0380)	1.2872 (1.5268)
β	2.2666 (1.8093)	2.0556 (1.3241)	1 (..)

In sum, these observations indicate a 'diversified' pattern of correlation between the estimates of the coefficients in the different part of the model:

- The estimates of the asymptotic budget shares s_i are weakly correlated with the estimates of the coefficients of the price index function, i.e. t_i and β.
- The estimated weights of the price index function, $t_1, \ldots, t_5$, are strongly intercorrelated.
- The estimates of the t_i's are strongly correlated with the income response coefficients a and b, and this correlation is more pronounced than the correlation between the t_i's and β.

Tables 6.16 and 6.17, which are excerpts from the complete *correlation matrices* of the coefficient vector[29], confirm these conjectures. Futhermore, we make the following additional observations:

- In the model variant with no restrictions imposed on the price index function, no coefficient of correlation between t_i and s_i coefficients exceeds 0.25 in absolute value (table 6.16, part A). We interpret this as an indication that our data have no difficulties in 'discerning' the effects going through s_i from the effects through t_i. (Recall that both coefficients have a 'budget share dimension', s_i being the asymptotic budget share of commodity i, and t_i its weight in the price index function in its basic year.)
- When the t_i's are unrestricted, all their estimates are highly intercorrelated. This obviously reflects the high degree of collinearity of the price variables across the sample and the ensuing problems in discerning empirically the effect of the different prices in the price index function. The collinearity is particularly strong in the model variant with a linear price index function, i.e. when $\beta=1$. Then all coefficients of correlation between the estimates of the t_i's exceed 0.99 in absolute value! The correlation between the t_i's and the coefficients of the income response function, i.e. a and b, is of the same order of magnitude (table 6.17, part C). This gives ample evidence of the sort of information on the price responses that we can hope to extract from our data.

---

29) Confer table A.6 in the table annex. These correlation matrices are calculated on the basis of the formulae for the covariance matrix (G.11) in appendix G.

- When t_4 is fixed at its boundary value, zero, the correlation pattern changes drastically (table 6.17, part B). Most compelling is the reduction of the correlation between the estimates of the price index coefficients t_i and β on the one hand and the income response coefficients a and b on the other.

Table 6.16. Coefficients of correlation between the estimates of t_i and s_i. No background variables included. Σ_μ unrestricted

(A) β and t_i unrestricted

	s_1	s_2	s_3	s_4
t_1	-0.007	-0.015	+0.051	-0.023
t_2	-0.015	-0.238	+0.000	+0.111
t_3	+0.054	-0.003	-0.063	+0.000
t_4	-0.015	+0.040	-0.011	-0.008

(B) β unrestricted, $t_4 = 0$

	s_1	s_2	s_3	s_4
t_1	-0.211	+0.241	+0.390	-0.307
t_2	-0.002	-0.690	+0.026	+0.296
t_3	+0.258	+0.243	-0.494	-0.043

(C) $\beta = 1$, t_i unrestricted

	s_1	s_2	s_3	s_4
t_1	+0.015	+0.004	+0.011	-0.020
t_2	-0.018	-0.044	-0.003	+0.033
t_3	+0.025	+0.006	-0.011	-0.014
t_4	-0.018	+0.001	-0.005	+0.014

Table 6.17. Coefficients of correlation between the estimates of coefficients in the income response function C(u) and the price index function P.
No background variables included.
Σ_μ unrestricted

(A) β and t_i unrestricted

	a	b	t_1	t_2	t_3	t_4	β
a	1.000						
b	+0.992	1.000					
t_1	-0.986	-0.996	1.000				
t_2	+0.928	+0.937	-0.941	1.000			
t_3	-0.977	-0.986	+0.976	-0.924	1.000		
t_4	+0.987	+0.997	-0.995	+0.925	-0.989	1.000	
β	-0.587	-0.594	+0.548	-0.482	+0.658	-0.598	1.000

(B) β unrestricted, $t_4 = 0$

	a	b	t_1	t_2	t_3	β
a	1.000					
b	+0.614	1.000				
t_1	-0.239	-0.473	1.000			
t_2	+0.230	+0.467	-0.564	1.000		
t_3	-0.024	-0.076	-0.596	-0.133	1.000	
β	-0.020	-0.076	-0.579	+0.264	+0.576	1.000

(C) $\beta = 1$, t_i unrestricted

	a	b	t_1	t_2	t_3	t_4
a	1.000					
b	+0.997	1.000				
t_1	-0.997	-0.999	1.000			
t_2	+0.990	+0.993	-0.993	1.000		
t_3	-0.996	-0.998	+0.998	-0.992	1.000	
t_4	+0.997	+0.999	-0.999	+0.991	-0.999	1.000

The imposition of restrictions on the price index function, of course, also affects the implied estimates of the *Engel and Cournot elasticities*. The main results are recorded in table 6.18. Not surprisingly, it is the Cournot elasticities which are most strongly affected; the sample mean estimates of the Engel elasticities undergo changes in their third decimal only.[30)] The change in the direct Cournot elasticity is most pronounced when the linearity constraint on the price index function, $\beta=1$, is imposed. This, for instance, decreases the estimate for Food, beverages and tobacco from -0.253 to -0.416 and increases the estimate for Travel and recreation from -1.093 to -0.809. Finally, when both the linearity restriction $\beta=1$ and the non-negativity restriction $t_j \geqq 0$ are imposed, all the estimated direct Cournot elasticities are less than (or equal to) one in absolute value (the last column of table 6.18).

Table 6.18. Estimates of Engel elasticities and direct Cournot elasticities with and without constraints imposed on the coefficients of the price index function $P = (\Sigma_i t_i p_i^{\beta})^{\frac{1}{\beta}}$.
Σ_μ unrestricted. u=50 000 Nkr, and all prices equal to their sample means

Commodity group i	Engel elasticity (E_i)				Direct Cournot elasticity (e_{ii})			
	Restrictions				Restrictions			
	None	$t_j \geqq 0, \forall j$	$\beta=1$	$t_j \geqq 0, \forall j$, and $\beta=1$	None	$t_j \geqq 0, \forall j$	$\beta=1$	$t_j \geqq 0, \forall j$, and $\beta=1$
	No background variables included							
1. Food, beverages and tobacco	0.677	0.677	0.676	0.676	-0.253	-0.242	-0.416	-0.416
2. Clothing and footwear	1.125	1.126	1.128	1.128	-0.736	-0.691	-0.627	-0.627
3. Housing, fuel and furniture	0.851	0.851	0.850	0.850	-0.361	-0.366	-0.520	-0.520
4. Travel and recreation ...	1.286	1.286	1.289	1.289	-1.093	-1.000	-0.809	-0.809
5. Other goods and services .	1.167	1.168	1.168	1.168	-0.826	-0.766	-0.648	-0.648
	Both background variables included; n=3 persons, A=50 years							
1. Food, beverages and tobacco	..	..	0.532	0.527	..	..	-0.533	-0.406
2. Clothing and footwear	..	..	1.137	1.144	..	..	-0.987	-0.834
3. Housing, fuel and furniture	..	..	0.877	0.881	..	..	-0.771	-0.673
4. Travel and recreation ...	..	..	1.338	1.341	..	..	-1.167	-1.000
5. Other goods and services .	..	..	1.240	1.243	..	..	-1.097	-0.917

30) For households outside the sample mean, however, the changes are more pronounced, but by no means substantial - cf. table A.7 in the table annex.

VII. EMPIRICAL RESULTS: SINGLE EQUATION ESTIMATION OF 28 DISAGGREGATED COMMODITY GROUPS

The analysis in chapter VI, concentrated on five aggregated commodity groups only, gives a sort of 'overall' picture of the demand structure. The resulting estimates and test conclusions are interesting as far as they go, but they certainly reveal only a fraction of the information contained in our data. Several questions of considerable practical relevance remain unanswered and they could only be examined within the framework of a more disaggregated commodity specification. In particular, we should like to know more details concerning (i) the effect of the background variables on the composition of consumption, (ii) the significance and relative importance of unobserved individual disturbance components, and (iii) the income and price responses. This motivates a reexamination of the model on the basis of data for the 28 disaggregated commodity groups.

This disaggregation, however, has its price: With 28 commodities and a corresponding number of demand equations involved, simultaneous estimation of the complete model is out of the question. When $N=28$, the total number of independent coefficients would amount to $4N+3=115$ in the model variant with the demographic variables n and A included, and even if both these background variables were omitted it would be as large as $2N+1=57$. (Cf. section 3.1.) Given the non-linearity of the model and the limited variation of the relative prices in our data, attempts at FIML estimation of a coefficient vector of that dimension could almost certainly be predicted to fail, with today's computer technology. We have to fall back on a simpler single equation estimation strategy and accept the reformulation - and possibly reinterpretation - of the model that this approach necessitates.

In section 7.1, we present our adaptation of the Fourgeaud-Nataf model to make it suited to single equation estimation. An account of the coefficient estimates is given in sections 7.2 and 7.3. Testing results for background variables are reported in section 7.4, whereas section 7.5 is concerned with estimates and tests of individual disturbance components. The final section 7.6 deals with the implied estimates of the demand elasticities - with main emphasis on the Engel elasticities - in the different model variants. In most of this chapter, single equation results for the 28 disaggregated commodity groups will be reported along with corresponding results for the 5 aggregated groups. We will also, to some extent, compare the latter with the simultaneous equation results in the previous chapter.

7.1. Adaptation of the model to single equation estimation

Single equation methods for estimating complete systems of demand equations cannot take account of the *cross-equational* coefficient restrictions which are implied by the underlying structural model. In our case, this becomes clear if we rewrite our basic demand model, (3.29), as follows:

$$(7.1)\qquad w_i = s_i + \left(t_i\left(\frac{p_i}{P}\right)^{\beta} - s_i\right)\frac{a\log\frac{y}{P}+b}{\frac{y}{P}}\qquad (i=1,\dots,N),$$

where $s_i = s_{io} + s_{in}n + s_{iA}A \qquad (i=1,\dots,N),$

$a = a_o + a_n n + a_A A,$

$b = b_o + b_n n + b_A A,$

and $P = \left(\sum_{j=1}^{N} t_j\, p_j^{\beta}\right)^{1/\beta},$

dropping, for convenience, the subscripts h and t, and the disturbance terms. Separate estimation of each of the N equations in this system by single equation methods implies that we ignore information of two kinds:

(i) The statistical implications of the adding-up constraint $\sum_{i=1}^{N} w_i = 1$, (i.e. eqs. (3.8) and (3.30)), and

(ii) the restrictions that a_o, a_n, a_A, b_o, b_n, b_A, $t_1,\dots,t_N$, and β are common coefficients in all the demand equations.

One of the restrictions of type (ii) says that β and t_i appear in the i'th demand equation in two places: (a) as coefficients in the expression for P, the common deflator of y and p_i, and (b) as "ordinary" structural coefficients. In addition, the "extraneous" coefficients, i.e. those with subscripts $j \neq i$, are part of the i'th equation only for the reason that they occur in the price index function P. An implication of this is that if the value of P were known *a priori*, the number of (free) coefficients in each equation in (7.1) would be reduced by N-2=26.[1)] Attempts to modify and simplify the model for single equation estimation purposes should therefore primarily be concentrated on these two points.

---

1) When taking account of the restriction $\sum_{i=1}^{N} t_i = 1$. Otherwise, the reduction would be equal to N-1.

Our first simplification is that we replace the price index formula above by

$$(7.2) \qquad \hat{P} = \sum_{j=1}^{N} \bar{w}_j p_j ,$$

where $\bar{w}_j$ denotes the average budget share of commodity j as calculated from the Surveys of Consumer Expenditure for the years 1973 - 1977.[2)] This is equivalent to setting $\beta=1$ and using $\hat{t}_j = \bar{w}_j$ $(j=1,\ldots,N)$ as an *a priori* estimate of t_j when constructing the deflator of the nominal price, p_i, and the total (nominal) expenditure, y, in the model. However, in situations where the individual prices are closely collinear, P is unlikely to be very sensitive to the choice of weights, i.e. $\hat{P}$ will be close to P.[3)] In that case, inconsistencies of this kind may be of minor importance and they are at any rate a price we have to pay for applying single equation methods on large demand systems.

Our attempts to estimate the N=28 demand equations (7.1), with the modifications described above, by means of the iterative single equation procedure outlined in section 4.5, were not successful. With 11 free parameters (s_{io}, s_{in}, s_{iA}, a_o, a_n, a_A, b_o, b_n, b_A, t_i, and β) to be estimated in each equation in each run of step 3 of this algorithm, convergence turned out to be very slow. The estimation of β proved particularly troublesome, and in the cases where convergence was eventually obtained, it gave very poor estimates for this coefficient.[4)]

These experiences motivated our second simplification of the model for single equation estimation, viz. to fix the value of β *a priori*, and we decided to concentrate on two alternative values, $\beta=1$ and $\beta=0$. Both lie within the approximate 95 per cent confidence interval for β, calculated in section 6.1 on the basis of the simultaneous equation estimation results for the five aggregated commodities.

After these modifications, the model (7.1) reads

$$(7.3) \qquad w_i = s_i + (t_i v_i - s_i) \frac{a_i \log u + b_i}{u} \qquad (\beta=1),$$

$$(7.4) \qquad w_i = s_i + (t_i - s_i) \frac{a_i \log u + b_i}{u} \qquad (\beta=0),$$

---

2) See Biørn and Jansen (1980, table 10)
3) A similar observation is made by Deaton and Muellbauer (1980a, p.316) in connection with the AIDS model. Confer also table 6.4 and our remarks on the relationship between the Fourgeaud-Nataf model and the AIDS model in section 2.3 above.
4) The geometrical interpretation of this is that the concentrated log-likelihood function is (or seems to be) extremely flat in the "β direction".

where $v_i = \frac{p_i}{\hat{P}}$, $u = \frac{y}{\hat{P}}$ (with $\hat{P}$ defined as in eq. (7.2)), and

$$s_i = s_{io} + s_{in}n + s_{iA}A,$$

$$(7.5) \quad a_i = a_{io} + a_{in}n + a_{iA}A,$$

$$b_i = b_{io} + b_{in}n + b_{iA}A \qquad (i=1,\ldots,N).$$

Note that we have furnished the coefficients a and b with the commodity subscript i to indicate that their values are not restricted to be the same for all commodities.

7.2. Estimation results when β=1 is imposed a priori

The restriction β=1 implies that the price index function is linear and it is in this respect consistent with the formula used to generate the deflator $\hat{P}$, eq. (7.2). A complete record of the resulting maximum likelihood estimates, confined to the model variant where all background variable are excluded, is given in table B.2 of the table annex. The estimated s_i and t_i coefficients are reproduced in table 7.1 below.

Recalling that s_i can be interpreted as the asymptotic value of the budget share of commodity i (i.e. the limit of w_i when real income goes to infinity, cf. eq. (6.5)), we should expect the estimates of all s_i's to be positive. This is the case for all commodity groups, except 14 Fuel and power.

Table 7.1. Estimates of the commodity specific demand coefficients (t_i and s_i) and corresponding sample mean estimates of the budget shares.
A priori restrictions: β=1. No background variables included

Commodity group	s_i	t_i	Estimated budget share[a)]	Sample mean of budget share
1. Flour and bread	0.0075	0.0246	0.0199	0.0263
2. Meat and eggs	0.0489	0.0578	0.0642	0.0670
3. Fish	0.0036	0.0013	0.0138	0.0160
4. Canned meat and fish ...	0.0006	0.0003	0.0050	0.0061
5. Dairy products	0.0092	0.0303	0.0286	0.0373
6. Butter and margarine ...	0.0004	-0.0005	0.0074	0.0098
7. Potatoes and vegetables	0.0197	-0.0068	0.0439	0.0513

a) Evaluated at u = 50 000 Nkr, and with prices equal to their sample mean values.

Table 7.1 (cont.). Estimates of the commodity specific demand coefficients (t_i and s_i) and corresponding sample mean estimates of the budget shares.
A priori restrictions: $\beta=1$. No background variables included.

Commodity group	s_i	t_i	Estimated budget share a)	Sample mean of budget share
8. Other food	0.0161	0.0031	0.0349	0.0429
9. Beverages	0.0279	0.0271	0.0251	0.0235
10. Tobacco	0.0050	0.0022	0.0138	0.0174
11. Clothing	0.0859	0.0782	0.0809	0.0758
12. Footwear	0.0261	0.0315	0.0188	0.0172
13. Housing	0.0809	0.0760	0.1049	0.1104
14. Fuel and power	-0.0012	-0.0021	0.0322	0.0478
15. Furniture	0.0669	0.1040	0.0511	0.0445
16. Household equipment ...	0.0306	0.0323	0.0286	0.0293
17. Misc. household goods .	0.0124	0.0125	0.0225	0.0260
18. Medical care	0.0128	0.0130	0.0167	0.0171
19. Motorcars, bicycles ...	0.1508	0.2345	0.0686	0.0495
20. Running costs of vehicles	0.1282	-0.6954	0.0836	0.0691
21. Public transport	0.0267	0.0283	0.0266	0.0254
22. P T T charges	0.0128	0.0140	0.0150	0.0147
23. Recreation	0.0924	0.2330	0.0664	0.0589
24. Public entertainment ..	0.0184	0.0231	0.0305	0.0310
25. Books and newspapers ..	0.0101	0.0809	0.0211	0.0226
26. Personal care	0.0131	0.0124	0.0195	0.0200
27. Misc. goods and services	0.0360	0.0887	0.0189	0.0149
28. Restaurants, hotels etc.	0.0415	0.0527	0.0319	0.0282
Sum across disaggregated commodities	0.9833	0.5570	0.9944	1.0000
I Food, beverages and tobacco	0.1357	0.2252	0.2564	0.2975
II Clothing and footwear .	0.1085	0.0940	0.0996	0.0930
III Housing, fuel and furniture	0.1901	0.1516	0.2397	0.2581
IV Travel and recreation .	0.4657	-0.1926	0.3163	0.2712
V Other goods and services	0.0992	0.1001	0.0872	0.0803
Sum across aggregated commodities	0.9992	0.3783	0.9992	1.0001

a) See note a), page 133.

Furthermore, the interpretation of the coefficient t_i as the weight assigned to commodity i in the price index function implies that t_i also ought to be positive. This requirement, however, is not fulfilled for four of the disaggregated commodity groups: 6 Butter and margarine, 7 Potatoes and vegetables, 14 Fuel and power, and 20 Running costs of vehicles, and for the aggregated group IV Travel and recreation.

In table 7.1, we have also included the estimated sample mean values of the budget shares as well as their observed sample meens, recalling that both s_i and t_i may be said to have a "budget share dimension". As expected, we find that the estimates of s_i and t_i are in most cases of the same order of magnitude as the average budget shares. To give a compact characterization of this, we have computed the *rank correlation*[5] between the four series in table 7.1 on the basis of the statistics for the disaggregated commodities. The result is given in table 7.2.

Table 7.2. Rank correlation coefficients between the estimates of the s_i coefficients, the t_i coefficients, the estimated sample mean of the budget shares ($\hat{w}_i$), and the sample mean of the observed budget shares ($\bar{w}_i$) [a)]

	s_i	t_i	$\hat{w}_i$	$\bar{w}_i$
s_i	1			
t_i	0.620	1		
$\hat{w}_i$	0.754	0.384	1	
$\bar{w}_i$	0.634	0.279	0.972	1

a) Based on ranking numbers of the 28 disaggregated commodity groups, derived from table 7.1.

The estimates of s_i and t_i are highly correlated. They both show a significant correlation with the (observed or estimated) sample mean of the budget shares - i.e. the large commodity groups have on the average higher s_i and t_i estimates than the smaller ones - a quite sensible result. This correlation is stronger for the s_i estimates than for the t_i's.

5) The rank correlation coefficient between two series is defined as the coefficient of correlation between the ranking numbers of the elements in the two series (arranged in ascending or descending order).

In chapter II, we established $\sum_{i=1}^{N} s_i = 1$ (eq. (2.5)) as a necessary condition for the model to be consistent with utility maximization. From table 7.1 we observe that this adding-up condition holds approximately for the estimated s_i coefficients both at the aggregated level ($\Sigma s_i = 0.9992$) and for the disaggregated commodity grouping, where the sum of the estimates adds to 0.9833. The corresponding restrictions for the t_i coefficients (eq. (2.6)) are, however, far from being satisfied.

Using equations (2.27) and (2.30), we have calculated the *Engel elasticities* together with the *approximate Slutsky elasticities*[6] for three different values of the real income, see table B.7 of the table annex. The extract given in table 7.3 shows that the single equation estimates of the Engel elasticities for the five aggregated commodities are very close to those obtained by simultaneous equation methods in chapter VI. For the approximate Slutsky elasticities we observe large differences for all commodities except II Clothing and footwear. The single equation estimates of the approximate Slutsky elasticity of I Food, beverages and tobacco must be rejected as implausible on *a priori* grounds since this elasticity comes out with positive estimates for real incomes up to ca. 90 000 Nkr (measured at 1974 prices), which is approximately twice the average real income in our sample.

At the disaggregated level - where S_{ii}^* provides a better approximation to the Slutsky elasticity S_{ii} (and the Cournot elasticity e_{ii}) than in the aggregated case, since the budget shares are smaller (cf. eqs. (2.28)-(2.30)) - we find 7 commodities with an estimated Slutsky elasticity greater than zero (within the income interval 30 000-100 000 Nkr): 1 Flour and bread, 2 Meat and eggs, 5 Dairy products, 14 Fuel and power, 17 Misc. household goods, 18 Medical care, and 24 Public entertainment (see table B.7). In addition, for at least 3 commodity groups (9 Beverages, 19 Motorcars, bicycles, and 26 Personal care) the estimates are implausibly high in absolute value. The estimated Engel elasticities seem on the whole more reasonable.

There may be several reasons why our attempts to estimate price elasticities by single equation methods did not bring much success. In the first place, there is a limited variation in the real prices in our

6) This interpretation of S_{ii}^* in eq. (2.30) is discussed in appendix A, see text below eq. (A.9).

Table 7.3. Estimates of Engel elasticities and the approximate direct Slutsky elasticities based on single equation estimation, as compared with corresponding simultaneous equation estimates.a)
A priori restrictions: $\beta=1$. All prices equal to their sample means. No background variables included. (Σ_μ (ρ) unrestricted)

Commodity group	Single equation estimation			Simultaneous equation estimation		
	Real income, u					
	30 000	50 000	100 000	30 000	50 000	100 000
	Engel elasticities (E_i)					
I Food, beverages and tobacco	0.648	0.675	0.739	0.649	0.676	0.712
II Clothing and footwear	1.137	1.082	1.042	1.178	1.128	1.078
III Housing, fuel and furniture	0.822	0.853	0.897	0.822	0.850	0.893
IV Travel and recreation	1.482	1.316	1.227	1.448	1.289	1.162
V Other goods and services	1.184	1.115	1.062	1.240	1.168	1.100
	Approximate direct Slutsky elasticities (S_{ii}^*)					
I Food, beverages and tobacco	0.333	0.130	-0.039	-0.128	-0.262	-0.452
II Clothing and footwear	-0.292	-0.579	-0.786	-0.366	-0.583	-0.764
III Housing, fuel and furniture	-1.999	-1.765	-1.501	-0.206	-0.385	-0.593
IV Travel and recreation	-1.231	-1.136	-1.071	-0.487	-0.697	-0.842
V Other goods and services	-6.052	-4.081	-2.624	-0.394	-0.611	-0.783

a) Confer tables A.7 and B.7 of the table annex.

data, covering the period 1975 - 1977 only. This is especially the case for some of the disaggregated commodities (see Biørn and Jansen (1980, appendix 2)). Second, we have disregarded all cross-equational constraints implied by the model. The effect of this neglect of a priori information on the estimated price responses is probably reinforced by the limited variation of the real prices in our data. (Confer in this connection our general remarks on the modelization of price responses in section 2.1.)

7.3. Main estimation results when $\beta=0$ is imposed a priori

Imposition of the restriction $\beta=0$ implies that the price variables v_i vanish from the budget share demand functions, and the model takes the simple form (7.4), or

$$w_i = s_i + a_i' \frac{\log u}{u} + b_i' \frac{1}{u} \qquad (i=1,\dots,N), \tag{7.6}$$

where

$$(7.7)\qquad \begin{aligned} a_i' &= a_i(t_i - s_i), \\ b_i' &= b_i(t_i - s_i) \qquad\qquad (i=1,\ldots,N). \end{aligned}$$

Its implicit expenditure functions are of a composite linear and semi-logarithmic form.[7] Moreover, the model has the econometrically attractive property of being linear in the transformed coefficients s_i, a_i', and b_i'.

However, as pointed out in chapter II[8], $\beta=0$ implies that the price index P underlying the complete demand model is a *log-linear* function of the commodity prices, whereas our approximative pre-selected index $\hat{P}$ (as defined in (7.2)) is *linear*. Interpreted within the framework of the Fourgeaud-Nataf class of models, this is, of course, a theoretical inconsistency - and represents a stronger violation of the basic specification than the simplification made in section 7.2. Nevertheless, we think that, empirically, it is not very important, considering the limited relative variation of the prices.

From (7.6) and (7.7) we observe that - unless additional *a priori* restrictions are imposed[9] - the original coefficients a_i, b_i, and t_i $(i=1,\ldots,N)$ cannot be identified in this version of the model, since only their products appear in the expressions for the composite coefficients a_i' and b_i'.

In the model to be estimated, we have chosen to introduce the background variables by letting s_i, a_i', and b_i' depend *linearly* on the family size (n) and the age of the main income earner (A):

$$(7.8)\qquad \begin{aligned} s_i &= s_{io} + s_{in}n + s_{iA}A, \\ a_i' &= a_{io}' + a_{in}'n + a_{iA}'A, \\ b_i' &= b_{io}' + b_{in}'n + b_{iA}'A \qquad\qquad (i=1,\ldots,N). \end{aligned}$$

---

7) Multiplication through (7.6) by u gives an equation expressing $w_i u = v_i x_i = p_i x_i/\hat{P}$ (cf. (2.20)) as the sum of a linear and a log-linear function of u. With constant prices, this implies a relation of the same form between $p_i x_i$ and y.

8) Cf. note 10 of that chapter.

9) For instance, the ones imposed when introducing background variables; cf. below.

In general, the model (7.6)-(7.8) is *not* equivalent to one which follows by inserting (7.5) into (7.4). The latter model would imply *quadratic* terms in the background variables, which the former does not. However, if we introduce the additional restriction

$$(7.9) \qquad a_{in} = a_{iA} = b_{in} = b_{iA} = 0 \qquad\qquad (i=1,\dots,N),$$

eqs. (7.4) and (7.5) may be written as

$$(7.10) \qquad w_i = s_{io} + s_{in}n + s_{iA}A + (t_i - s_{io} - s_{in}n - s_{iA}A)\, a_{io} \frac{\log u}{u}$$
$$+ (t_i - s_{io} - s_{in}n - s_{iA}A)\, b_{io} \frac{1}{u} \qquad (i=1,\dots,N).$$

The correspondence between the two models is then

$$(7.11) \qquad \begin{aligned} a'_{io} &= (t_i - s_{io})\, a_{io}, \\ a'_{in} &= -s_{in}\, a_{io}, \\ a'_{iA} &= -s_{iA}\, a_{io}, \\ b'_{io} &= (t_i - s_{io})\, b_{io}, \\ b'_{in} &= -s_{in}\, b_{io}, \\ b'_{iA} &= -s_{iA}\, b_{io} \qquad\qquad (i=1,\dots,N). \end{aligned}$$

It is easily seen that the structural coefficients s_{io}, s_{in}, s_{iA}, a_{io}, and b_{io} in eq. (7.10) are 'overidentified'. Since we neglect the 'overidentifying' restrictions (7.11) when estimating the model (7.6)-(7.8) by single equation regression methods, it is not possible to establish a one-to-one correspondence between our estimates and the 'original' structural coefficients.

It may well be objected that, after all these modifications, there is not much left of our original starting point, the Fourgeaud-Nataf model in section 3.4. What is retained is essentially only the s_i coefficients and the functional form of income response in the demand equations.

Since the transformed model is linear in the unknown coefficients s_{io}, s_{in}, s_{iA}, a'_{io}, a'_{in}, a'_{iA}, b'_{io}, b'_{in} and b'_{iA}, the stepwise estimation procedure described in section 4.5 for obtaining maximum likelihood estimates consists of repeated application of OLS estimation, conditional on the value of ρ. The coefficient estimates in four different model variants are reported in tables B.1, B.3, B.4, and B.5 of the table annex. The estimates of the coefficients s_i in the variant without background variables are very close to the analogous estimates in section 7.2; i.e. it does not matter very much for the estimated asymptotic budget shares whether β is set equal to one or zero *a priori*, cf. table 7.4. This conclusion agrees with the conclusion established in section 6.6 on the basis of the simultaneous equation results that the estimate of β is weakly correlated with the estimates of s_i. (Confer, in particular, table 6.15.)

Table 7.4. Estimates of the s_i coefficients. A comparison between the model variants with β=1 and β=0. No background variables included

Commodity group	A priori restrictions β=1	A priori restrictions β=0
1. Flour and bread	0.0075	0.0077
2. Meat and eggs	0.0489	0.0497
3. Fish	0.0036	0.0036
4. Canned meat and fish	0.0006	0.0006
5. Dairy products	0.0092	0.0091
6. Butter and margarine	0.0004	0.0004
7. Potatoes and vegetables	0.0197	0.0198
8. Other food	0.0161	0.0158
9. Beverages	0.0279	0.0247
10. Tobacco	0.0050	0.0048
11. Clothing	0.0859	0.0826
12. Footwear	0.0261	0.0257
13. Housing	0.0809	0.0816
14. Fuel and power	-0.0012	-0.0022
15. Furniture	0.0669	0.0671
16. Household equipment	0.0306	0.0299
17. Misc. household goods	0.0124	0.0130
18. Medical care	0.0128	0.0121
19. Motorcars, bicycles	0.1508	0.1502
20. Running costs of vehicles	0.1282	0.1281
21. Public transport	0.0267	0.0285
22. PTT charges	0.0128	0.0131

Table 7.4 (cont.). Estimates of the s_i coefficients. A comparison between the model variants with $\beta=1$ and $\beta=0$. No background variables included

Commodity group	A priori restrictions $\beta=1$	$\beta=0$
23. Recreation	0.0924	0.0926
24. Public entertainment	0.0184	0.0178
25. Books and newspapers	0.0101	0.0097
26. Personal care	0.0131	0.0136
27. Misc. goods and services	0.0360	0.0361
28. Restaurants, hotels, etc.	0.0415	0.0412
I Food, beverages and tobacco	0.1357	0.1365
II Clothing and footwear	0.1085	0.1063
III Housing, fuel and furniture	0.1901	0.1902
IV Travel and recreation	0.4657	0.4658
V Other goods and services	0.0992	0.0988

7.4. Test of background variables

A main conclusion in section 6.1 is that the number of household members (n) is a significant background variable in the demand system at the aggregated 5 commodity level, while the age of the head person (A) is insignificant. This statement, considered as a summary characterization, is interesting enough, but it may have limited appeal to a practitioner. In this section, we perform similar tests for each of the 28 commodity groups to investigate how the above overall conclusion should be qualified at the disaggregated level.

From the model variant with $\beta=0$ (i.e. eqs. (7.6) - (7.8)), we formulate the following four hypotheses:

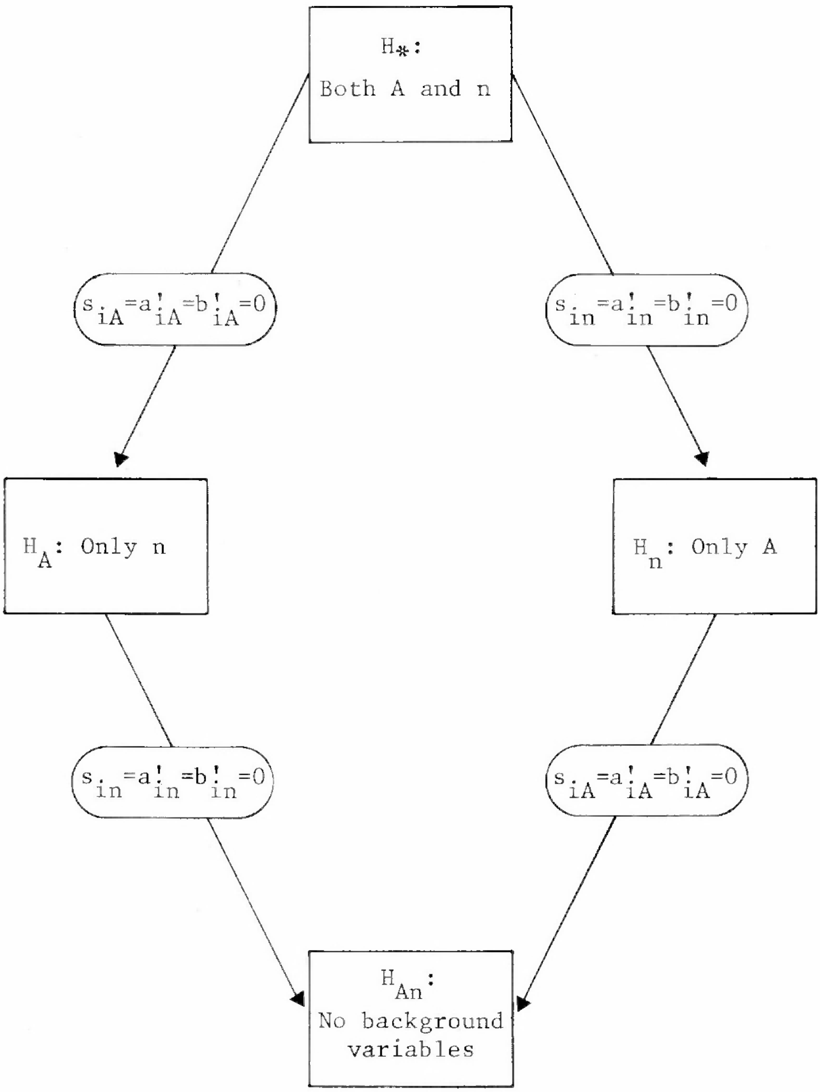
H_* : Both age (A) and number of persons (n) are significant background variables.

H_A : Age (A) is insignificant: $s_{iA} = a'_{iA} = b'_{iA} = 0$.

H_n : Number of persons (n) is insignificant:
$s_{in} = a'_{in} = b'_{in} = 0$.

H_{An} : Both age and number of persons are insignificant:
$s_{iA} = s_{in} = a'_{iA} = a'_{in} = b'_{iA} = b'_{in} = 0$.

Figure 7.1. Testing scheme for background variables in the single equation model



These hypotheses are tested separately for each commodity group - both the 28 disaggregated and the 5 aggregated ones - by means of the Likelihood Ratio principle within the framework of a multiple testing scheme, as illustrated in figure 7.1. The testing procedure is similar to that in section 6.1, so we do not reproduce the details here. (Confer also appendix E, section E.1.)[10] Suffice it to say that since four subtests are involved, with a 1 per cent level assigned to each of them, the total level of significance - i.e. the probability of making at least one false rejection - will have an upper bound of 4 per cent for each commodity group.

A complete record of the test statistics and the final result are given in table B.14 in the table annex. Table 7.5 summarizes the main findings for the 28 commodity grouping. We find at least one significant background variable for exactly half of the commodity groups. For four commodities - including three food commodities and tobacco - both background variables are significant.

Table 7.5 also shows the *sign* of the effect on the estimated budget shares by increasing the value of a *significant* background variable. In general, the results seem plausible, and a closer look gives only some minor surprises. The budget share of 10 Tobacco decreases with increasing age, and family size comes out as a significant background variable for all food commodity groups (groups 1-8) except 3 Fish (indicating that Norwegian children do not fancy fish very much). Moreover, for all the food commodities we find the same sign of the effect of the significant background variables on the estimated budget shares. We also notice that the negative effect of increasing age on the budget share of 20 Running costs of vehicles finds its counterpart in the positive effect for 21 Public transport.

Neither do the five aggregated groups show a uniform picture when tested on the basis of the single equation results. For group I Food, beverages and tobacco and IV Travel and recreation, the number of household members comes out as the only significant background variable, thus supporting the conclusion in section 6.1. (Confer the bottom section of table B.14 in the table annex.) For group III Housing, fuel and furniture, however, both variables are significant, whereas the budget shares for II Clothing and footwear and V Other goods and services are not significantly affected by the background variables at all.

So we may conclude that the overall analysis in section 6.1 conceal a lot of interesting details in the demand structure.

---

10) Note, however, that these tests are conditional on the estimated value of ρ.

Table 7.5. The preferred specification of background variables.[a)]
28 commodity groups

Commodity group, accepted hypothesis	Effect on estimated budget shares of increasing a significant background variable: Number of persons n	Age A
$\underline{H_*}$: 4 commodities		
1 Flour and bread	+	+
5 Dairy products	+	+
8 Other food	+	+
10 Tobacco	−	−
$\underline{H_n}$: 3 commodities		
3 Fish		+
17 Misc. household goods		−
20 Running costs of vehicles		−
$\underline{H_A}$: 7 commodities		
2 Meat and eggs	+	
4 Canned meat and fish	+	
6 Butter and margarine	+	
7 Potatoes and vegetables	+	
15 Furniture	−	
26 Personal care	+	
28 Restaurants, hotels, etc.	−	
$\underline{H_n}$ or $\underline{H_A}$: 1 commodity		
21 Public transport H_A	−	
21 Public transport H_n		+
$\underline{H_{An}}$:		
The remaining 13 commodity groups ..		

a) Based on the results from Likelihood Ratio tests of significance of the background variables and the corresponding coefficient estimates. See tables B.1, B.3, B.4, B.5, B.11, and B.14 in the table annex.

7.5. Individual disturbance components. Estimates and tests

Perhaps the most striking results we have obtained from the single equation analysis of the disaggregated commodity groups are the estimates of ρ, the individual share of the total disturbance variance. Our test of the simultaneous equation model in section 6.1 - and the corresponding estimates reported in section 6.3 - indicated that individual components are present, but gave little information of which commodities contributed most to this result. It is to this question that we now turn.

Table B.12 in the table annex contains single equation maximum likelihood estimates of ρ when $\beta=0$ for all the model variants considered.[11] We have performed Likelihood Ratio tests of the significance of this parameter, confining attention to the equations including the preferred specification of background variables[12] only. In appendix E, we show that the statistic for testing $\rho=0$ against $\rho\neq0$ ($\rho>0$) is

$$(7.12)\quad -2 \log \lambda = 2M\{\log \tilde{\sigma}^2 - \log \hat{\sigma}^2 - \log(1-\hat{\rho}^2)\},$$

where $\tilde{\sigma}^2$ and $\hat{\sigma}^2$ are the Maximum Likelihood estimate of the total disturbance variance σ^2 when $\rho=0$ and $\rho\neq0$, respectively, and $\hat{\rho}$ is the estimate of ρ. This statistic is approximately distributed as $\chi^2(1)$ under the null hypothesis $\rho=0$. The results for all the 28 disaggregated and the 5 aggregated groups are given in table B.15.[13] In table 7.6, we have recorded the main findings for the disaggregated groups, ranking, for convenience, the commodities by decreasing values of the estimate of ρ.

We observe that $\hat{\rho}$ is significantly different from zero at the 1 per cent level for all the 5 aggregated commodities, while 20 of the 28 disaggregated commodities have a significant $\hat{\rho}$ at the 1 per cent level and 2 additional commodities are significant at the 5 per cent level. These 22 commodities account for about 85 per cent of the budget of the average consumer. The 6 commodities for which no "individual idiosyncrasies" can be detected are 15 Furniture, 12 Footwear, 27 Miscellaneous goods and services, 16 Household equipment, 18 Medical care, and 21 Public transport - all having an estimated $|\rho|$ value less than 0.05. For the last 2 commodities at least, this is somewhat unexpected. It comes as no surprise, however, that we find commodity group 10 Tobacco on the top of the list; not less than 70 per cent of the total disturbance variance can be attributed to the individual component for this commodity. In general, we find that food commodities have a high $\hat{\rho}$. Nine of the 17 commodities with estimates exceeding 0.20 belong to the aggregated group I Food, beverages and tobacco. But also recreational commodities as 24 Public entertainment, 25 Books and newspapers, 20 Running costs of vehicles, and 28 Restaurants, hotels etc., as well as 13 Housing and 11 Clothing, occupy a high position on this ranking list.

---

11) Selected results for commodity groups 5 and 10 have been published earlier, in Biørn (1981a, section 6).
12) I.e. the specification which contains the significant background variables. Cf. table 7.5.
13) The critical $\chi^2(1)$ value is 6.64 at the 1 per cent level, and 3.84 at the 5 per cent level.

Table 7.6. Estimates of the individual share of the total disturbance variance, ranked in descending order, and corresponding Likelihood Ratio test statistics.[a)]
Based on preferred specification of background variables.
β=0.
28 commodity groups

Commodity group	$\hat{\rho}$	$-2 \log \lambda$[b)]	Preferred specification of background variables[c)]
10. Tobacco	0.707	567.3	H_*
3. Fish	0.439	177.9	H_n
24. Public entertainment ...	0.424	164.7	H_{An}
25. Books and newspapers ...	0.420	161.9	H_{An}
1. Flour and bread	0.383	129.8	H_*
5. Dairy products	0.369	121.4	H_*
13. Housing	0.362	117.0	H_{An}
7. Potatoes and vegetables	0.333	97.3	H_A
9. Beverages	0.312	85.1	H_{An}
20. Running costs of vehicles	0.296	76.7	H_n
28. Restaurants, hotels, etc.	0.291	73.6	H_A
11. Clothing	0.287	71.9	H_{An}
17. Misc. household goods ..	0.255	56.1	H_n
8. Other food	0.251	53.7	H_*
26. Personal care	0.250	53.3	H_A
6. Butter and margarine ...	0.238	48.8	H_A
2. Meat and eggs	0.217	40.1	H_A
4. Canned meat and fish ...	0.196	32.6	H_A
22. PTT charges	0.175	25.8	H_{An}
14. Fuel and power	0.108	9.6	H_{An}
23. Recreation	0.071	4.2	H_{An}
21. Public transport	0.049/0.050	2.01/2.05	H_A or H_n
16. Household equipment	0.043	1.5	H_{An}
27. Misc. goods and services	0.035	1.0	H_{An}
18. Medical care	0.007	0.04	H_{An}
12. Footwear	-0.009	0.07	H_{An}
15. Furniture	-0.009	0.06	H_A
19. Motorcars, bicycles	-0.081	5.4	H_{An}

a) Extracted from tables B.12, B.14, and B.15.
b) See appendix E, section E.3.
c) Confer figure 7.1 and table 7.5.

Commodity 19 Motorcars, bicycles takes the bottom position; its estimated ρ is -0.08, which is significantly *negative* at the 5 per cent level. This, of course, is no statistically meaningful result if interpreted as an estimate of the ratio between the variance of the individual disturbance component and the variance of the total disturbance. But if interpreted as an estimate of the coefficient of correlation between the two disturbances from the same individual, the result is quite sensible: For durables like motorcars, our maintained hypothesis of zero correlation between the two disturbance components μ_{ih} and ν_{iht} is questionable; a household which buys a new car in one year is unlikely to buy another one the next year - a fact that suggests negative correlation as a more reasonable hypothesis. As we noticed in section 3.2, the disturbances ε_{ih1} and ε_{ih2} may well be negatively correlated if this possibility is taken into account. That the negative correlation is not stronger can be explained by the fact that the majority of households report no expenditure at all on this item in any of the years. Note, incidentally, that the other two commodities for which the estimated ρ is negative, 12 Footwear and 15 Furniture, also have the basic characteristics of durables. This supports the interpretation given above.[14)]

The estimates of ρ may be influenced by the omission of some relevant explanatory variables from the structural part of the model.[15)] Seasonal variables are notable examples. Since each household is observed *in the same 14 days' period* in both years, and no seasonals are included in the model, one may say that we *individualize* the seasonal variation. The estimated values of ρ for commodities which are known to exhibit a pronounced seasonal pattern - e.g. 7 Potatoes and vegetables and 24 Public entertainment (confer diagrams W7 and W24 in Biørn and Jansen (1980, pp. 88 and 105)) - should be interpreted with this in mind.

On this background, it may be rewarding to compare the estimates of ρ for all specifications of background variables, reported in table B.12. The table also includes the *marginal* estimates of ρ, i.e. the coefficients of correlation between the observed budget shares from the two sets of reports. (Column M in table B.12.) *A priori*, we should

---

14) It should be admitted, of course, that the static theory underlying our model is not a particularly good theory for durables. If the dynamics of the purchase pattern for these commodities were accounted for (or if purchase figures were replaced by estimated service values), we should expect the disturbances to behave in a more 'normal' way.
15) The problem is of the same nature as the problem with autocorrelated disturbances that may be the consequence of omitting important cyclical variables in a model estimated from pure time series data.

expect that the estimated ρ would drop when we include further explanatory variables characterizing the household, e.g. the background variables. This hypothesis is confirmed for some goods - 5 Dairy products and 1 Flour and bread - while others exhibit a different pattern. For a commodity like 10 Tobacco the estimate of ρ is fairly stable - the lowest $\hat{\rho}$ is indeed found in the marginal case - i.e. the variance of the individual component of the total disturbance variance is reduced in approximately equal proportion to the reduction in the total disturbance variance.

7.6. Estimates of Engel elasticities based on single equation estimation of model variants with β=0

Since the model variants with $\beta=0$ imposed *a priori* imply that the price variables v_i vanish from the demand functions, the elasticity of w_i with respect to v_i will be zero. This means that the approximate Slutsky elasticities S_{ii}^* are constant and equal to -1 for all commodities. (Cf. eq. (2.30).) Besides, this value provides a first order approximation to the direct Cournot elasticity since all estimates of t_j and π_j are small at the disaggregated level of classification (cf. eqs. (2.28) and (2.29)). This is all we can say about the price responses in this case.

The Engel elasticities calculated from (7.6) are

$$(7.13)\qquad E_i = \frac{s_i u + a_i'}{s_i u + a_i' \log u + b_i'}, \qquad (i=1,\dots,N).$$

In table 7.7, we have recorded their estimated values for the 28 disaggregated commodity groups from the four model variants with $\beta=0$ along with the analogous results from the model variant with $\beta=1$ (and no background variables included). All estimates are evaluated at the approximate sample mean values of the explanatory variables (u=50 000 Nkr, n=3 persons, A=50 years, and all nominal prices set equal to their sample means). We observe that the estimates from the model variants with no background variables included (H_{An}) and with $\beta=0$ are almost identical to those obtained when β is set equal to 1. This means that the curvature assumed for the price index function has a very slight effect on the Engel elasticities. An analogous conclusion was obtained for the s_i coefficients in section 7.3.

Table 7.7. Estimates of Engel elasticities for the 28 disaggregated commodities, based on single equation estimation with different specifications of background variables.a)
A priori restrictions: β=0 or β=1. u=50 000 Nkr, A=50 years and n=3 persons (when included in the model variant).
ρ unrestricted

Commodity group	Model variant b)				
	β=1	β=0			
	H_{An}	H_{An}	H_A	H_n	H_*
1. Flour and bread	0.509	0.513	0.248	0.429	0.242*
2. Meat and eggs	0.856	0.863	0.760*	0.870	0.773
3. Fish	0.540	0.541	0.513	0.663*	0.626
4. Canned meat and fish .	0.433	0.432	0.353*	0.499	0.435
5. Dairy products	0.492	0.486	0.136	0.379	0.133*
6. Butter and margarine .	0.350	0.346	0.038*	0.268	0.019
7. Potatoes and vegetables	0.629	0.631	0.454*	0.607	0.471
8. Other food	0.601	0.597	0.408	0.564	0.429*
9. Beverages	1.092	1.028*	1.121	0.956	1.050
10. Tobacco	0.537	0.526	0.550	0.220	0.285*
11. Clothing	1.066	1.046*	0.999	1.068	1.022
12. Footwear	1.243	1.231*	1.136	1.198	1.120
13. Housing	0.855	0.859*	0.871	0.840	0.864
14. Fuel and power	0.208	0.194*	0.082	0.219	0.145
15. Furniture	1.227	1.229	1.328*	1.327	1.414
16. Household equipment ..	1.022	1.009*	1.007	1.060	1.036
17. Misc. household goods	0.684	0.703	0.674	0.846*	0.807
18. Medical care	0.872	0.850*	0.922	1.167	1.125
19. Motorcars, bicycles ..	1.756	1.751*	1.924	1.875	1.960
20. Running costs of vehicles	1.365	1.364	1.307	1.208*	1.195
21. Public transport	1.029	1.061	1.177*	1.069*	1.179
22. PTT charges	0.936	0.948*	1.238	1.086	1.277
23. Recreation	1.258	1.260*	1.317	1.162	1.236
24. Public entertainment .	0.793	0.769*	0.780	0.804	0.760
25. Books and newspapers .	0.701	0.687*	0.788	0.844	0.872
26. Personal care	0.830	0.835	0.664*	0.769	0.666
27. Misc. goods and services	1.569	1.572*	1.780	1.828	1.909
28. Restaurants, hotels etc.	1.215	1.211	1.388*	1.108	1.260

a) Cf. tables B.7, B.6, B.8, B.9, and B.10, respectively, in the table annex.

b) H_{An}, H_A, H_n, and H_* refer to the following hypotheses:

H_* : Both n and A included.

H_A : A excluded.

H_n : n excluded.

H_{An}: No background variables included.

An asterisk (*) indicates which hypothesis was accepted, confer table 7.4.

On the other hand, introduction of background variables in the model has a pronounced effect on the Engel elasticity estimates. This is particularly the case for the family size variable, which, of course, reflects the fact that the number of household members and total expenditure are positively correlated variables. When the former is omitted, the estimated Engel elasticity will have the character of a "gross elasticity" representing the joint effect of income and family size.

We conclude by giving some brief comments on the specific estimates for selected commodities, confining attention to the hypothesis which comes out as the preferred one from the significance test of background variables (marked with an asterisk (*) in table 7.7):

- All food commodities are necessities ($E_i<1$) and the individual estimates include no implausibilities. The estimates for 3 Butter and margarine (0.038), 5 Dairy products (0.133) and 1 Flour and bread (0.242) are the lowest within this group, whereas 2 Meat and eggs (0.760) and 3 Fish (0.663) come on top.

- The estimated Engel elasticity for 10 Tobacco (0.285) is relatively low as compared with findings from other studies, confer our remarks in subsection 6.4.1 and the references made there. However, the consumption pattern for tobacco in Norway has undergone rapid changes during the 1970s[16]. Many people have quitted smoking as a result of the public anti-smoking campaign which has been launched by Norwegian health authorities in that period. The conjecture that this campaign has been more successful in the higher income brackets than in the lower ones finds support in our Engel elasticity estimate.

- The commodity groups which distinguish themselves as 'luxuries' ($E_i>1$) comprise such items as 19 Motorcars, bicycles (1.751), 27 Misc. goods and services, i.e. jewellery, watches, travel goods, etc. (1.572), 28 Restaurants, hotels etc. (1.388), 15 Furniture (1.328), and 23 Recreation (1.260). All these commodities have the common sense characteristics of luxuries.

---

16) See Biørn and Jansen (1980, p.45).

VIII. CONCLUSION

To our knowledge, this is the first study of its kind based on cross-section/time-series data from household budget surveys. For us, the challenge has been to solve - with data from *individual* households - the problem of simultaneous FIML estimation of a complete demand system incorporating demographic variables and a stochastic specification with an error component structure. The solution is obtained by developing an iterative step-wise algorithm which simplifies the numerical maximization of a relatively complex likelihood function.

The stochastic disturbance terms in the demand equations are decomposed into two independent parts: an individual component and a remainder. A main conclusion is that the unobservable individual factors are significant explanatory variables, and for several commodities their order of magnitude is substantial. No previous study has, as far as we know, provided information of this kind.

The Fourgeaud-Nataf specification, which forms the basis of the formulation of the structural part of our model, is shown to possess several attractive properties for analyzing household budget data. First, under certain conditions, it is compatible with utility maximization, and it contains several interesting demand systems as special cases. Second, from a computational point of view it is a major advantage with this model that even if it admits non-linear Engel curves, it is parsimonious in terms of the number of coefficients to be estimated. Third, it is easy to incorporate demographic variables characterizing type of household into the model structure.

This choice of parametrization is also motivated by our particular data situation. With data spanning three years only, the relative variation of the prices is substantially less than that of the other explanatory variables. This is matched with the fact that the model offers a fairly flexible treatment of income effects and a relatively restrictive parametrization of the price responses - although less restrictive than, say, the well known LES system. It is thus not surprising that the coefficients characterizing price responses turn out to cause the most serious estimation problems and that the model renders more reliable estimates for the Engel elasticities than for the Cournot and Slutsky elasticities.

A highly interesting by-product from the simultaneous equation estimation is the implied estimates of the income flexibility. Our results support Frisch's famous conjecture that its absolute value is a decreasing function of income.

The model incorporates two demographic background variables to represent different types of households. The simultaneous estimation of demand equations for 5 aggregated commodity groups discloses that the first, the number of household members, is a significant explanatory variable, whereas the second, the age of the main income earner, is not.

Although we regard the simultaneous FIML estimates of the demand system for 5 aggregated commodities as our main result, we have also found it worthwhile to exploit the detailed information in our data base at a lower level of commodity aggregation. With 28 commodities, while making use of single equation estimation methods, a considerably richer and more differentiated picture of the demand structure emerges. A significant part of the disturbance variance can be ascribed to individual differences for the great majority of commodity groups, whereas at least one of the background variables are significant for one half of the commodities. The implied estimates of the demand elasticities are also interesting and may have several practical applications. The Engel elasticities, for instance, have already been used in updating the demand coefficients of the Norwegian medium term planning model *MODIS IV*, see Bjerkholt and Longva (1980).

As is always the case with empirical studies, the investigators are more or less forced to make a number of simplifying assumptions and to treat some aspects of their analysis less rigorously than others. Our study is no exception:

- Our analysis is static, whereas the existence of durables calls for a dynamic approach. A proper treatment of the demand for durable goods should in some way or another pay regard to habit formation (e.g. by including lagged real income, previous stocks or purchases, etc. as explanatory variables). Apart from the increased complexity of the analytical problem that would follow, our data base hardly contains sufficient information for extending the analysis in that direction.

- Zero-reporting is a serious problem attached to the use of individual reports as the data base for demand analysis, in particular at a detailed level of commodity classification. This problem is discussed in Biørn and Jansen (1980, pp. 34-37) in the context of testing the marginal distributions of the budget shares for normality. These tests indicated significant deviations from normality. This fact and its possible harmfull effects are not pursued in the present analysis.

- There are several relevant modifications of the stochastic specification of our model which we had to disregard in this study. *A priori*, good reasons could be given for relaxing the assumption of zero correlation between the individual disturbance components and the structural variables included. Omitted background variables (like seasonals), for instance, may cause correlation between the individual disturbance term and the specified explanatory variables and hence violate one of the basic assumptions of our model.

These problems definitely deserve further research. Research directed more systematically towards the utilization of estimation results from error component models for practical *prediction* purposes may also be rewarding. Finally, an augmentation of the data base along the 'time dimension', i.e. including data for a longer time span than three years will tend to extend the variation of the relative prices and thus increase the chances of getting more reliable estimates of the price responses.

LIST OF SYMBOLS AND NOTATIONAL CONVENTIONS

Important variables and coefficients have a common notation in the main text and in appendices A, D, and E of this study. In the following list, reference is made to the section in which a particular symbol first occurs.

Symbol	Description	Section
A	age of the main income earner in the household	3.1
a	coefficient of $\log u$ in the income response function $C(u)$	2.3
a_o	constant term in the linear demographic translating of the coefficient a	3.1
a_n	coefficient of n in the linear demographic translating of the coefficient a	3.1
a_A	coefficient of A in the linear demographic translating of the coefficient a	3.1
a_i'	transformed coefficient, equal to $a(t_i - s_i)$	7.3
$a_{io}', a_{in}', a_{iA}'$	coefficients in the linear demographic translating of the coefficient a_i'	7.3
b	constant term in the income response function $C(u)$	2.3
b_o	constant term in the linear demographic translating of the coefficient b	3.1
b_n	coefficient of n in the linear demographic translating of the coefficient b	3.1
b_A	coefficient of A in the linear demographic translating of the coefficient b	3.1
b_i'	transformed coefficient, equal to $b(t_i - s_i)$	7.3
$b_{io}', b_{in}', b_{iA}'$	coefficients in the linear demographic translating of the coefficient b_i'	7.3
C, $C(u)$	value of the income response function in the Fourgeaud-Nataf model	2.2
E_n	$n \times n$ matrix with all elements equal to one	3.2
E_i	Engel elasticity of commodity i	2.4
E_{iu}	partial elasticity of the budget share of commodity i with respect to real income	Appendix A
E_{iv}	partial elasticity of the budget share of commodity i with respect to own real price	Appendix A
e_{ij}	Cournot elasticity of commodity i with respect to the price of commodity j	2.4
g	short-hand expression for the non-constant part of the log-likelihood function, i.e. $\log\|\Omega\| + \varepsilon'\Omega^{-1}\varepsilon$	4.1

		Section
h	subscript for household number	3.2
I_n	n x n identity matrix	3.2
i,j	subscripts for commodity group	2.2
M	number of households	3.2
N	number of commodities	2.2
n	family size, i.e. number of household members	3.1
P	total price index (function)	2.2
$\hat{P}$	approximation to P, used for single equation estimation of the model	7.1
P_{ht}	total price index in the period when household h gives its t'th report	3.4
p_i	price (index) of commodity i	2.2
p_{iht}	price of commodity i in the period when household h gives its t'th report	3.4
Q	short-hand expression for the quadratic form $\varepsilon'\Omega^{-1}\varepsilon$ in the log-likelihood function	4.1
S_{ij}	Slutsky elasticity of commodity i with respect to the price of commodity j	2.4
S_{ii}^*	approximate direct Slutsky elasticity of commodity i	2.4
s_i	commodity specific structural coefficient in the Fourgeaud-Nataf model (the asymptotic budget share of commodity i)	2.2
s_{io}	constant term in the linear demographic translating of the coefficient s_i	3.1
s_{in}	coefficient of n in the linear demographic translating of the coefficient s_i	3.1
s_{iA}	coefficient of A in the linear demographic translating of the coefficient s_i	3.1
t	subscript for number of report	3.2
t_i	weight of commodity i in the price index function	2.2
u	total real income (consumption expenditure)	2.2
u_{ht}	total real income, household h, report t	3.4
V, V^*	indirect utility function expressed in terms of nominal expenditure and nominal prices ($V = \log V^*$)	2.2
v_i	real price of commodity i	2.2
v_{iht}	real price of commodity i, in the period when household h gives its t'th report	3.4
W, W^*	indirect utility function expressed in terms of real expenditure and real prices ($W = \log W^*$)	2.2
w	2NM x 1 vector of observations of budget shares	4.1
w_i	budget share of commodity i	2.2
$\bar{w}_i$	average budget share of commodity i	6.2
w_{iht}	budget share of commodity i, household h, report t	3.4
x_i	quantity consumed of commodity i	2.2
y	total nominal value of consumption expenditure	2.2
Z	short-hand expression for the matrix of observations on exogenous variables	4.1

		Section
α	short-hand expression for the vector of structural demand coefficients	4.1
β	structural coefficient of the Fourgeaud-Nataf model	2.2
δ_{ij}	Kronecker delta (δ_{ij} = 1 for i=j, 0 otherwise)	3.2
ε	2NM x 1 vector of all disturbance terms	3.2
ε_h	vector of disturbance terms from household h	3.2
ε_{ih}	vector of disturbance terms for commodity i, household h	3.2
ε_{iht}	disturbance term for commodity i, household h, report t	3.2
η_h	2N x 1 vector of transformed residuals from household h	4.3
Λ	value of the likelihood function	4.1
λ	Likelihood Ratio	Appendix E
μ_{ih}	household-specific component in the disturbance term of commodity i	3.2
ν_{iht}	component of the disturbance term for commodity i, household h and report t, which is not due to individual differences (remainder component)	3.2
π_i	elasticity of the price index function with respect to the price of commodity i	2.2
ρ	individual part in the variance of the disturbance term in the single equation case	4.5
Σ	'total' N x N covariance matrix ($=\Sigma_\mu+\Sigma_\nu$)	4.3
Σ_μ	N x N matrix with typical element σ^μ_{ij}	3.2
Σ_ν	N x N matrix with typical element σ^ν_{ij}	3.2
Σ_{ij}	2 x 2 matrix equal to $\sigma^\mu_{ij}\ E_2 + \sigma^\nu_{ij}\ I_2$	3.2
σ^2	variance of the disturbance term in the single equation case	4.5
σ_{ij}	covariance between the disturbance terms of commodity i and commodity j	3.2
σ^μ_{ij}	covariance between the individual components of the disturbances of commodity i and commodity j	3.2
σ^ν_{ij}	covariance between the remainder components of the disturbance terms of commodity i and commodity j	3.2
$\Psi(u)$	auxiliary function in the indirect utility function V	2.2
Ω	2MN x 2MN complete covariance matrix of the vector ε	3.2
Ω_*	2N x 2N covariance matrix of the vector ε_h	3.2
ω	the Frisch parameter, i.e. the income flexibility	2.4

TABLE ANNEX*)

Part A. Simultaneous equation estimation of the complete 5 commodity model.

Part B. Single equation estimation of the model (28+5 commodity groups).

---

*) The aggregated commodities are, for typographical reasons, numbered by Arabic numerals (1,2,...,5) in part A of this table annex. In part B, we use Arabic numerals (1,2,...,28) for the disaggregated commodities and Roman numerals (I,II,...,V) for the aggregated ones.

CONTENTS

Table A.1. FIML estimates of model coefficients with different specifications of background variables.a)
A priori restrictions: $\beta=1$, $t_i \in [0,1]$ $(i=1,2,\ldots,5)$

Coefficient	Both n and A		Only n		Only A b)		No background variables b)	
	Σ_μ unrestr.	$\Sigma_\mu=0$	Σ_μ unrestr.	$\Sigma_\mu=0$	Σ_μ unrestr.	$\Sigma_\mu=0$	Σ_μ unrestr.	$\Sigma_\mu=0$
s_{1o}	-0.0251 (0.0768)	-0.0361 (0.0752)	0.0577 (0.0321)	0.0823 (0.0303)	0.0980 (0.0602)	0.1068 (0.0571)	0.1366 (0.0148)	0.1568 (0.0137)
$s_{1n}\cdot 10^2$	1.4978 (0.9446)	1.2918 (0.8984)	1.0647 (0.8593)	0.8014 (0.8020)	-	-	-	-
$s_{1A}\cdot 10^2$	0.1301 (0.1105)	0.1964 (0.1083)	-	-	0.0474 (0.1082)	0.0738 (0.1011)	-	-
s_{2o}	0.0896 (0.0322)	0.0928 (0.0290)	0.0939 (0.0169)	0.0943 (0.0152)	0.1329 (0.0340)	0.1420 (0.0331)	0.1177 (0.0087)	0.1187 (0.0079)
$s_{2n}\cdot 10^2$	0.8055 (0.4478)	0.8078 (0.3995)	0.7798 (0.4180)	0.7738 (0.3709)	-	-	-	-
$s_{2A}\cdot 10^2$	0.0088 (0.0451)	0.0037 (0.0399)	-	-	-0.0256 (0.0624)	-0.0407 (0.0597)	-	-
s_{3o}	0.3122 (0.0480)	0.3106 (0.0442)	0.2749 (0.0247)	0.2749 (0.0227)	0.2123 (0.0533)	0.2433 (0.0520)	0.1880 (0.0132)	0.1894 (0.0123)
$s_{3n}\cdot 10^2$	-2.5426 (0.6870)	-2.5469 (0.6297)	-2.2637 (0.6383)	-2.2202 (0.5810)	-	-	-	-
$s_{3A}\cdot 10^2$	-0.0592 (0.0676)	-0.0514 (0.0613)	-	-	-0.0568 (0.0968)	-0.1092 (0.0938)	-	-
s_{4o}	0.5029 (0.0828)	0.5037 (0.0778)	0.4452 (0.0370)	0.4190 (0.0340)	0.4746 (0.0738)	0.4183 (0.0704)	0.4488 (0.0184)	0.4273 (0.0171)
$s_{4n}\cdot 10^2$	0.4962 (1.0843)	0.6973 (0.9964)	0.7428 (0.9954)	0.9541 (0.9065)	-	-	-	-
$s_{4A}\cdot 10^2$	-0.0943 (0.1182)	-0.1496 (0.1105)	-	-	-0.0225 (0.1337)	0.0365 (0.1270)	-	-
s_{5o}	0.1204 (..)	0.1290 (..)	0.1283 (..)	0.1295 (..)	0.0822 (..)	0.0896 (..)	0.1089 (..)	0.1078 (..)
$s_{5n}\cdot 10^2$	-0.2569 (..)	-0.2500 (..)	-0.3236 (..)	-0.3091 (..)	-	-	-	-
$s_{5A}\cdot 10^2$	0.0146 (..)	0.0009 (..)	-	-	0.0575 (..)	0.0396 (..)	-	-
a_o	0.5218 (0.2025)	0.5210 (0.2086)	0.3029 (0.0677)	0.2569 (0.0696)	1.1481 (0.5153)	1.1450 (0.4054)	0.7786 (0.9373)	0.4754 (0.8148)
$a_n\cdot 10^2$	2.4599 (3.7683)	3.1909 (3.7203)	3.8914 (3.4389)	4.6449 (3.3731)	-	-	-	-
$a_A\cdot 10^2$	-0.3106 (0.2730)	-0.4022 (0.2844)	-	-	-0.5927 (0.4425)	-0.4517 (0.4696)	-	-
b_o	0.4775 (0.1267)	0.5073 (0.1262)	0.5102 (0.0605)	0.4929 (0.0602)	1.3553 (0.5746)	1.5337 (0.4490)	1.2872 (1.5268)	0.8268 (1.3948)
$b_n\cdot 10^2$	13.381 (3.053)	11.866 (2.969)	12.883 (2.983)	11.351 (2.870)	-	-	-	-

a) Note that the unit of measurement for real income is 10 000 Nkr.
b) The result is identical with the corresponding alternative in table A.2, i.e. none of the restrictions on the t coefficients are effective.

Tabell A.1 (cont.). FIML estimates of model coefficients with different specifications of background variables.[a]
A priori restrictions: $\beta=1$, $t_i \in [0,1]$ $(i=1,2,\ldots,5)$

Coefficient	Both n and A		Only n		Only A[b]		No background variables[b]	
	Σ_μ unrestr.	$\Sigma_\mu=0$	Σ_μ unrestr.	$\Sigma_\mu=0$	Σ_μ unrestr.	$\Sigma_\mu=0$	Σ_μ unrestr.	$\Sigma_\mu=0$
$b_A \cdot 10^2$	0.0234 (0.1726)	-0.0706 (0.1764)	-	-	-0.1603 (0.2472)	-0.1484 (0.2945)	-	-
t_1	0.5727 (0.0231)	0.5817 (0.0247)	0.5715 (0.0227)	0.5779 (0.0236)	0.3644 (0.1017)	0.3315 (0.0503)	0.3609 (0.2751)	0.4690 (0.5430)
t_2	0.0607 (0.0148)	0.0531 (0.0158)	0.0619 (0.0145)	0.0570 (0.0150)	0.0851 (0.0164)	0.0885 (0.0101)	0.0859 (0.0441)	0.0631 (0.1036)
t_3	0.3415 (0.0169)	0.3473 (0.0179)	0.3397 (0.0167)	0.3458 (0.0173)	0.2944 (0.0496)	0.2792 (0.0280)	0.2948 (0.1272)	0.3548 (0.2809)
t_4	0 (..)	0 (..)	0 (..)	0 (..)	0.1909 (0.1155)	0.2305 (0.0585)	0.1936 (0.3117)	0.0701 (0.6181)
t_5	0.025 (..)	0.0179 (..)	0.0269 (..)	0.0193 (..)	0.0652 (..)	0.0703 (..)	0.0648 (..)	0.0430 (..)
β	1	1	1	1	1	1	1	1
No. of free coef.[c]	21	21	15	15	16	16	10	10
log Λ	2864.8	2798.1	2862.4	2794.3	2821.1	2750.2	2818.1	2747.8

a) See note a, page 161.
b) See note b, page 161.
c) In optimum.

Table A.2. FIML estimates of model coefficients with different specifications of background variables.[a)]
A priori restriction: $\beta=1$ ($t_1,\ldots,t_5$ unrestricted)

Coefficient	Both n and A		Only n		Only A		No background variables	
	Σ_μ unrestr.	$\Sigma_\mu=0$	Σ_μ unrestr.	$\Sigma_\mu=0$	Σ_μ unrestr.	$\Sigma_\mu=0$	Σ_μ unrestr.	$\Sigma_\mu=0$
s_{1o}	0.0127 (0.0779)	0.0083 (0.0076)	0.0747 (0.0326)	0.1021 (0.0305)	0.0980 (0.0602)	0.1068 (0.0571)	0.1366 (0.0148)	0.1568 (0.0137)
$s_{1n}\cdot 10^2$	0.8253 (1.0267)	0.6266 (0.9827)	0.4491 (0.9327)	0.0848 (0.8754)	–	–	–	–
$s_{1A}\cdot 10^2$	0.0949 (0.1094)	0.1480 (0.1068)	–	–	0.0474 (0.1082)	0.0738 (0.1011)	–	–
s_{2o}	0.0957 (0.0264)	0.0975 (0.0246)	0.0988 (0.0146)	0.0997 (0.0129)	0.1329 (0.0340)	0.1420 (0.0331)	0.1177 (0.0087)	0.1187 (0.0079)
$s_{2n}\cdot 10^2$	0.6281 (0.3774)	0.6468 (0.3480)	0.5985 (0.3421)	0.5921 (0.2954)	–	–	–	–
$s_{2A}\cdot 10^2$	0.0067 (0.0360)	0.0039 (0.0325)	–	–	-0.0256 (0.0624)	-0.0407 (0.0597)	–	–
s_{3o}	0.2935 (0.0463)	0.2894 (0.0446)	0.2645 (0.0256)	0.2634 (0.0227)	0.2123 (0.0533)	0.2433 (0.0520)	0.1880 (0.0132)	0.1894 (0.0123)
$s_{3n}\cdot 10^2$	-2.1693 (0.7553)	-2.1435 (0.7180)	-1.9506 (0.6766)	-1.8737 (0.5879)	–	–	–	–
$s_{3A}\cdot 10^2$	-0.0498 (0.0570)	-0.0394 (0.0534)	–	–	-0.0568 (0.0968)	-0.1092 (0.0938)	–	–
s_{4o}	0.4823 (0.0804)	0.4822 (0.0760)	0.4399 (0.0356)	0.4130 (0.0326)	0.4746 (0.0738)	0.4183 (0.0704)	0.4488 (0.0184)	0.4273 (0.0171)
$s_{4n}\cdot 10^2$	0.7983 (1.0796)	0.9327 (1.0017)	1.0316 (0.9789)	1.2697 (0.8955)	–	–	–	–
$s_{4A}\cdot 10^2$	-0.0652 (0.1130)	-0.1152 (0.1061)	–	–	-0.0225 (0.1337)	0.0365 (0.1270)	–	–
s_{5o}	0.1158 (..)	0.1225 (..)	0.1221 (..)	0.1218 (..)	0.0822 (..)	0.0896 (..)	0.1089 (..)	0.1078 (..)
$s_{5n}\cdot 10^2$	-0.0824 (..)	-0.0626 (..)	-0.1286 (..)	-0.0729 (..)	–	–	–	–
$s_{5A}\cdot 10^2$	0.0134 (..)	0.0027 (..)	–	–	0.0575 (..)	0.0396 (..)	–	–
a_o	0.2473 (0.2238)	0.2336 (0.2324)	0.1353 (0.1244)	0.0846 (0.0984)	1.1481 (0.5153)	1.1450 (0.4054)	0.7786 (0.9373)	0.4754 (0.8148)
$a_n\cdot 10^2$	2.2097 (2.8604)	2.6048 (3.0833)	2.7652 (2.9703)	2.7823 (3.1736)	–	–	–	–
$a_A\cdot 10^2$	-0.1446 (0.1928)	-0.1818 (0.2160)	–	–	-0.5927 (0.4425)	-0.4517 (0.4696)	–	–
b_o	0.2579 (0.2069)	0.2675 (0.2292)	0.2548 (0.2143)	0.1946 (0.2029)	1.3553 (0.5746)	1.5337 (0.4490)	1.2872 (1.5268)	0.8268 (1.3948)

a) See note a, table A.1.

Table A.2 (cont.). FIML estimates of model coefficients with different specifications of background variables.[a)]
A priori restriction: $\beta=1$ ($t_1,\ldots,t_5$ unrestricted)

Coefficient	Both n and A		Only n		Only A		No background variables	
	Σ_μ unrestr.	$\Sigma_\mu=0$	Σ_μ unrestr.	$\Sigma_\mu=0$	Σ_μ unrestr.	$\Sigma_\mu=0$	Σ_μ unrestr.	$\Sigma_\mu=0$
$b_n\cdot10^2$	7.0317 (5.5645)	5.9858 (5.2988)	6.6136 (5.4029)	4.8510 (4.7536)	–	–	–	–
$b_A\cdot10^2$	0.0065 (0.1285)	-0.0519 (0.1233)	–	–	-0.1603 (0.2472)	-0.1484 (0.2945)	–	–
t_1	1.0039 (0.7182)	1.0421 (0.8265)	1.0527 (0.8049)	1.2791 (1.2108)	0.3644 (0.1017)	0.3315 (0.0503)	0.3609 (0.2751)	0.4690 (0.5430)
t_2	0.0087 (0.0959)	-0.0085 (0.1230)	0.0071 (0.1009)	-0.0338 (0.1664)	0.0851 (0.0164)	0.0885 (0.0101)	0.0859 (0.0441)	0.0631 (0.1036)
t_3	0.4694 (0.2182)	0.4905 (0.2617)	0.4709 (0.2259)	0.5475 (0.3558)	0.2944 (0.0496)	0.2792 (0.0280)	0.2948 (0.1272)	0.3548 (0.2809)
t_4	-0.4279 (0.7093)	-0.4429 (0.7916)	-0.4748 (0.7917)	-0.6750 (1.1641)	0.1909 (0.1155)	0.2305 (0.0585)	0.1936 (0.3117)	0.0701 (0.6181)
t_5	-0.0541 (..)	-0.0812 (..)	-0.0559 (..)	-0.1178 (..)	0.0652 (..)	0.0703 (..)	0.0648 (..)	0.0430 (..)
β	1	1	1	1	1	1	1	1
No. of free coef.	22	22	16	16	16	16	10	10
log Λ	2865.9	2799.0	2863.6	2795.6	2821.1	2750.2	2818.1	2747.8

a) See note a, table A.1.

Table A.3. FIML estimates of model coefficients with different restrictions on β and t_i.[a] No background variables included

Coefficient	β and t_i unrestricted		β unrestricted; $t_i \in [0,1]$		$\beta=1$; t_i unrestricted	
	Σ_μ unrestricted	$\Sigma_\mu=0$	Σ_μ unrestricted	$\Sigma_\mu=0$	Σ_μ unrestricted	$\Sigma_\mu=0$
s_{1o}	0.1368 (0.0148)	0.1572 (0.0137)	0.1367 (0.0148)	0.1570 (0.0137)	0.1366 (0.0148)	0.1568 (0.0137)
s_{2o}	0.1175 (0.0087)	0.1183 (0.0079)	0.1176 (0.0087)	0.1186 (0.0079)	0.1177 (0.0087)	0.1187 (0.0079)
s_{3o}	0.1882 (0.0132)	0.1894 (0.0123)	0.1882 (0.0132)	0.1895 (0.0123)	0.1880 (0.0132)	0.1894 (0.0123)
s_{4o}	0.4486 (0.0184)	0.4270 (0.0170)	0.4486 (0.0184)	0.4269 (0.0170)	0.4488 (0.0184)	0.4273 (0.0171)
s_{5o}	0.1089 (..)	0.1081 (..)	0.1089 (..)	0.1080 (..)	0.1089 (..)	0.1078 (..)
a_o	0.3954 (0.2736)	0.2746 (0.1373)	0.4459 (0.0482)	0.3976 (0.0502)	0.7786 (0.9373)	0.4754 (0.8148)
b_o	0.6597 (0.4415)	0.4828 (0.2266)	0.7421 (0.0380)	0.6932 (0.0381)	1.2872 (1.5268)	0.8268 (1.3948)
t_1	0.5687 (0.2997)	0.6820 (0.2710)	0.5198 (0.0248)	0.5182 (0.0269)	0.3609 (0.2751)	0.4690 (0.5430)
t_2	0.0553 (0.0503)	0.0231 (0.0593)	0.0628 (0.0160)	0.0549 (0.0167)	0.0859 (0.0441)	0.0631 (0.1036)
t_3	0.4103 (0.1668)	0.5041 (0.1664)	0.3830 (0.0203)	0.3968 (0.0225)	0.2948 (0.1272)	0.3548 (0.2809)
t_4	−0.0602 (0.3661)	−0.2080 (0.3304)	0 (..)	0 (..)	0.1936 (0.3117)	0.0701 (0.6181)
t_5	0.0259 (..)	−0.0012 (..)	0.0344 (..)	0.0301 (..)	0.0648 (..)	0.0430 (..)
β	2.2666 (1.8093)	2.9298 (1.2466)	2.0556 (1.3241)	2.1213 (1.5975)	1	1
No. of free coef.[b]	11	11	10	10	10	10
log Λ	2818.3	2748.2	2818.3	2748.0	2818.1	2747.8

a) See note a, table A.1.
b) In optimum.

Table A.4. FIML estimates of the 'total' disturbance covariance matrix Σ and the 'individual' matrix Σ_μ.
A priori restrictions: $\beta=1$, $t_i \in [0,1]$

Both n and A included

$\hat{\sigma}_{ij} \cdot 10^2$

i	j=1	2	3	4	5
1	1.386				
2	-0.113	0.867			
3	-0.365	-0.196	1.936		
4	-0.794	-0.481	-1.159	2.776	
5	-0.114	-0.077	-0.216	-0.342	0.749

$\hat{\sigma}^{\mu}_{ij} \cdot 10^2$

i	j=1	2	3	4	5
1	0.550				
2	-0.011	0.212			
3	-0.127	-0.016	0.348		
4	-0.358	-0.142	-0.240	0.823	
5	-0.054	-0.043	0.035	-0.083	0.145

Only n included

$\hat{\sigma}_{ij} \cdot 10^2$

i	j=1	2	3	4	5
1	1.397				
2	-0.112	0.867			
3	-0.376	-0.196	1.940		
4	-0.795	-0.482	-1.151	2.771	
5	-0.114	-0.077	-0.217	-0.343	0.751

$\hat{\sigma}^{\mu}_{ij} \cdot 10^2$

i	j=1	2	3	4	5
1	0.562				
2	-0.007	0.212			
3	-0.143	-0.015	0.351		
4	-0.358	-0.146	-0.228	0.816	
5	-0.054	-0.044	0.035	-0.084	0.147

Only A included

$\hat{\sigma}_{ij} \cdot 10^2$

i	j=1	2	3	4	5
1	1.555				
2	-0.112	0.869			
3	-0.412	-0.201	1.938		
4	-0.884	-0.480	-1.120	2.810	
5	-0.147	-0.076	-0.205	-0.326	0.754

$\hat{\sigma}^{\mu}_{ij} \cdot 10^2$

i	j=1	2	3	4	5
1	0.643				
2	-0.001	0.213			
3	-0.166	-0.020	0.351		
4	-0.390	-0.149	-0.215	0.827	
5	-0.086	-0.043	0.050	-0.073	0.152

No background variables included

$\hat{\sigma}_{ij} \cdot 10^2$

i	j=1	2	3	4	5
1	1.563				
2	-0.113	0.870			
3	-0.416	-0.199	1.937		
4	-0.885	-0.481	-1.114	2.803	
5	-0.149	-0.077	-0.208	-0.323	0.757

$\hat{\sigma}^{\mu}_{ij} \cdot 10^2$

i	j=1	2	3	4	5
1	0.643				
2	-0.002	0.214			
3	-0.177	-0.017	0.349		
4	-0.378	-0.151	-0.203	0.802	
5	-0.086	-0.044	0.048	-0.070	0.152

Table A.5. FIML estimates of the disturbance covariance matrix Σ when all individual disturbance components are omitted ($\Sigma_\mu=0$). A priori restrictions: $\beta=1$, $t_i \in [0,1]$.
The figures reported are $\hat{\sigma}_{ij} \cdot 10^2$

Both n and A included

i	j				
	1	2	3	4	5
1	1.382				
2	-0.112	0.866			
3	-0.365	-0.195	1.935		
4	-0.791	-0.482	-1.159	2.774	
5	-0.114	-0.077	-0.216	-0.342	0.749

Only n included

i	j				
	1	2	3	4	5
1	1.396				
2	-0.111	0.867			
3	-0.379	-0.195	1.937		
4	-0.792	-0.483	-1.147	2.765	
5	-0.114	-0.078	-0.216	-0.343	0.751

Only A included

i	j				
	1	2	3	4	5
1	1.551				
2	-0.112	0.869			
3	-0.413	-0.201	1.935		
4	-0.879	-0.480	-1.116	2.801	
5	-0.147	-0.076	-0.205	-0.326	0.754

No background variables included

i	j				
	1	2	3	4	5
1	1.559				
2	-0.113	0.870			
3	-0.416	-0.199	1.936		
4	-0.880	-0.481	-1.114	2.799	
5	-0.150	-0.077	-0.207	-0.324	0.758

Table A.6. Correlation matrix of model coefficient estimates.
No background variables included. Σ_μ unrestricted

(i) β and t_i unrestricted

	s_{1o}	s_{2o}	s_{3o}	s_{4o}	a_o	b_o	t_1	t_2	t_3	t_4	β
s_{1o}	+1.000										
s_{2o}	-0.162	+1.000									
s_{3o}	-0.018	-0.180	+1.000								
s_{4o}	-0.612	-0.181	-0.537	+1.000							
a_o	-0.119	+0.043	-0.081	+0.118	+1.000						
b_o	-0.046	+0.024	-0.044	+0.053	+0.992	+1.000					
t_1	-0.007	-0.015	+0.051	-0.023	-0.986	-0.996	+1.000				
t_2	-0.015	-0.238	+0.000	+0.111	+0.928	+0.937	-0.941	+1.000			
t_3	+0.054	-0.003	-0.063	+0.000	-0.977	-0.986	+0.976	-0.924	+1.000		
t_4	-0.015	+0.040	-0.011	-0.008	+0.987	+0.997	-0.995	+0.925	-0.989	+1.000	
β	+0.029	-0.006	+0.030	-0.038	-0.587	-0.594	+0.548	-0.482	+0.658	-0.598	+1.000

(ii) β unrestricted, $t_4=0$

	s_{1o}	s_{2o}	s_{3o}	s_{4o}	a_o	b_o	t_1	t_2	t_3	β
s_{1o}	+1.000									
s_{2o}	-0.161	+1.000								
s_{3o}	-0.018	-0.180	+1.000							
s_{4o}	-0.612	-0.181	-0.537	+1.000						
a_o	-0.641	+0.024	-0.422	+0.765	+1.000					
b_o	-0.349	-0.204	-0.387	+0.712	+0.614	+1.000				
t_1	-0.211	+0.241	+0.390	-0.307	-0.239	-0.473	+1.000			
t_2	-0.002	-0.690	+0.026	+0.296	+0.230	+0.467	-0.564	+1.000		
t_3	+0.258	+0.243	-0.494	-0.043	-0.024	-0.076	-0.596	-0.133	+1.000	
β	+0.023	+0.020	+0.020	-0.043	-0.020	-0.076	-0.579	+0.264	+0.576	+1.000

Table A.6 (cont.). Correlation matrix of model coefficient estimates. No background variables included. Σ_μ unrestricted

(iii) $\beta=1$, t_i unrestricted

	s_{1o}	s_{2o}	s_{3o}	s_{4o}	a_o	b_o	t_1	t_2	t_3	t_4
s_{1o}	+1.000									
s_{2o}	-0.162	+1.000								
s_{3o}	-0.018	-0.180	+1.000							
s_{4o}	-0.612	-0.180	-0.537	+1.000						
a_o	-0.060	+0.008	-0.029	+0.058	+1.000					
b_o	-0.017	-0.004	-0.008	+0.019	+0.997	+1.000				
t_1	+0.015	+0.004	+0.011	-0.020	-0.997	-0.999	+1.000			
t_2	-0.018	-0.044	-0.003	+0.033	+0.990	+0.993	-0.993	+1.000		
t_3	+0.025	+0.006	-0.011	-0.014	-0.996	-0.998	+0.998	-0.992	+1.000	
t_4	-0.018	+0.001	-0.005	+0.014	+0.997	+0.999	-0.999	+0.991	-0.999	+1.000

Table A.7. Estimates of Engel elasticities[a)] for different values of real income and background variables.
A priori restrictions: $\beta=1$. All prices are set equal to their sample means

Case: Background variables	Case: Restrictions on t_i	n=3, A=50: u= 30 000	n=3, A=50: u= 50 000	n=3, A=50: u= 100 000	u=50 000, A=50: n=1	u=50 000, A=50: n=3	u=50 000, A=50: n=5	u=50 000, n=3: A=30	u=50 000, n=3: A=50	u=50 000, n=3: A=70
		Commodity 1 Food, beverages and tobacco								
n,A	$0\leq t_i\leq 1$	0.504	0.527	0.594	0.514	0.527	0.540	0.471	0.527	0.579
n,A	None	0.511	0.532	0.597	0.555	0.532	0.519	0.494	0.532	0.567
n	$0\leq t_i\leq 1$	0.507	0.535	0.608	0.543	0.535	0.534	-	-	-
n	None	0.510	0.536	0.606	0.576	0.536	0.510	-	-	-
A	None	0.628	0.644	0.703	-	-	-	0.620	0.644	0.699
None	None	0.649	0.676	0.712	-	-	-	-	-	-
		Commodity 2 Clothing and footwear								
n,A	$0\leq t_i\leq 1$	1.211	1.144	1.084	1.085	1.144	1.216	1.147	1.144	1.140
n,A	None	1.200	1.137	1.081	1.095	1.137	1.185	1.142	1.137	1.133
n	$0\leq t_i\leq 1$	1.204	1.138	1.080	1.081	1.138	1.210	-	-	-
n	None	1.191	1.131	1.077	1.088	1.131	1.177	-	-	-
A	None	1.195	1.142	1.087	-	-	-	1.169	1.142	1.117
None	None	1.178	1.128	1.078	-	-	-	-	-	-
		Commodity 3 Housing, fuel and furniture								
n,A	$0\leq t_i\leq 1$	0.851	0.881	0.920	0.953	0.881	0.770	0.894	0.881	0.868
n,A	None	0.847	0.877	0.916	0.933	0.877	0.796	0.883	0.877	0.869
n	$0\leq t_i\leq 1$	0.853	0.884	0.922	0.951	0.884	0.783	-	-	-
n	None	0.853	0.884	0.922	0.936	0.884	0.811	-	-	-
A	None	0.817	0.842	0.884	-	-	-	0.856	0.842	0.827
None	None	0.822	0.850	0.893	-	-	-	-	-	-
		Commodity 4 Travel and recreation								
n,A	$0\leq t_i\leq 1$	1.575	1.341	1.181	1.230	1.341	1.477	1.361	1.341	1.322
n,A	None	1.564	1.338	1.180	1.226	1.338	1.473	1.349	1.338	1.327
n	$0\leq t_i\leq 1$	1.571	1.336	1.177	1.222	1.336	1.479	-	-	-
n	None	1.563	1.334	1.177	1.219	1.334	1.474	-	-	-
A	None	1.536	1.314	1.177	-	-	-	1.349	1.314	1.281
None	None	1.448	1.289	1.162	-	-	-	-	-	-
		Commodity 5 Other goods and services								
n,A	$0\leq t_i\leq 1$	1.382	1.243	1.135	1.171	1.243	1.324	1.253	1.243	1.234
n,A	None	1.374	1.240	1.134	1.161	1.240	1.334	1.253	1.240	1.228
n	$0\leq t_i\leq 1$	1.368	1.233	1.129	1.162	1.233	1.312	-	-	-
n	None	1.354	1.227	1.127	1.148	1.227	1.321	-	-	-
A	None	1.249	1.177	1.107	-	-	-	1.150	1.177	1.194
None	None	1.240	1.168	1.100	-	-	-	-	-	-

a) Based on the final estimates of the model coefficients, when Σ_μ is unrestricted.

Table A.8. Estimates of Engel elasticities[a] for different values of real income and background variables.
A priori restrictions: $\beta=1$. Own price is reduced by 15 per cent, other prices are set equal to their sample means[b]

Case: Background variables	Case: Restrictions on t_i	n=3, A=50: u=30 000	n=3, A=50: u=50 000	n=3, A=50: u=100 000	u=50 000, A=50: n=1	u=50 000, A=50: n=3	u=50 000, A=50: n=5	u=50 000, n=3: A=30	u=50 000, n=3: A=50	u=50 000, n=3: A=70
		Commodity 1 Food, beverages and tobacco								
n,A	$0\le t_i\le 1$	0.513	0.537	0.606	0.522	0.537	0.553	0.479	0.537	0.592
n,A	None	0.511	0.532	0.597	0.555	0.532	0.518	0.494	0.532	0.567
n	$0\le t_i\le 1$	0.516	0.547	0.620	0.553	0.547	0.546	-	-	-
n	None	0.509	0.535	0.605	0.575	0.535	0.509	-	-	-
A	None	0.645	0.665	0.724	-	-	-	0.639	0.665	0.691
None	None	0.670	0.699	0.763	-	-	-	-	-	-
		Commodity 2 Clothing and footwear								
n,A	$0\le t_i\le 1$	1.244	1.164	1.095	1.101	1.164	1.242	1.168	1.164	1.159
n,A	None	1.202	1.139	1.082	1.096	1.139	1.187	1.144	1.139	1.134
n	$0\le t_i\le 1$	1.237	1.158	1.091	1.096	1.158	1.236	-	-	-
n	None	1.193	1.132	1.077	1.089	1.132	1.179	-	-	-
A	None	1.264	1.187	1.112	-	-	-	1.218	1.187	1.158
None	None	1.245	1.171	1.102	-	-	-	-	-	-
		Commodity 3 Housing, fuel and furniture								
n,A	$0\le t_i\le 1$	0.879	0.905	0.936	0.970	0.905	0.797	0.917	0.905	0.892
n,A	None	0.863	0.891	0.926	0.943	0.891	0.813	0.897	0.891	0.883
n	$0\le t_i\le 1$	0.881	0.907	0.939	0.968	0.907	0.811	-	-	-
n	None	0.869	0.897	0.931	0.946	0.897	0.828	-	-	-
A	None	0.852	0.874	0.910	-	-	-	0.890	0.874	0.858
None	None	0.858	0.882	0.918	-	-	-	-	-	-
		Commodity 4 Travel and recreation								
n,A	$0\le t_i\le 1$	1.575	1.341	1.181	1.230	1.341	1.477	1.361	1.341	1.322
n,A	None	1.489	1.301	1.163	1.203	1.301	1.417	1.311	1.301	1.291
n	$0\le t_i\le 1$	1.571	1.336	1.177	1.222	1.336	1.479	-	-	-
n	None	1.483	1.295	1.159	1.195	1.295	1.413	-	-	-
A	None	1.548	1.347	1.192	-	-	-	1.387	1.347	1.311
None	None	1.510	1.321	1.177	-	-	-	-	-	-
		Commodity 5 Other goods and services								
n,A	$0\le t_i\le 1$	1.401	1.253	1.140	1.177	1.253	1.339	1.264	1.253	1.243
n,A	None	1.352	1.228	1.128	1.153	1.228	1.316	1.240	1.228	1.216
n	$0\le t_i\le 1$	1.388	1.244	1.135	1.169	1.244	1.329	-	-	-
n	None	1.333	1.215	1.121	1.141	1.215	1.303	-	-	-
A	None	1.315	1.218	1.129	-	-	-	1.195	1.218	1.229
None	None	1.304	1.207	1.121	-	-	-	-	-	-

a) Based on the final estimates of the model coefficients, when Σ_{μ} is unrestricted.
b) For commodity i, $p_j=\bar{p}_j$ $(j\neq i)$ and $p_j=\bar{p}_j-0.15$ $(j=i)$, where $\bar{p}_j$ is the sample mean of commodity price j.

Table A.9. Estimates of budget shares[a)] for different values of real income and background variables.
A priori restrictions: $\beta=1$. All prices are set equal to their sample means

Case		n=3, A=50			u=50 000, A=50			u=50 000, n=3		
Background variables	Restrictions on t_i	u= 30 000	u= 50 000	u= 100 000	n=1	n=3	n=5	A=30	A=50	A=70
		Commodity 1 Food, beverages and tobacco (Sample mean of budget shares: 29.75)								
n,A	$0\le t_i\le 1$	31.36	24.46	18.00	18.75	24.46	29.75	23.69	24.46	25.33
n,A	None	31.06	24.29	17.92	18.97	24.29	29.49	23.58	24.29	25.06
n	$0\le t_i\le 1$	31.30	24.48	18.16	19.08	24.48	29.55	-	-	-
n	None	31.03	24.29	18.01	19.24	24.29	29.26	-	-	-
A	None	30.87	25.60	20.38	-	-	-	26.32	25.60	24.95
None	None	30.47	25.62	20.91	-	-	-	-	-	-
		Commodity 2 Clothing and footwear (Sample mean of budget shares: 9.30)								
n,A	$0\le t_i\le 1$	9.02	9.86	10.66	9.08	9.86	10.42	9.63	9.86	10.10
n,A	None	9.09	9.90	10.66	9.27	9.90	10.42	9.69	9.90	10.11
n	$0\le t_i\le 1$	9.05	9.87	10.63	9.13	9.87	10.37	-	-	-
n	None	9.13	9.90	10.62	9.33	9.90	10.37	-	-	-
A	None	9.12	9.93	10.74	-	-	-	9.98	9.93	9.84
None	None	9.20	9.94	10.66	-	-	-	-	-	-
		Commodity 3 Housing, fuel and furniture (Sample mean of budget shares: 25.81)								
n,A	$0\le t_i\le 1$	26.43	24.68	23.04	27.62	24.68	22.45	25.71	24.68	23.61
n,A	None	26.37	24.56	22.87	27.30	24.56	22.19	25.60	24.56	23.51
n	$0\le t_i\le 1$	26.32	24.60	23.01	27.10	24.60	22.78	-	-	-
n	None	26.20	24.50	22.92	26.87	24.50	22.47	-	-	-
A	None	26.19	23.99	21.82	-	-	-	24.95	23.99	22.94
None	None	26.10	24.00	21.95	-	-	-	-	-	-
		Commodity 4 Travel and recreation (Sample mean of budget shares: 27.12)								
n,A	$0\le t_i\le 1$	25.50	32.01	38.10	34.53	32.01	29.35	32.36	32.01	31.59
n,A	None	25.76	32.24	38.33	34.57	32.24	29.77	32.51	32.24	31.94
n	$0\le t_i\le 1$	25.59	32.05	38.04	34.50	32.05	29.37	-	-	-
n	None	25.84	32.29	38.30	34.54	32.29	29.86	-	-	-
A	None	25.93	31.69	37.38	-	-	-	30.64	31.69	32.69
None	None	26.33	31.69	36.88	-	-	-	-	-	-
		Commodity 5 Other goods and services (Sample mean of budget shares: 8.03)								
n,A	$0\le t_i\le 1$	7.68	8.98	10.20	10.02	8.98	8.01	8.61	8.98	9.37
n,A	None	7.72	9.01	10.22	9.89	9.01	8.14	8.63	9.01	9.39
n	$0\le t_i\le 1$	7.74	9.00	10.16	10.19	9.00	7.91	-	-	-
n	None	7.79	9.02	10.16	10.02	9.02	8.04	-	-	-
A	None	7.89	8.79	9.69	-	-	-	8.11	8.79	8.79
None	None	7.90	8.76	9.60	-	-	-	-	-	-

a) Based on the final estimates of the model coefficients, when Σ_μ is unrestricted.

Table A.10. Estimates of Engel elasticities for different values of real income, different price sets and different restrictions on the t_i coefficients. No background variables included, β and Σ_μ are unrestricted a priori

Case		Real income		
Price set[a)]	Restrictions on t_i	u=30 000	u=50 000	u=100 000
	Commodity 1 Food, beverages and tobacco			
P_1	$0\le t_i\le 1$	0.649	0.677	0.742
P_2	$0\le t_i\le 1$	0.677	0.707	0.771
P_1	None	0.649	0.677	0.743
P_2	None	0.676	0.706	0.770
	Commodity 2 Clothing and footwear			
P_1	$0\le t_i\le 1$	1.177	1.126	1.077
P_2	$0\le t_i\le 1$	1.227	1.159	1.095
P_1	None	1.175	1.125	1.077
P_2	None	1.217	1.153	1.092
	Commodity 3 Housing, fuel and furniture			
P_1	$0\le t_i\le 1$	0.823	0.851	0.894
P_2	$0\le t_i\le 1$	0.871	0.894	0.927
P_1	None	0.823	0.851	0.894
P_2	None	0.872	0.895	0.927
	Commodity 4 Travel and recreation			
P_1	$0\le t_i\le 1$	1.444	1.286	1.161
P_2	$0\le t_i\le 1$	1.444	1.286	1.161
P_1	None	1.444	1.286	1.160
P_2	None	1.453	1.291	1.163
	Commodity 5 Other goods and services			
P_1	$0\le t_i\le 1$	1.240	1.168	1.100
P_2	$0\le t_i\le 1$	1.278	1.191	1.113
P_1	None	1.240	1.167	1.100
P_2	None	1.267	1.184	1.109

a) The following abbreviations are used:

P_1 - all prices equal to sample means.
P_2 - own price is reduced by 15 per cent, other prices equal to sample means.

Table A.11. Cournot elasticities e_{ij} derived from FIML estimates of the model with no background variable.
A priori restrictions: $\beta=1$ ($t_1,\ldots,t_5$ and Σ_μ unrestricted).
All prices equal to sample means

Commodity i	Commodity j				
	1	2	3	4	5
			(i) u=30 000 Nkr		
1	-0.322	-0.043	-0.151	-0.098	-0.035
2	-0.302	-0.432	-0.236	-0.154	-0.054
3	-0.229	-0.050	-0.385	-0.117	-0.041
4	-0.358	-0.078	-0.279	-0.669	-0.064
5	-0.315	-0.069	-0.245	-0.160	-0.451
			(ii) u=50 000 Nkr		
1	-0.416	-0.034	-0.120	-0.078	-0.028
2	-0.203	-0.627	-0.158	-0.103	-0.036
3	-0.173	-0.038	-0.520	-0.088	-0.031
4	-0.220	-0.048	-0.172	-0.809	-0.040
5	-0.207	-0.045	-0.161	-0.105	-0.648
			(iii) u=100 000 Nkr		
1	-0.560	-0.024	-0.084	-0.055	-0.019
2	-0.117	-0.790	-0.091	-0.059	-0.021
3	-0.112	-0.024	-0.680	-0.057	-0.020
4	-0.119	-0.026	-0.093	-0.903	-0.021
5	-0.117	-0.026	-0.092	-0.060	-0.806

Table A.12. Cournot elasticities e_{ij} derived from FIML estimates of the model with A as background variable.
A priori restrictions: $\beta=1$ ($t_1,\ldots,t_5$ and Σ_μ unrestricted).
All prices equal to sample means

Commodity i	Commodity j				
	1	2	3	4	5
			(i) u=50 000, A=30		
1	-0.346	-0.035	-0.127	-0.082	-0.030
2	-0.238	-0.586	-0.184	-0.118	-0.043
3	-0.196	-0.042	-0.485	-0.097	-0.035
4	-0.262	-0.056	-0.202	-0.781	-0.047
5	-0.235	-0.051	-0.182	-0.117	-0.567
			(ii) u=50 000, A=50		
1	-0.382	-0.034	-0.122	-0.078	-0.028
2	-0.215	-0.616	-0.166	-0.107	-0.039
3	-0.181	-0.039	-0.500	-0.090	-0.032
4	-0.235	-0.051	-0.181	-0.805	-0.042
5	-0.219	-0.047	-0.169	-0.109	-0.633
			(iii) u=50 000, A=70		
1	-0.420	-0.032	-0.115	-0.074	-0.027
2	-0.193	-0.644	-0.149	-0.096	-0.035
3	-0.165	-0.036	-0.515	-0.082	-0.029
4	-0.210	-0.045	-0.162	-0.828	-0.038
5	-0.201	-0.043	-0.155	-0.100	-0.695

Table A.13. Cournot elasticities e_{ij} derived from FIML estimation of the model with n as background variable.
A priori restrictions: $\beta=1$ ($t_1,\ldots,t_5$ and Σ_μ unrestricted).
All prices equal to sample means

Commodity i	Commodity j 1	2	3	4	5
	(i) u=50 000, n=1				
1	-0.587	-0.001	-0.095	0.095	0.012
2	-0.101	-0.992	-0.043	0.043	0.005
3	-0.138	-0.001	-0.864	0.059	0.007
4	-0.070	-0.000	-0.030	-1.121	0.004
5	-0.087	-0.001	-0.037	0.037	-1.061
	(ii) u=50 000, n=3				
1	-0.547	-0.002	-0.104	0.104	0.013
2	-0.148	-0.990	-0.063	0.063	0.008
3	-0.187	-0.001	-0.785	0.080	0.010
4	-0.116	-0.001	-0.049	-1.174	0.006
5	-0.133	-0.001	-0.057	0.057	-1.088
	(iii) u=50 000, n=5				
1	-0.522	-0.002	-0.110	0.111	0.014
2	-0.200	-0.988	-0.085	0.085	0.011
3	-0.232	-0.001	-0.690	0.099	0.012
4	-0.174	-0.001	-0.074	-1.234	0.009
5	-0.187	-0.001	-0.080	0.080	-1.133

Table A.14. Cournot elasticities e_{ij} derived from FIML estimates of the model with n as background variable.
A priori restrictions: $\beta=1$ and $0 \leq t_i \leq 1$ $(i=1,\ldots,5)$. Σ_μ unrestricted. All prices equal to sample means

Commodity i	Commodity j				
	1	2	3	4	5
			(i) u=50 000, n=1		
1	-0.430	-0.016	-0.089	0.000	-0.007
2	-0.135	-0.862	-0.077	0.000	-0.006
3	-0.141	-0.014	-0.789	0.000	-0.007
4	-0.130	-0.013	-0.074	-1.000	-0.006
5	-0.132	-0.013	-0.075	0.000	-0.942
			(ii) u=50 000, n=3		
1	-0.416	-0.017	-0.094	0.000	-0.008
2	-0.188	-0.833	-0.107	0.000	-0.009
3	-0.179	-0.018	-0.678	0.000	-0.008
4	-0.196	-0.020	-0.111	-1.000	-0.009
5	-0.192	-0.019	-0.109	0.000	-0.913
			(iii) u=50 000, n=5		
1	-0.407	-0.018	-0.101	0.000	-0.008
2	-0.250	-0.805	-0.142	0.000	-0.012
3	-0.205	-0.020	-0.548	0.000	-0.010
4	-0.279	-0.028	-0.158	-1.000	-0.013
5	-0.261	-0.026	-0.148	0.000	-0.876

Table A.15. Cournot elasticities e_{ij} derived from FIML estimates of the model with n and A as background variables.
A priori restrictions: $\beta=1$. ($t_1,\ldots,t_5$ and Σ_μ unrestricted.)
All prices equal to sample means

Commodity i	Commodity j				
	1	2	3	4	5
			(i) u=30 000, A=50, n=3		
1	-0.512	-0.002	-0.137	0.124	0.017
2	-0.222	-0.980	-0.099	0.090	0.012
3	-0.265	-0.002	-0.701	0.107	0.014
4	-0.178	-0.001	-0.079	-1.315	0.010
5	-0.201	-0.002	-0.090	0.081	-1.162
			(ii) u=50 000, A=50, n=3		
1	-0.533	-0.002	-0.109	0.098	0.013
2	-0.152	-0.987	-0.068	0.061	0.008
3	-0.191	-0.002	-0.771	0.077	0.010
4	-0.122	-0.001	-0.054	-1.167	0.007
5	-0.136	-0.001	-0.061	0.055	-1.097
			(iii) u=100 000, A=50, n=3		
1	-0.598	-0.001	-0.077	0.070	0.009
2	-0.089	-0.993	-0.040	0.036	0.005
3	-0.118	-0.001	-0.851	0.048	0.006
4	-0.072	-0.001	-0.032	-1.080	0.004
5	-0.080	-0.001	-0.036	0.032	-1.050

Table A.16. Cournot elasticities e_{ij} derived from FIML estimates of the model with n and A as background variables. A priori restrictions: $\beta=1$ and $0\le t_i\le 1$ $(i=1,\ldots,5)$. (Σ_μ unrestricted.) All prices equal to sample means

Commodity i	Commodity j 1	2	3	4	5
	(i) u=30 000, A=50, n=3				
1	-0.355	-0.020	-0.119	0.000	-0.009
2	-0.292	-0.739	-0.167	0.000	-0.013
3	-0.250	-0.024	-0.565	0.000	-0.011
4	-0.336	-0.033	-0.191	-1.000	-0.015
5	-0.313	-0.031	-0.178	0.000	-0.860
	(ii) u=50 000, A=50, n=3				
1	-0.406	-0.017	-0.097	0.000	-0.007
2	-0.192	-0.834	-0.109	0.000	-0.008
3	-0.182	-0.018	-0.673	0.000	-0.008
4	-0.199	-0.019	-0.113	-1.000	-0.009
5	-0.196	-0.019	-0.111	0.000	-0.917
	(iii) u=100 000, A=50, n=3				
1	-0.505	-0.012	-0.070	0.000	-0.005
2	-0.109	-0.909	-0.062	0.000	-0.005
3	-0.114	-0.011	-0.790	0.000	-0.005
4	-0.106	-0.010	-0.060	-1.000	-0.005
5	-0.107	-0.010	-0.061	0.000	-0.957
	(iv) u=50 000, A=30, n=3				
1	-0.344	-0.018	-0.102	0.000	-0.008
2	-0.203	-0.819	-0.116	0.000	-0.009
3	-0.194	-0.019	-0.672	0.000	-0.009
4	-0.211	-0.021	-0.120	-1.000	-0.009
5	-0.206	-0.020	-0.118	0.000	-0.908
	(v) u= 50 000, A=50, n=3				
1	-0.406	-0.017	-0.097	0.000	-0.007
2	-0.192	-0.834	-0.109	0.000	-0.008
3	-0.182	-0.018	-0.673	0.000	-0.008
4	-0.199	-0.019	-0.113	-1.000	-0.009
5	-0.196	-0.019	-0.111	0.000	-0.917

Table A.16 (cont.). Cournot elasticities e_{ij} derived from FIML estimates of the model with n and A as background variables. A priori restrictions: $\beta=1$ and $0 \leq t_i \leq 1$ $(i=1,\ldots,5)$. Σ_μ unrestricted. All prices equal to sample means

Commodity i	Commodity j				
	1	2	3	4	5
	(vi) u=50 000, A=70, n=3				
1	-0.466	-0.016	-0.091	0.000	-0.007
2	-0.181	-0.848	-0.104	0.000	-0.008
3	-0.171	-0.017	-0.673	0.000	-0.008
4	-0.188	-0.018	-0.107	-1.000	-0.008
5	-0.185	-0.018	-0.105	0.000	-0.925
	(vii) u=50 000, A=50, n=1				
1	-0.391	-0.017	-0.097	0.000	-0.008
2	-0.142	-0.856	-0.081	0.000	-0.006
3	-0.149	-0.015	-0.783	0.000	-0.007
4	-0.134	-0.013	-0.077	0.000	-0.006
5	-0.137	-0.013	-0.078	0.000	-0.942
	(viii) u=50 000, A=50, n=3				
1	-0.406	-0.017	-0.097	0.000	-0.007
2	-0.192	-0.834	-0.109	0.000	-0.008
3	-0.182	-0.018	-0.673	0.000	-0.008
4	-0.199	-0.019	-0.113	-1.000	-0.009
5	-0.196	-0.019	-0.111	0.000	-0.917
	(ix) u=50 000, A=50, n=5				
1	-0.414	-0.017	-0.101	0.000	-0.008
2	-0.251	-0.811	-0.143	0.000	-0.011
3	-0.203	-0.020	-0.538	0.000	-0.009
4	-0.278	-0.027	-0.159	-1.000	-0.012
5	-0.262	-0.026	-0.149	0.000	-0.887

Table A.17. Cournot elasticities e_{ij} derived from FIML estimates of the model with no background variables.
A priori restrictions: None. (β, t_1, $t_2,\ldots,t_5$ and Σ_μ unrestricted.) All prices equal to sample means

Commodity i	Commodity j 1	2	3	4	5
	(i) u=30 000 Nkr				
1	-0.135	-0.061	-0.489	0.071	-0.035
2	-0.360	-0.598	-0.234	0.034	-0.016
3	-0.623	-0.050	-0.179	0.058	-0.029
4	-0.160	-0.013	-0.104	-1.159	-0.007
5	-0.312	-0.025	-0.203	0.029	-0.728
	(ii) u=50 000 Nkr				
1	-0.253	-0.050	-0.403	0.058	-0.029
2	-0.243	-0.736	-0.158	0.023	-0.011
3	-0.474	-0.038	-0.361	0.044	-0.022
4	-0.109	-0.009	-0.071	-1.093	-0.005
5	-0.208	-0.017	-0.135	0.020	-0.826
	(iii) u=100 000 Nkr				
1	-0.436	-0.036	-0.292	0.042	-0.021
2	-0.141	-0.851	-0.092	0.013	-0.006
3	-0.309	-0.025	-0.574	0.029	-0.014
4	-0.063	-0.005	-0.041	-1.048	-0.003
5	-0.119	-0.010	-0.078	0.011	-0.904

Table A.18. Cournot elasticities e_{ij} derived from FIML estimates of the model with no background variables.
A priori restrictions: None. (β, t_1, t_2,...,t_5, and Σ_μ unrestricted.) u=50 000 Nkr

Commodity i	Commodity j				
	1	2	3	4	5
	(i) Price of commodity 1 is reduced by 15 per cent, all other prices are equal to sample means				
1	-0.228	-0.057	-0.455	0.066	-0.032
2	-0.227	-0.699	-0.196	0.028	-0.014
3	-0.445	-0.048	-0.349	0.056	-0.027
4	-0.090	-0.010	-0.078	-1.111	-0.006
5	-0.192	-0.021	-0.167	0.024	-0.799
	(ii) Price of commodity 2 is reduced by 15 per cent, all other prices are equal to sample means				
1	-0.257	-0.037	-0.410	0.059	-0.029
2	-0.222	-0.796	-0.145	0.021	-0.010
3	-0.483	-0.029	-0.360	0.045	-0.022
4	-0.110	-0.006	-0.071	-1.094	-0.005
5	-0.212	-0.013	-0.138	0.020	-0.824
	(iii) Price of commodity 3 is reduced by 15 per cent, all other prices are equal to sample means				
1	-0.288	-0.058	-0.342	0.067	-0.033
2	-0.281	-0.711	-0.135	0.026	-0.013
3	-0.486	-0.039	-0.392	0.046	-0.022
4	-0.116	-0.009	-0.056	-1.104	-0.005
5	-0.239	-0.019	-0.115	0.022	-0.809
	(iv) Price of commodity 4 is reduced by 15 per cent, all other prices are equal to sample means				
1	-0.250	-0.049	-0.396	0.042	-0.028
2	-0.238	-0.739	-0.155	0.017	-0.011
3	-0.465	-0.038	-0.363	0.032	-0.021
4	-0.017	-0.009	-0.076	-1.045	-0.005
5	-0.205	-0.017	-0.133	0.014	-0.828
	(v) Price of commodity 5 is reduced by 15 per cent, all other prices are equal to sample means				
1	-0.255	-0.051	-0.407	0.059	-0.022
2	-0.245	-0.735	-0.160	0.023	-0.009
3	-0.479	-0.039	-0.360	0.045	-0.017
4	-0.109	-0.009	-0.071	-1.093	-0.004
5	-0.195	-0.016	-0.127	0.018	-0.865

Table A.19. Cournot elasticities e_{ij} derived from FIML estimates of the model with no background variables.
A priori restrictions: $0 \leq t_i \leq 1$ $(i=1,\ldots,5)$. (β and Σ_μ unrestricted.) All prices equal to sample means

Commodity i	Commodity j				
	1	2	3	4	5
	(i) u=30 000 Nkr				
1	-0.120	-0.064	-0.423	0.000	-0.042
2	-0.373	-0.529	-0.250	0.000	-0.025
3	-0.545	-0.056	-0.185	0.000	-0.037
4	-0.242	-0.025	-0.162	-1.000	-0.016
5	-0.352	-0.035	-0.229	0.000	-0.635
	(ii) u=50 000 Nkr				
1	-0.242	-0.053	-0.348	0.000	-0.035
2	-0.251	-0.691	-0.168	0.000	-0.017
3	-0.415	-0.042	-0.366	0.000	-0.028
4	-0.156	-0.016	-0.104	-1.000	-0.010
5	-0.226	-0.023	-0.152	0.000	-0.766
	(iii) u=100 000 Nkr				
1	-0.428	-0.038	-0.251	0.000	-0.025
2	-0.145	-0.825	-0.097	0.000	-0.010
3	-0.271	-0.028	-0.577	0.000	-0.018
4	-0.087	-0.009	-0.058	-1.000	-0.006
5	-0.129	-0.013	-0.086	0.000	-0.871

Table B.1. Coefficient estimates for the model specification without background variables.a)
A priori restrictions: $\beta=0$, $s_{in}=s_{iA}=a'_{in}=a'_{iA}=b'_{in}=b'_{iA}=0$.

$$w_i = s_i + a'_i \frac{\log u}{u} + b'_i \frac{1}{u}$$

Commodity group i	s_i	a'_i	b'_i	$\hat{\rho}$	$\hat{\sigma}_\varepsilon$
	0.01196	0.00494	0.03456	0	0.02103
1. Flour and bread ...	0.00766	0.01285	0.04067	0.462	0.02107
	(0.00267)	(0.00473)	(0.00385)		
	0.04949	0.02972	0.02639	0	0.06221
2. Meat and eggs	0.04971	0.02920	0.02614	0.225	0.06221
	(0.00787)	(0.01439)	(0.01125)		
	0.00376	0.01996	0.01896	0	0.02210
3. Fish	0.00357	0.01941	0.01983	0.453	0.02211
	(0.00280)	(0.00497)	(0.00404)		
	0.00079	0.00774	0.00896	0	0.00889
4. Canned meat and fish	0.00056	0.00799	0.00938	0.200	0.00889
	(0.00112)	(0.00205)	(0.00160)		
	0.01132	0.02188	0.05419	0	0.02424
5. Dairy products	0.00911	0.02366	0.05886	0.473	0.02427
	(0.00307)	(0.00542)	(0.00443)		
	0.00093	0.01018	0.01657	0	0.00996
6. Butter and margarine	0.00035	0.01112	0.01749	0.262	0.00996
	(0.00126)	(0.00230)	(0.00181)		
	0.02203	0.03779	0.05223	0	0.03960
7. Potatoes and vegetables	0.01980	0.03935	0.05709	0.334	0.03963
	(0.00505)	(0.00912)	(0.00725)		
	0.01808	0.02151	0.05131	0	0.03069
8. Other food	0.01579	0.02518	0.05492	0.284	0.03070
	(0.00390)	(0.00709)	(0.00559)		
	0.02711	0.00013	-0.00967	0	0.03468
9. Beverages	0.02474	0.00374	-0.00580	0.312	0.03469
	(0.00441)	(0.00800)	(0.00634)		
	0.00974	0.00517	0.01671	0	0.02648
10. Tobacco	0.00475	0.01247	0.02503	0.720	0.02654
	(0.00305)	(0.00507)	(0.00444)		
	0.08261	0.00952	-0.02434	0	0.07915
11. Clothing	0.08262	0.00876	-0.02388	0.287	0.07915
	(0.01006)	(0.01828)	(0.01442)		
	0.02560	-0.01268	-0.01379	0	0.03968
12. Footwear	0.02566	-0.01283	-0.01385	-0.009	0.03968
	(0.00478)	(0.00892)	(0.00677)		
	0.08455	0.03830	0.04282	0	0.10123
13. Housing	0.08156	0.04301	0.04760	0.362	0.10123
	(0.01289)	(0.02321)	(0.01855)		
	-0.00232	0.04328	0.10378	0	0.03100
14. Fuel and power	-0.00220	0.04221	0.10420	0.108	0.03100
	(0.00385)	(0.00711)	(0.00547)		

a) Asymptotic standard errors of estimates are given in parentheses. The unit of measurement of real income is 10 000 Nkr.

Table B.1 (cont.). Coefficient estimates for the model specification without background variables.[a)]
A priori restrictions: $\beta=0$, $s_{in}=s_{iA}=a'_{in}=a'_{iA}=b'_{in}=b'_{iA}=0$.

$$w_i = s_i + a'_i \frac{\log u}{u} + b'_i \frac{1}{u}$$

Commodity group i	s_i	a'_i	b'_i	$\hat{\rho}$	$\hat{\sigma}_\varepsilon$
	0.06712	-0.02126	-0.04562	0	0.07315
15. Furniture	0.06711	-0.02125	-0.04562	0.003	0.07315
	(0.00884)	(0.01649)	(0.01253)		
	0.03006	-0.00664	0.00240	0	0.05606
16. Household equipment	0.02988	-0.00614	0.00254	0.043	0.05606
	(0.00685)	(0.01274)	(0.00973)		
17. Misc. household	0.01262	0.01560	0.02507	0	0.03611
goods	0.01304	0.01401	0.02502	0.269	0.03611
	(0.00459)	(0.00835)	(0.00657)		
	0.01225	0.00940	0.00642	0	0.04695
18. Medical care	0.01213	0.00973	0.00653	0.007	0.04695
	(0.00569)	(0.01060)	(0.00806)		
	0.15508	-0.15785	-0.17400	0	0.11971
19. Motorcars, bicycles	0.15019	-0.15012	-0.16623	-0.081	0.11972
	(0.01409)	(0.02642)	(0.01991)		
20. Running costs of	0.13163	-0.07618	-0.11453	0	0.09213
vehicles	0.12812	-0.07054	-0.10900	0.311	0.09214
	(0.01173)	(0.02125)	(0.01684)		
	0.02772	0.00142	-0.00711	0	0.05406
21. Public transport ..	0.02849	-0.00017	-0.00808	0.058	0.05406
	(0.00664)	(0.01232)	(0.00942)		
	0.01094	0.00938	0.00361	0	0.05594
22. PTT charges	0.01309	0.00548	0.00052	0.175	0.05595
	(0.00703)	(0.01292)	(0.01002)		
	0.09373	-0.04645	-0.06102	0	0.08419
23. Recreation	0.09257	-0.04457	-0.05920	0.071	0.08419
	(0.01036)	(0.01923)	(0.01473)		
	0.01200	0.04272	0.02181	0	0.05366
24. Public entertainment	0.01709	0.03113	0.01607	0.424	0.05369
	(0.00682)	(0.01216)	(0.00984)		
	0.00893	0.02432	0.01986	0	0.04456
25. Books and newpapers	0.00967	0.02356	0.01842	0.420	0.04456
	(0.00566)	(0.01010)	(0.00817)		
	0.01414	0.01390	0.00630	0	0.02302
26. Personal care	0.01364	0.01358	0.00785	0.258	0.02302
	(0.00292)	(0.00532)	(0.00418)		
27. Misc. goods and	0.03625	-0.03193	-0.03508	0	0.04797
services	0.03613	-0.03170	-0.03493	0.035	0.04797
	(0.00585)	(0.01089)	(0.00830)		
28. Restaurants,	0.04187	-0.01387	-0.02680	0	0.05448
hotels, etc.	0.04121	-0.01328	-0.02544	0.314	0.05448
	(0.00693)	(0.01256)	(0.00995)		

a) See note a, page 184.

Table B.1 (cont.). Coefficient estimates for the model specification without background variables.[a)]
A priori restrictions: $\beta=0$, $s_{in}=s_{iA}=a'_{in}=a'_{iA}=b'_{in}=b'_{iA}=0$.

$$w_i = s_i + a'_i \frac{\log u}{u} + b'_i \frac{1}{u}$$

Commodity group i	s_i	a'_i	b'_i	$\hat{\rho}$	$\hat{\sigma}_\varepsilon$
I Food, beverages and tobacco	0.15521	0.15902	0.27021	0	0.12495
	0.13645	0.18401	0.30315	0.409	0.12516
	(0.01592)	(0.02845)	(0.02294)		
II Clothing and footwear	0.10821	-0.00316	-0.03813	0	0.09316
	0.10628	0.00101	-0.03581	0.248	0.09316
	(0.01181)	(0.02155)	(0.01690)		
III Housing, fuel and furniture	0.19203	0.06928	0.12846	0	0.13914
	0.19016	0.07218	0.13148	0.181	0.13914
	(0.01749)	(0.03214)	(0.02496)		
IV Travel and recreation	0.44003	-0.20264	-0.31138	0	0.16720
	0.46577	-0.24695	-0.34990	0.292	0.16739
	(0.02129)	(0.03866)	(0.03053)		
V Other goods and services	0.10452	-0.02250	-0.04916	0	0.08704
	0.09883	-0.00986	-0.04255	0.211	0.08707
	(0.01099)	(0.02013)	(0.01570)		

a) See note a, page 184.

Table B.2. Coefficient estimates for the model specification with no background variables included.[a)]
A priori restrictions: $\beta=1$. $s_{in}=s_{iA}=a_{in}=a_{iA}=b_{in}=b_{iA}=0$.

$$w_i = s_i + (t_i v_i - s_i)(a_i \log u + b_i)/u$$

Commodity group i	s_i	a_i	b_i	t_i	$\hat{\rho}$	$\overline{\hat{\sigma}}_{\varepsilon}$
1. Flour and bread	0.008 (0.003)	0.822 (1.739)	2.508 (5.291)	0.025 (0.036)	0.458	0.02102
2. Meat and eggs ..	0.049 (0.008)	2.287 (2.047)	2.095 (1.822)	0.058 (0.011)	0.224	0.06212
3. Fish	0.004 (0.003)	-7.862 (9.007)	-8.063 (8.901)	0.001 (0.002)	0.454	0.02211
4. Canned meat and fish	0.0006 (0.0011)	-29.921 (77.041)	-35.345 (88.665)	0.0003 (0.0008)	0.200	0.00889
5. Dairy products .	0.009 (0.003)	1.059 (0.705)	2.554 (1.595)	0.030 (0.014)	0.482	0.02388
6. Butter and margarine	0.0004 (0.0013)	-13.582 (52.509)	-21.355 (80.638)	-0.0005 (0.0021)	0.261	0.00995
7. Potatoes and vegetables	0.020 (0.005)	-1.478 (1.423)	-2.119 (1.743)	-0.007 (0.021)	0.332	0.03961
8. Other food	0.016 (0.004)	-1.937 (1.181)	-4.278 (2.039)	0.003 (0.005)	0.283	0.03069
9. Beverages	0.028 (0.004)	-3.670 (6.261)	-13.701 (8.182)	0.027 (0.004)	0.305	0.03457
10. Tobacco	0.005 (0.003)	-4.134 (5.106)	-8.660 (9.379)	0.002 (0.003)	0.720	0.02653
11. Clothing	0.086 (0.010)	-0.152 (1.495)	2.342 (3.161)	0.078 (0.018)	0.288	0.07916
12. Footwear	0.026 (0.005)	-2.764 (2.511)	-3.053 (2.859)	0.032 (0.009)	-0.008	0.03963
13. Housing	0.081 (0.013)	-5.190 (4.590)	-5.761 (4.725)	0.076 (0.016)	-0.361	0.10113
14. Fuel and power .	-0.001 (0.004)	-44.345 140.386	-115.852 (379.735)	-0.003 (0.007)	0.122	0.03092
15. Furniture	0.067 (0.009)	-0.586 (2.562)	-1.278 (5.635)	0.104 (0.163)	0.002	0.07314
16. Household equipment	0.031 (0.004)	-7.952 (8.548)	1.231 (5.453)	0.032 (0.004)	0.043	0.05603
17. Misc. household goods	0.012 (0.005)	13.483 (11.460)	23.298 (15.004)	0.013 (0.004)	0.274	0.03605
18. Medical care ...	0.013 (0.006)	12.353 (20.900)	7.158 (12.986)	0.013 (0.005)	0.010	0.04694
19. Motorcars, bicycles	0.151 (0.014)	-1.576 (3.545)	-1.741 (3.919)	0.244 (0.215)	-0.083	0.11969
20. Running costs of vehicles	0.128 (0.012)	0.089 (1.592)	0.137 (2.458)	-0.695 (14.847)	0.310	0.09211

a) Asymptotic standard errors of estimates are given in parentheses. The unit of measurement of real income is 10 000 Nkr.

Table B.2 (cont.). Coefficient estimates for the model specification with no background variables included.a)
A priori restrictions: $\beta=1$. $s_{in}=s_{iA}=a_{in}=a_{iA}=b_{in}=b_{iA}=0$.

$$w_i = s_i + (t_i v_i - s_i)(a_i \log u + b_i)/u$$

Commodity group i	s_i	a_i	b_i	t_i	$\hat{\rho}$	$\hat{\sigma}_\varepsilon$
21. Public transport	0.027 (0.005)	1.099 (4.117)	-1.889 (3.257)	0.028 (0.009)	0.055	0.05405
22. PTT charges	0.013 (0.005)	-9.669 (10.136)	-1.592 (6.934)	0.014 (0.006)	0.176	0.05592
23. Recreation	0.092 (0.010)	-0.338) (1.806)	-0.448 (2.405)	0.233 (0.743)	0.071	0.08418
24. Public entertainment	0.018 (0.007)	10.183 (8.277)	4.965 (4.483)	0.023 (0.007)	0.427	0.05366
25. Books and newspapers	0.010 (0.006)	-14.270 (13.422)	-10.506 (10.160)	0.008 (0.005)	0.423	0.04456
26. Personal care ..	0.014 (0.003)	-23.473 (19.472)	-14.412 (12.632)	0.012 (0.003)	0.261	0.02301
27. Misc. goods and services	0.036 (0.006)	-0.557 (3.825)	-0.615 (4.227)	0.089 (0.376)	0.036	0.04796
28. Restaurant, hotels, etc. ...	0.042 (0.007)	-1.047 (2.231)	-1.994 (4.256)	0.053 (0.030)	0.313	0.05447
I Food, beverages and tobacco	0.136 (0.016)	1.921 (1.845)	3.135 (2.960)	0.225 (0.090)	0.412	0.12500
II Clothing and footwear	0.108 (0.011)	0.203 (1.069)	2.015 (3.080)	0.094 (0.031)	0.248	0.09316
III Housing, fuel and furniture ..	0.190 (0.018)	-1.783 (3.092)	-3.256 (5.437)	0.152 (0.070)	0.182	0.13913
IV Travel and recreation	0.466 (0.021)	0.377 (1.412)	0.535 (1.997)	-0.193 (2.496)	0.292	0.16738
V Other goods and services	0.099 (0.011)	-2.205 (3.762)	-9.379 (9.060)	0.100 (0.014)	0.207	0.08696

a) See note a, page 187.

Table B.3. Coefficient estimates for the model specification with family size as the only background variable.[a)]
A priori restrictions: $\beta=0$. $s_{iA}=a'_{iA}=b'_{iA}=0$.

$$w_i = s_{io} + s_{in}n + (a'_{io}+a'_{in}n)\frac{\log u}{u} + (b'_{io}+b'_{in}n)\frac{1}{u}$$

Commodity group i	$s_{io}\cdot 10^2$	$s_{in}\cdot 10^2$	$a'_{io}\cdot 10^2$	$a'_{in}\cdot 10^2$	$b'_{io}\cdot 10^2$	$b'_{in}\cdot 10^2$	$\hat{\rho}$
	1.098	-0.228	-1.350	0.831	1.624	1.393	0
1. Flour and bread	0.629	-0.148	-0.269	0.550	1.975	1.452	0.433
	(0.554)	(0.169)	(0.880)	(0.442)	(0.792)	(0.293)	
	1.811	0.528	4.250	0.589	6.006	-0.643	0
2. Meat and eggs .	1.877	0.579	4.334	0.324	5.626	-0.495	0.217
	(1.701)	(0.520)	(2.771)	(1.386)	(2.433)	(0.899)	
	1.544	-0.449	-0.062	1.013	0.726	0.315	0
3. Fish	1.908	-0.390	0.222	0.774	0.693	0.433	0.450
	(0.610)	(0.186)	(0.097)	(0.487)	(0.873)	(0.323)	
4. Canned meat	-0.181	0.124	1.299	-0.429	0.684	0.231	0
and fish	-0.199	0.113	1.253	-0.377	0.715	0.238	0.196
	(0.242)	(0.074)	(0.395)	(0.197)	(0.346)	(0.128)	
	0.912	-0.542	-1.777	2.414	3.425	1.735	0
5. Dairy products	0.914	-0.515	-1.575	2.205	3.227	1.874	0.393
	(0.600)	(0.183)	(0.959)	(0.482)	(0.859	(0.318)	
6. Butter and	-0.223	-0.008	0.681	0.271	1.550	0.263	0
margarine	-0.262	-0.008	0.744	0.274	1.596	0.267	0.238
	(0.268)	(0.082)	(0.436)	(0.218)	(0.384)	(0.142)	
7. Potatoes and	0.624	0.152	3.375	0.493	5.048	0.976	0
vegetables	0.731	0.089	3.102	0.473	4.579	1.372	0.333
	(1.078)	(0.329)	(1.735)	(0.871)	(1.543)	(0.571)	
	1.320	-0.167	0.430	0.927	3.704	1.324	0
8. Other food	0.986	-0.071	1.181	0.592	3.854	1.421	0.264
	(0.824)	(0.252)	(1.336)	(0.669)	(1.179)	(0.436)	
	3.794	-0.147	-0.153	-0.317	-1.831	0.006	0
9. Beverages	3.235	-0.081	0.441	-0.363	-1.212	0.023	0.299
	(0.959)	(0.293)	(1.549)	(0.776)	(1.372)	(0.507)	
	1.238	0.289	2.102	-2.114	-1.018	1.478	0
10. Tobacco	0.473	0.153	1.983	-0.978	1.021	0.862	0.710
	(0.665)	(0.202)	(0.996)	(0.504)	(0.947)	(0.350)	
	6.138	0.320	1.925	0.949	1.813	-1.724	0
11. Clothing	6.539	0.057	0.365	2.010	2.069	-1.977	0.293
	(2.192)	(0.670)	(3.544)	(1.777)	(3.137)	(1.160)	
	1.572	0.045	-1.335	1.187	1.658	-1.419	0
12. Footwear	1.538	0.039	-1.391	1.207	1.642	-1.419	-0.019
	(1.031)	(0.315)	(1.711)	(0.850)	(1.473)	(0.542)	
	9.356	-0.398	2.460	1.622	6.504	-1.804	0
13. Housing	9.185	-0.512	2.302	1.977	6.764	-1.635	0.359
	(2.808)	(0.858)	(4.505)	(2.263)	(4.020)	(1.487)	
14. Fuel and	0.074	-0.354	2.241	1.349	9.772	0.403	0
furniture	0.052	-0.331	2.235	1.263	9.745	0.449	0.103
	(0.831)	(0.254)	(1.368)	(0.682)	(1.189)	(0.438)	

a) Asymptotic standard errors of estimates are given in parentheses. The unit of measurement of real income is 10 000 Nkr.

Table B.3 (cont.). Coefficient estimates for the model specification with family size as the only background variable.[a)]
A priori restrictions: $\beta=0$. $s_{iA}=a'_{iA}=b'_{iA}=0$.

$$w_i = s_{io} + s_{in} n + (a'_{io} + a'_{in} n) \frac{\log u}{u} + (b'_{io} + b'_{in} n) \frac{1}{u}$$

Commodity group i	$s_{io} \cdot 10^2$	$s_{in} \cdot 10^2$	$a'_{io} \cdot 10^2$	$a'_{in} \cdot 10^2$	$b'_{io} \cdot 10^2$	$b'_{in} \cdot 10^2$	$\hat{\rho}$
15. Furniture	11.64	-1.138	-6.063	0.588	-9.721	1.083	0
	11.64	-1.134	-6.056	0.578	-9.711	1.078	-0.009
	(1.90)	(0.580)	(3.151)	(1.566)	(2.716)	(1.000)	
16. Household equipment	2.767	0.073	-0.320	-0.069	0.729	-0.221	0
	2.744	0.043	-0.425	0.113	0.933	-0.328	0.047
	(1.492)	(0.456)	(0.247)	(1.227)	(2.133)	(0.786)	
17. Misc. household goods	0.375	0.240	2.525	-0.234	3.644	-0.351	0
	0.562	0.207	2.187	-0.236	3.321	-0.216	0.267
	(0.999)	(0.305)	(1.620)	(0.811)	(1.430)	(0.528)	
18. Medical care ..	0.406	0.331	2.568	-0.698	1.940	-0.546	0
	0.337	0.350	2.734	-0.752	1.999	-0.554	0.015
	(1.238)	(0.378)	(2.049)	(1.019)	(1.769)	(0.652)	
19. Motorcars, bicycles	14.17	1.263	-8.384	-4.452	-14.34	-1.890	0
	13.52	1.272	-7.776	-4.271	-13.42	-1.924	-0.073
	(3.05)	(0.932)	(5.085)	(2.522)	(4.36)	(1.605)	
20. Running costs of vehicles ...	11.80	0.143	-7.889	0.374	-11.58	0.822	0
	11.23	0.215	-7.121	0.395	-10.53	0.529	0.307
	(2.55)	(0.780)	(4.120)	(2.066)	(3.65)	(1.351)	
21. Public transport	6.764	-0.916	-3.246	0.188	-6.260	1.784	0
	6.866	-0.942	-3.469	0.271	-6.322	1.767	0.049
	(1.429)	(0.437)	(2.360)	(1.175)	(2.043)	(0.752)	
22. PTT charges ...	3.527	-0.457	-0.212	-0.076	-1.445	-0.011	0
	4.062	-0.582	-1.072	0.135	-2.151	0.155	0.170
	(1.519)	(0.464)	(2.485)	(1.241)	(2.172)	(0.802)	
23. Recreation	10.86	-0.254	-5.105	-0.072	-6.711	-0.347	0
	10.43	-0.141	-4.407	-0.309	-6.197	-0.436	0.065
	(2.25)	(0.687)	(3.708)	(1.846)	(3.213)	(1.184)	
24. Public entertainment	-0.886	0.587	6.958	-0.372	6.398	-1.862	0
	0.034	0.454	5.026	-0.166	5.078	-1.533	0.419
	(1.486)	(0.454)	(2.365)	(1.190)	(2.127)	(0.787)	
25. Books and newspapers	-0.068	0.663	5.823	-2.421	2.038	0.194	0
	0.187	0.455	4.665	-1.477	2.401	-0.130	0.413
	(1.234)	(0.377)	(1.966)	(0.989)	(1.766)	(0.654)	
26. Personal care .	1.336	-0.212	-0.055	1.048	0.429	0.290	0
	1.282	-0.193	-0.017	0.965	0.466	0.338	0.250
	(0.630)	(0.193)	(1.023)	(0.512)	(0.901)	(0.333)	
27. Misc. goods and services ..	2.870	0.544	-0.299	-1.923	-2.986	-0.179	0
	2.856	0.544	-0.279	-1.922	-2.970	-1.776	0.031
	(1.268)	(0.387)	(0.210)	(1.043)	(1.813)	(0.668)	
28. Restaurants, hotels, etc. ..	5.418	-0.031	-0.388	-0.665	-1.799	-1.300	0
	5.298	-0.057	-0.458	-0.599	-1.770	-1.131	0.291
	(1.492)	(0.456)	(2.412)	(1.209)	(2.135)	(0.789)	

a) See note a, page 189.

Table B.3 (cont.). Coefficient estimates for the model specification with family size as the only background variable.[a)]
A priori restrictions: $\beta=0$. $s_{iA}=a'_{iA}=b'_{iA}=0$.

$$w_i = s_{io} + s_{in}n + (a'_{io}+a'_{in}n)\frac{\log u}{u} + (b'_{io}+b'_{in}n)\frac{1}{u}$$

Commodity group i	$s_{io}\cdot 10^2$	$s_{in}\cdot 10^2$	$a'_{io}\cdot 10^2$	$a'_{in}\cdot 10^2$	$b'_{io}\cdot 10^2$	$b'_{in}\cdot 10^2$	$\hat{\rho}$
I Food, beverages and tobacco	11.94	-0.448	8.80	3.678	19.92	7.078	0
	9.83	-0.097	12.15	2.436	20.80	8.145	0.404
	(3.29)	(1.005)	(5.25)	(2.640)	(4.71)	(1.743)	
II Clothing and footwear	7.656	0.365	0.591	2.136	3.471	-3.142	0
	7.698	0.226	0.114	2.734	3.874	-3.336	0.250
	(2.564)	(0.784)	(4.163)	(2.084)	(3.668)	(1.356)	
III Housing, fuel and furniture ..	24.21	-1.578	0.843	3.256	10.93	-0.890	0
	24.30	-1.838	-0.108	4.243	11.40	-0.990	0.181
	(3.80)	(1.160)	(6.204)	(3.100)	(5.43)	(2.005)	
IV Travel and recreation	46.16	1.030	-12.06	-6.832	-31.90	-1.311	0
	49.26	0.983	-15.99	-7.181	-35.32	-1.729	0.300
	(4.60)	(1.407)	(7.44)	(3.730)	(6.59)	(2.436)	
V Other goods and services	10.03	0.632	1.826	-2.238	-2.416	-1.735	0
	8.80	0.921	4.405	-3.077	-1.427	-1.674	0.208
	(2.38)	(0.728)	(3.882)	(1.941)	(3.406)	(1.258)	

a) See note a, page 189.

Table B.4. Coefficient estimates for the model specification with age of the main income earner as the only background variable.[a)]
A priori restrictions: $\beta=0$. $s_{in}=a'_{in}=b'_{in}=0$.

$$w_i = s_{io} + s_{iA} + (a'_{io}+a'_{iA}A)\,\frac{\log u}{u} + (b'_{io}+b'_{iA}A)\,\frac{1}{u}$$

Commodity group i	$s_{io}\cdot 10^2$	$s_{iA}\cdot 10^2$	$a'_{io}\cdot 10^2$	$a'_{iA}\cdot 10^2$	$b'_{io}\cdot 10^2$	$b'_{iA}\cdot 10^2$	$\hat{\rho}$
	-2.545	5.947	11.52	-16.60	3.960	-1.459	0
1. Flour and bread	-2.039	4.474	10.26	-13.71	4.116	-0.670	0.418
	(1.012)	(1.806)	(1.96)	(3.12)	(1.545)	(2.618)	
	5.315	-4.552	1.326	2.311	3.010	-0.608	0
2. Meat and eggs .	5.574	-1.012	1.415	2.367	2.221	0.754	0.225
	(3.105)	(5.517)	(6.110)	(9.703)	(4.704)	(7.976)	
	-1.224	3.774	2.530	-1.816	2.416	-2.223	0
3. Fish	-0.655	2.555	1.544	-0.012	1.728	-0.603	0.439
	(1.083)	(1.934)	(2.090)	(3.330)	(1.655)	(2.804)	
	0.908	-1.596	-0.208	1.949	-0.768	2.916	0
4. Canned meat and fish	0.921	-1.670	-0.194	1.967	-0.733	2.951	0.197
	(0.440)	(0.782)	(0.868)	(1.378)	(0.666)	(1.130)	
	-3.841	7.999	15.75	-20.86	8.336	-5.406	0
5. Dairy products	-3.128	6.467	14.06	-18.10	8.130	-4.242	0.451
	(1.167)	(2.085)	(2.25)	(3.58)	(1.784)	(3.023)	
	-0.899	1.744	3.038	-3.336	2.805	-2.067	0
6. Butter and margarine	-0.874	1.549	3.181	-3.336	2.679	-1.669	0.263
	(0.497)	(0.883)	(0.975)	(1.549)	(0.754)	(1.278)	
	3.514	-2.256	-0.491	6.106	7.839	-3.532	0
7. Potatoes and vegetables	2.936	-1.690	0.111	5.309	9.788	-5.835	0.343
	(1.971)	(3.512)	(3.842)	(6.112)	(3.000)	(5.085)	
	0.022	2.431	9.308	-10.20	3.836	1.994	0
8. Other food	0.663	0.964	7.932	-7.52	3.374	3.447	0.267
	(1.521)	(2.705)	(2.983)	(4.74)	(2.307)	(3.911)	
	0.741	3.293	4.340	-7.082	2.000	-4.984	0
9. Beverages	0.979	2.379	4.059	-5.938	1.910	-4.053	0.311
	(1.734)	(3.087)	(3.389)	(5.389)	(2.636)	(4.468)	
	-1.874	4.641	5.435	-8.94	10.88	-14.50	0
10. Tobacco	-2.977	5.555	9.475	-13.76	11.04	-13.89	0.718
	(1.137)	(2.051)	(2.081)	(3.33)	(1.75)	(2.96)	
	10.73	-4.434	-2.745	6.84	-8.111	9.48	0
11. Clothing	10.01	-3.028	-2.235	5.44	-6.159	6.25	0.285
	(3.96)	(7.042)	(7.754)	(12.32)	(6.011)	(10.19)	
	2.785	-0.939	0.729	-2.209	-3.500	3.774	0
12. Footwear	2.752	-0.867	0.820	-2.391	-3.493	3 736	-0.018
	(1.891)	(3.347)	(3.775)	(5.977)	(2.840)	(4.821)	
	16.05	-15.11	-7.386	19.79	0.008	10.67	0
13. Housing	13.12	-10.47	-0.468	9.39	2.323	6.85	0.354
	(5.04)	(8.98)	(9.822)	(15.61)	(7.678)	(13.01)	
	0.540	-1.039	1.636	4.003	9.683	1.015	0
14. Fuel and furniture	0.316	-0.742	2.474	2.661	9.716	1.052	0.100
	(1.522)	(2.698)	(3.018)	(4.785)	(2.294)	(3.892)	

a) Asymptotic standard errors are given in parentheses. The unit of measurement of real income is 10 000 Nkr.

Table B.4 (cont.). Coefficient estimates for the model specification with age of the main income earner as the only background variable.[a)]
A priori restrictions: $\beta=0$. $s_{in}=a'_{in}=b'_{in}=0$.

$$w_i = s_{io} + s_{iA} + (a'_{io}+a'_{iA}A)\frac{\log u}{u} + (b'_{io}+b'_{iA}A)\frac{1}{u}$$

Commodity group i	$s_{io}\cdot 10^2$	$s_{iA}\cdot 10^2$	$a'_{io}\cdot 10^2$	$a'_{iA}\cdot 10^2$	$b'_{io}\cdot 10^2$	$b'_{iA}\cdot 10^2$	$\hat{\rho}$
	11.47	-7.976	-12.14	16.62	-12.65	13.45	0
15. Furniture	11.51	-8.046	-12.21	16.73	-12.71	13.56	-0.007
	(3.50)	(6.191)	(6.98)	(11.05)	(5.25)	(8.92)	
	3.900	-0.807	-6.037	7.281	2.127	-3.243	0
16. Household equipment	3.711	-5.092	-5.657	6.735	2.369	-3.640	0.039
	(2.712)	(4.802)	(5.397)	(8.550)	(4.079)	(6.922)	
	6.668	-9.805	-7.430	15.85	-6.589	15.83	0
17. Misc. household goods	6.282	-9.021	-6.901	14.57	-5.752	14.39	0.255
	(1.791)	(3.185)	(3.517)	(5.59)	(2.717)	(4.61)	
18. Medical care[b)] .	3.103	-1.931	-6.600	11.22	-3.853	6.138	0
	(2.239)	(3.963)	(4.465)	(7.07)	(3.364)	(5.709)	
	25.84	-17.95	-36.40	34.46	-32.44	25.98	0
19. Motorcars, bicycles	24.60	-16.67	-33.79	31.32	-30.19	23.48	-0.071
	(5.61)	(9.93)	(11.23)	(17.77)	(8.41)	(14.28)	
	5.162	11.76	12.57	-32.72	8.838	-30.40	0
20. Running costs of vehicles	5.863	9.94	12.01	-30.65	6.875	-27.25	0.296
	(4.556)	(8.11)	(8.91)	(14.17)	(6.921)	(11.73)	
	-3.654	13.20	7.084	-13.78	6.225	-14.41	0
21. Public transport	-3.701	13.38	7.119	-14.05	6.332	-14.71	0.050
	(2.606)	(4.62)	(5.182)	(8.21)	(3.921)	(6.65)	
	-2.507	8.179	2.660	-4.866	3.015	-6.946	0
22. P T T charges .	-2.519	8.495	2.682	-5.419	2.995	-7.326	0.167
	(2.759)	(4.897)	(5.451)	(8.649)	(4.170)	(7.073)	
	7.059	2.040	6.130	-15.63	-1.597	-5.68	0
23. Recreation	7.384	1.310	5.413	-14.23	-1.835	-4.99	0.063
	(4.090)	(7.246)	(8.129)	(12.88)	(6.158)	(10.45)	
	-0.789	4.734	4.261	-1.523	4.336	-5.076	0
24. Public entertainment	-1.064	6.199	3.283	-2.148	6.111	-8.874	0.426
	(2.660)	(4.748)	(5.140)	(8.187)	(4.062)	(6.883)	
	3.582	-4.221	-3.673	10.08	-4.054	9.545	0
25. Books and newspapers	3.060	-3.183	-2.905	8.59	-3.164	7.872	0.415
	(2.208)	(3.940)	(4.272)	(6.80)	(3.371)	(5.711)	
	1.124	-0.022	3.860	-3.334	0.619	0.411	0
26. Personal care ..	1.313	-0.465	3.483	-2.793	0.631	0.699	0.258
	(1.148)	(2.042)	(2.254)	(3.581)	(1.742)	(2.953)	
	8.752	-8.839	-13.55	17.28	-11.25	13.26	0
27. Misc. goods and services	8.681	-8.731	-13.40	17.07	-11.16	13.11	0.024
	(2.308)	(4.086)	(4.60)	(7.28)	(3.47)	(5.89)	
	0.076	7.631	4.474	-10.91	5.375	-13.91	0
28. Restaurants, hotels, etc. ...	-0.521	8.525	5.554	-12.53	6.353	-15.34	0.315
	(2.720)	(4.842)	(5.315)	(8.45)	(4.136)	(7.01)	

a) See note a, page 192.
b) Convergence with respect to ρ obtained after one iteration.

Table B.4 (cont.). Coefficient estimates for the model specification with age of the main income earner as the only background variable.[a)]
A priori restrictions: $\beta=0$. $s_{in}=a'_{in}=b'_{in}=0$.

$$w_i = s_{io} + s_{iA} + (a'_{io}+a'_{iA}A)\,\frac{\log u}{u} + (b'_{io}+b'_{iA}A)\,\frac{1}{u}$$

Commodity group i	$s_{io}\cdot 10^2$	$s_{iA}\cdot 10^2$	$a'_{io}\cdot 10^2$	$a'_{iA}\cdot 10^2$	$b'_{io}\cdot 10^2$	$b'_{iA}\cdot 10^2$	$\hat{\rho}$
I Food, beverages and tobacco	0.116	25.52	52.54	-58.47	44.32	-29.87	0
	2.492	17.44	49.41	-48.71	43.59	-22.03	0.406
	(6.198)	(11.06)	(12.00)	(19.11)	(9.46)	(16.03)	
II Clothing and footwear	13.51	-5.373	-2.016	4.63	-11.61	13.26	0
	13.41	-5.273	-3.016	6.28	-10.02	11.01	0.243
	(4.65)	(8.264)	(9.136)	(14.51)	(7.05)	(11.95)	
III Housing, fuel and furniture	38.63	-34.73	-31.36	63.55	-7.42	37.22	0
	35.03	-28.86	-23.48	51.18	-3.30	30.86	0.157
	(6.85)	(12.16)	(13.55)	(21.49)	(10.35)	(17.56)	
IV Travel and recreation	34.69	17.74	-7.36	-23.97	-16.17	-27.00	0
	42.42	9.34	-24.24	-3.68	-28.14	-13.57	0.292
	(8.37)	(14.90)	(16.39)	(26.05)	(12.72)	(21.56)	
V Other goods and services	13.05	-3.160	-11.81	14.26	-9.111	5.90	0
	11.72	-1.946	-8.72	11.42	-7.640	4.57	0.207
	(4.33)	(7.692)	(8.53)	(13.55)	(6.555)	(11.12)	

a) See note a, page 192.

Table B.5. Coefficient estimates for the model specification with age of main income earner and family size as background variables.[a)]
A priori restriction: $\beta=0$.

$$w_i = s_{io} + s_{in} n + s_{iA} A + (a'_{io}+a'_{in} n+a'_{iA} A)\,\frac{\log u}{u} + (b'_{io}+b'_{in} n+b'_{iA} A)\,\frac{1}{u}$$

Commodity group i	$s_{io}\cdot 10^2$	$s_{in}\cdot 10$	$s_{iA}\cdot 10^4$	$a'_{io}\cdot 10^2$	$a'_{in}\cdot 10^2$	$a'_{iA}\cdot 10^4$	$b'_{io}\cdot 10^2$	$b'_{in}\cdot 10^2$	$b'_{iA}\cdot 10^4$	$\hat{\rho}$
1. Flour and bread	-2.868	0.676	5.581	9.297	-0.009	-13.96	1.088	1.380	0.456	0
	-2.287	0.573	4.232	8.043	-0.030	-11.28	0.627	1.478	1.594	0.383
	(1.354)	(1.836)	(1.892)	(2.577)	(0.477)	(3.36)	(1.952)	(0.295)	(2.720)	
2. Meat and eggs	1.287	4.86	1.636	1.839	0.887	2.28	7.260	-0.690	-2.572	0
	1.502	5.43	1.234	2.360	0.568	1.98	6.043	-0.508	-1.201	0.214
	(0.438)	(15.49)	(6.095)	(8.478)	(1.549)	(11.00)	(6.251)	(0.938)	(8.716)	
3. Fish	1.487	-5.477	1.210	-3.512	1.462	3.666	-0.818	0.389	1.220	0
	1.811	-5.027	0.298	-3.707	1.230	4.675	-1.682	0.548	2.712	0.437
	(1.522)	(2.069)	(2.130)	(2.876)	(0.535)	(3.753)	(2.201)	(0.334)	(3.067)	
4. Canned meat and fish ..	0.693	0.724	-1.398	0.336	-0.351	1.564	-1.228	0.315	2.909	0
	0.786	0.547	-1.560	0.092	-0.281	1.827	-1.316	0.328	3.090	0.194
	(0.621)	(0.835)	(0.865)	(1.205)	(0.220)	(1.563)	(0.886)	(0.133)	(1.236)	
5. Dairy products ..	-3.064	-2.720	5.879	7.616	1.705	-12.60	3.488	1.701	-0.632	0
	-2.156	-3.186	4.588	6.058	1.686	-10.38	2.185	1.892	1.049	0.369
	(1.486)	(2.014)	(2.076)	(2.833)	(0.523)	(3.69)	(2.140)	(0.323)	(2.984)	
6. Butter and margarine .	-1.378	0.576	1.851	2.479	0.145	-2.647	2.639	0.211	-1.817	0
	-1.315	0.541	1.659	2.548	0.144	-2.600	2.408	0.227	-1.376	0.235
	(0.688)	(0.926)	(0.958)	(1.330)	(0.243)	(1.726)	(0.983)	(0.148)	(1.371)	
7. Potatoes and vegetables	3.323	-0.952	-3.287	-5.759	1.285	11.53	5.409	0.990	-0.738	0
	3.062	-1.175	-2.880	-5.472	1.162	11.02	6.014	1.335	-2.186	0.336
	(2.733)	(3.698)	(3.810)	(5.230)	(0.964)	(6.81)	(3.930)	(0.593)	(5.478)	
8. Other food	0.258	-0.773	1.365	5.001	0.591	-5.625	2.288	1.456	4.936	0
	0.834	-0.541	0.094	3.962	0.439	-3.339	-0.680	1.604	6.505	0.251
	(2.091)	(2.818)	(2.914)	(4.036)	(0.739)	(5.241)	(2.992)	(0.449)	(4.172)	
9. Beverages .	0.663	1.210	4.033	7.736	-1.055	-10.41	2.819	-0.215	-6.490	0
	0.689	1.450	3.205	7.317	-1.009	-9.09	2.884	-0.176	-5.680	0.297
	(2.446)	(3.304)	(3.412)	(4.701)	(0.864)	(6.11)	(3.509)	(0.528)	(4.892)	
10. Tobacco ...	-4.635	7.727	7.792	14.56	-3.335	-16.97	11.74	0.901	-18.06	0
	-5.439	6.394	8.014	15.89	-2.275	-19.08	12.91	0.288	-17.19	0.707
	(1.572)	(2.169)	(2.220)	(2.82)	(0.537)	(3.70)	(2.31)	(0.355)	(3.21)	
11. Clothing ..	8.768	1.098	-3.555	-3.459	1.469	7.389	-3.756	-1.472	7.942	0
	9.083	-1.650	-3.240	-6.231	2.629	8.756	-2.225	-1.771	5.985	0.297
	(5.611)	(7.575)	(7.824)	(1.078)	(1.981)	(1.402)	(8.046)	(1.211)	(1.122)	
12. Footwear ..	1.312	0.940	-0.088	1.060	0.961	-2.721	0.413	-1.374	2.060	0
	1.264	0.929	0.013	1.183	0.969	-2.973	0.429	-1.376	2.009	-0.026
	(2.696)	(3.593)	(3.733)	(5.310)	(0.955)	(6.852)	(3.802)	(0.565)	(5.301)	
13. Housing ...	20.78	-10.02	-18.82	-14.72	2.712	25.58	-0.921	-1.430	1.37	0
	16.93	-8.46	-13.48	-6.20	2.354	13.75	2.401	-1.439	8.83	0.349
	(7.12)	(9.63)	(9.94)	(13.60)	(2.508)	(17.70)	(1.024)	(1.545)	(14.27)	
14. Fuel and power	3.445	-6.523	-4.232	-6.981	2.206	12.00	5.621	0.608	5.685	0
	3.028	-5.986	-3.754	-5.739	2.029	10.44	5.807	0.628	5.442	0.075
	(2.144)	(2.869)	(2.975)	(4.196)	(0.760)	(5.43)	(3.039)	(0.453)	(4.237)	

a) Asymptotic standard errors of estimates are given in parentheses. The unit of measurement of real income is 10 000 Nkr.

Table B.5 (cont.). Coefficient estimates for the model specification with age of main income earner and family size as background variables.[a)] A priori restriction: $\beta=0$.

$$w_i = s_{io} + s_{in}n + s_{iA}A + (a'_{io}+a'_{in}n+a'_{iA}A)\,\frac{\log u}{u} + (b'_{io}+b'_{in}n+b'_{iA}A)\,\frac{1}{u}$$

Commodity group i	$s_{io}\cdot10^2$	$s_{in}\cdot10$	$s_{iA}\cdot10^4$	$a'_{io}\cdot10^2$	$a'_{in}\cdot10^2$	$a'_{iA}\cdot10^4$	$b'_{io}\cdot10^2$	$b'_{in}\cdot10^2$	$b'_{iA}\cdot10^4$	$\hat{\rho}$
	23.02	-19.23	-16.76	-27.31	2.386	29.85	-25.81	1.831	24.22	0
15. Furniture .	23.20	-19.24	-17.04	-27.65	2.391	30.34	-26.07	1.832	24.64	-0.025
	(4.94)	(6.59)	(6.84)	(9.73)	(1.751)	(12.56)	(6.97)	(1.036)	(9.72)	
16. Household	4.111	-0.948	-1.143	-7.257	0.563	8.332	2.436	-0.271	-2.941	0
equipment .	3.850	-1.067	-0.798	-6.972	0.705	7.805	2.989	-0.349	-3.464	0.046
	(3.886)	(5.193)	(5.389)	(7.620)	(1.377)	(9.848)	(5.500)	(0.819)	(7.669)	
17. Misc. house-	7.895	-2.745	-11.12	-10.90	0.914	19.05	-8.151	0.191	17.68	0
hold goods	7.492	-2.650	-10.28	-10.18	0.822	17.58	-7.518	0.281	16.29	0.255
	(2.543)	(3.428)	(3.54)	(4.91)	(0.899)	(6.37)	(3.639)	(0.547)	(5.07)	
18. Medical	2.602	0.383	-1.680	-6.642	0.289	11.25	-2.851	-0.316	5.740	0
care	2.630	0.334	-1.702	-6.712	0.303	11.30	-2.875	-0.314	5.759	-0.003
	(3.205)	(0.428)	(4.441)	(6.305)	(1.136)	(8.14)	(4.526)	(0.673)	(6.311)	
19. Motorcars,	23.46	6.72	-14.20	-23.98	-3.202	22.49	-26.92	-1.308	19.42	0
bicycles ..	21.55	7.82	-12.42	-20.61	-3.290	18.68	-23.97	-1.437	16.42	-0.069
	(8.04)	(10.70)	(11.12)	(15.87)	(2.849)	(20.46)	(11.32)	(1.679)	(15.78)	
20. Running	-1.858	13.16	17.56	24.25	-2.754	-42.89	15.08	-0.403	-37.28	0
costs of	-0.850	12.73	15.42	22.73	-2.505	-39.89	13.68	-0.597	-33.90	0.293
vehicles ..	(6.451)	(8.71)	(9.00)	(12.40)	(2.278)	(16.12)	(9.25)	(1.392)	(12.90)	
21. Public	1.097	-6.660	9.886	2.515	-0.830	-9.695	-0.646	1.526	-9.98	0
transport .	1.103	-6.822	9.999	2.435	-0.024	-9.842	-0.545	1.505	-10.19	0.042
	(3.706)	(4.952)	(5.139)	(7.269)	(1.313)	(9.393)	(5.244)	(0.781)	(7.31)	
22. PTT	-0.183	-3.358	6.936	2.049	-0.063	-4.725	1.686	-0.153	-6.155	0
charges ...	0.456	-4.596	6.689	0.983	0.146	-4.293	0.844	0.015	-5.832	0.163
	(3.913)	(5.253)	(5.440)	(7.607)	(1.385)	(9.857)	(5.571)	(0.834)	(7.768)	
	6.259	3.193	3.947	13.41	-1.991	-22.74	1.166	-0.735	-9.34	0
23. Recreation	6.132	3.978	3.623	13.47	-2.145	-22.08	1.501	-0.808	-9.19	0.058
	(5.820)	(7.781)	(8.073)	(11.40)	(2.062)	(14.74)	(8.241)	(1.228)	(11.49)	
24. Public	-6.229	8.386	9.142	12.42	-0.675	-9.153	13.37	-2.177	-11.63	0
enter-	-5.603	7.182	9.690	9.36	-0.419	-7.713	13.82	-1.920	-14.21	0.422
tainment ..	(3.742)	(5.086)	(5.237)	(7.09)	(1.316)	(9.245)	(5.41)	(0.819)	(7.54)	
25. Books and	1.613	5.111	-2.083	2.246	-2.033	4.853	-3.031	0.419	7.047	0
newspapers	1.879	2.911	-2.003	0.219	-1.018	5.893	-2.226	0.090	6.370	0.410
	(3.116)	(4.231)	(4.359)	(5.911)	(1.096)	(7.706)	(4.499)	(0.681)	(6.271)	
26. Personal	2.003	-2.330	-1.262	-0.050	1.023	0.396	-1.087	0.354	2.466	0
care	2.187	-2.265	-1.689	-0.268	0.961	0.756	-1.266	0.413	2.820	0.251
	(1.618)	(2.181)	(2.254)	(3.123)	(0.572)	(4.055)	(2.315)	(0.348)	(3.228)	
27. Misc. goods	8.477	1.509	-8.180	-10.91	-1.011	14.85	-11.25	0.204	12.34	0
and ser-	8.373	1.578	-8.047	-10.71	-1.026	14.60	-11.11	0.199	12.15	0.023
vices	(3.301)	(4.408)	(4.576)	(6.48)	(1.170)	(8.37)	(4.67)	(0.695)	(6.51)	
28. Restau-	-2.346	5.450	10.98	14.69	-2.036	-20.96	12.02	-1.931	-20.16	0
rants, ho-	-2.817	5.399	11.57	15.12	-2.010	-21.76	12.86	-1.810	-21.43	0.291
tels, etc.	(3.798)	(5.127)	(5.30)	(7.30)	(1.341)	(9.49)	(5.45)	(0.819)	(7.59)	

a) See note a, page 195.

Table B.5 (cont.). Coefficient estimates for the model specification with age of main income earner and family size as background variables.[a)] A priori restriction: $\beta=0$.

$$w_i = s_{io} + s_{in}n + s_{iA}A + (a'_{io}+a'_{in}n+a'_{iA}A)\frac{\log u}{u} + (b'_{io}+b'_{in}n+b'_{iA}A)\frac{1}{u}$$

Commodity group i	$s_{io}\cdot 10^2$	$s_{in}\cdot 10$	$s_{iA}\cdot 10^4$	$a'_{io}\cdot 10^2$	$a'_{in}\cdot 10^2$	$a'_{iA}\cdot 10^4$	$b'_{io}\cdot 10^2$	$b'_{in}\cdot 10^2$	$b_{iA}\cdot 10^4$	$\hat{\rho}$
I Food, beve-	-4.234	5.86	24.66	39.59	1.323	-43.16	32.62	6.438	-20.79	0
rages and	-1.217	6.19	16.63	35.66	0.700	-32.70	26.52	7.819	-9.78	0.398
tobacco ...	(8.306)	(11.27)	(11.61)	(15.78)	(2.923)	(20.57)	(11.98)	(1.813)	(16.71)	
II Clothing	10.08	2.038	-3.643	-2.40	2.431	4.67	-3.342	-2.846	10.00	0
and foot-	10.88	-0.126	-4.488	-6.09	3.302	8.63	-2.207	-3.054	8.83	0.249
wear	(6.59)	(0.887)	(9.175)	(12.71)	(2.328)	(16.51)	(9.421)	(1.415)	(13.13)	
III Housing,	59.26	-39.46	-52.07	-67.18	8.782	94.81	-26.82	0.928	58.31	0
fuel and	54.76	-38.04	-45.68	-57.96	8.675	81.45	-21.29	0.597	50.68	0.145
furniture .	(9.67)	(12.98)	(13.45)	(18.84)	(3.426)	(24.41)	(13.77)	(2.059)	(19.19)	
IV Travel and	24.16	26.55	31.19	32.90	-10.80	-61.86	0.71	-2.831	-47.91	0
recreation	35.81	19.35	19.33	9.52	-9.39	-35.53	-13.82	-2.708	-31.69	0.289
	(11.75)	(15.86)	(16.38)	(22.59)	(4.15)	(29.36)	(16.84)	(2.535)	(23.48)	
V Other goods	10.74	5.011	-0.139	-2.92	-1.735	5.544	-3.162	-1.689	0.38	0
and services	7.25	9.637	2.985	4.65	-3.016	-1.143	0.536	-1.760	-3.56	0.207
	(6.14)	(8.259)	(8.546)	(11.90)	(2.172)	(1.543)	(0.876)	(1.314)	(12.22)	

a) See note a, page 195.

Table B.6. Estimates of Engel elasticities for the model specification with no background variables included.
A priori restrictions: $\beta=0$. $s_{in}=s_{iA}=a'_{in}=a'_{iA}=b'_{in}=b'_{iA}=0$

Commodity group i	ρ=0			ρ unrestricted		
	u= 30 000	u= 50 000	u= 100 000	u= 30 000	u= 50 000	u= 100 000
1. Flour and bread	0.5381	0.6328	0.7524	0.4608	0.5133	0.6091
2. Meat and eggs	0.8587	0.8617	0.8896	0.8601	0.8634	0.8913
3. Fish	0.5988	0.5546	0.5614	0.5808	0.5406	0.5498
4. Canned meat and fish ..	0.5097	0.4607	0.4508	0.4879	0.4315	0.4080
5. Dairy products	0.4977	0.5375	0.6202	0.4545	0.4856	0.5613
6. Butter and margarine ..	0.4247	0.3944	0.3950	0.3956	0.3463	0.3134
7. Potatoes and vegetables	0.6499	0.6628	0.7178	0.6182	0.6305	0.6865
8. Other food	0.5864	0.6347	0.7183	0.5583	0.5971	0.6761
9. Beverages	1.1345	1.0761	1.0363	1.0749	1.0284	1.0037
10. Tobacco	0.6663	0.7306	0.8139	0.5045	0.5262	0.5924
11. Clothing	1.1000	1.0459	1.0145	1.0985	1.0460	1.0152
12. Footwear	1.3063	1.2293	1.1422	1.3082	1.2310	1.1433
13. Housing	0.8624	0.8745	0.9051	0.8473	0.8593	0.8923
14. Fuel and power	0.2516	0.1958	0.1115	0.2473	0.1936	0.1125
15. Furniture	1.3605	1.2291	1.1272	1.3605	1.2290	1.1271
16. Household equipment ...	0.9795	1.0116	1.0217	0.9774	1.0085	1.0190
17. Misc. household goods .	0.6677	0.6947	0.7575	0.6681	0.7025	0.7695
18. Medical care	0.8627	0.8533	0.8761	0.8603	0.8496	0.8722
19. Motorcars, bicycles ...	2.6090	1.7779	1.3746	2.5160	1.7511	1.3654
20. Running costs of vehicles	1.6205	1.3823	1.2083	1.5860	1.3635	1.1989
21. Public transport	1.0898	1.0467	1.0193	1.1049	1.0611	1.0300
22. PTT charges	0.9030	0.8729	0.8824	0.9769	0.9484	0.9468
23. Recreation	1.3879	1.2684	1.1580	1.3752	1.2602	1.1535
24. Public entertainment ..	0.7515	0.6822	0.6774	0.8115	0.7688	0.7811
25. Books and newspapers ..	0.6967	0.6655	0.6880	0.7170	0.6869	0.7100
26. Personal care	0.8801	0.8513	0.8642	0.8558	0.8353	0.8545
27. Misc. goods and services	1.9908	1.5755	1.3020	1.9847	1.5727	1.3008
28. Restaurants, hotels etc.	1.3370	1.2200	1.1246	1.3199	1.2106	1.1200
I Food, beverages and tobacco	0.6860	0.7181	0.7819	0.6502	0.6777	0.7422
II Clothing and footwear .	1.1358	1.0805	1.0408	1.1257	1.0708	1.0335
III Housing, fuel and furniture	0.8267	0.8578	0.9010	0.8226	0.8536	0.8975
IV Travel and recreation .	1.4215	1.2783	1.1588	1.4822	1.3164	1.1796
V Other goods and services	1.2144	1.1438	1.0831	1.1790	1.1115	1.0600

Table B.7. Estimates of Engel elasticities and approximate Slutsky elasticities for the model specification with no background variables. All prices are set equal to their sample means. A priori restrictions: $\beta=1$. $s_{in}=s_{iA}=a_{in}=a_{iA}=b_{in}=b_{iA}=0$

Commodity group i	E_i			S^*_{ii}		
	u= 30 000	u= 50 000	u= 100 000	u= 30 000	u= 50 000	u= 100 000
1. Flour and bread	0.459	0.509	0.602	0.040	-0.088	-0.286
2. Meat and eggs	0.852	0.856	0.885	0.378	0.118	-0.085
3. Fish	0.580	0.540	0.550	-1.356	-1.332	-1.288
4. Canned meat and fish ..	0.488	0.433	0.411	-2.091	-2.057	-1.989
5. Dairy products	0.462	0.492	0.566	0.059	-0.048	-0.225
6. Butter and margarine ..	0.398	0.350	0.319	-0.490	-0.497	-0.513
7. Potatoes and vegetables	0.618	0.629	0.684	-0.829	-0.850	-0.883
8. Other food	0.561	0.601	0.681	-1.163	-1.140	-1.106
9. Beverages	1.144	1.092	1.051	-8.164	-5.479	-3.411
10. Tobacco	0.510	0.537	0.608	-1.545	-1.485	-1.388
11. Clothing	1.120	1.066	1.031	-0.306	-0.616	-0.824
12. Footwear	1.330	1.243	1.150	-4.859	-3.464	-2.353
13. Housing	0.843	0.855	0.889	-3.441	-2.949	-2.337
14. Fuel and power	0.251	0.208	0.149	1.447	1.477	1.547
15. Furniture	1.358	1.227	1.126	-2.485	-1.889	-1.467
16. Household equipment ...	0.995	1.022	1.027	-3.771	-3.548	-2.847
17. Misc. household goods .	0.653	0.684	0.751	5.422	4.393	2.965
18. Medical care	0.886	0.872	0.890	4.242	3.371	2.129
19. Motorcars, bicycles ...	2.532	1.756	1.367	-8.210	-4.077	-2.336
20. Running costs of vehicles	1.588	1.365	1.199	-1.793	-1.448	-1.226
21. Public transport	1.072	1.029	1.012	-1.261	-1.027	-0.929
22. PTT charges	0.965	0.936	0.937	-4.213	-3.781	-3.024
23. Recreation	1.372	1.258	1.153	-2.080	-1.668	-1.359
24. Public entertainment ..	0.833	0.793	0.805	2.398	1.974	1.281
25. Books and newspapers ..	0.733	0.701	0.722	-4.017	-3.681	-3.127
26. Personal care	0.847	0.830	0.839	-9.385	-8.096	-6.201
27. Misc. goods and services	1.976	1.569	1.299	-3.928	-2.476	-1.694
28. Restaurants, hotels, etc.	1.328	1.215	1.122	-3.054	-2.258	-1.672
I Food, beverages and tobacco	0.648	0.675	0.739	0.333	0.130	-0.039
II Clothing and footwear .	1.137	1.082	1.042	-0.292	-0.579	-0.786
III Housing, fuel and furniture	0.822	0.853	0.897	-1.999	-1.765	-1.501
IV Travel and recreation .	1.482	1.316	1.227	-1.231	-1.136	-1.071
V Other goods and services	1.184	1.115	1.062	-6.052	-4.081	-2.624

Table B.8. Estimates of Engel elasticities for the model specification with family size as the only background variable included. A priori restrictions: $\beta=0$. $s_{iA}=a'_{iA}=b'_{iA}=0$

Commodity group i	n=3			u=50 000		
	u= 30 000	u= 50 000	u= 100 000	n=1	n=3	n=5
1. Flour and bread	0.234	0.248	0.294	0.429	0.248	0.159
2. Meat and eggs	0.776	0.760	0.773	0.680	0.760	0.815
3. Fish	0.592	0.513	0.482	0.778	0.513	0.206
4. Canned meat and fish ..	0.273	0.353	0.489	0.230	0.353	0.440
5. Dairy products	0.252	0.136	-0.091	0.324	0.136	0.058
6. Butter and margarine ..	0.217	0.038	-0.414	-0.155	0.038	0.124
7. Potatoes and vegetables	0.451	0.454	0.498	0.486	0.454	0.434
8. Other food	0.386	0.408	0.472	0.500	0.408	0.351
9. Beverages	1.170	1.121	1.073	1.078	1.121	1.179
10. Tobacco	0.344	0.550	0.779	0.624	0.550	0.480
11. Clothing	1.139	0.999	0.943	0.958	0.999	1.034
12. Footwear	1.499	1.136	0.985	0.986	1.136	1.246
13. Housing	0.921	0.871	0.871	0.860	0.871	0.883
14. Fuel and power	0.215	0.082	-0.219	0.146	0.082	0.029
15. Furniture	1.512	1.328	1.184	1.341	1.328	1.306
16. Household equipment ...	1.007	1.007	1.006	0.970	1.007	1.043
17. Misc. household goods .	0.641	0.674	0.743	0.575	0.674	0.631
18. Medical care	0.924	0.922	0.937	0.671	0.922	1.174
19. Motorcars, bicycles ...	3.083	1.924	1.431	1.579	1.924	2.384
20. Running costs of vehicles	1.473	1.307	1.173	1.387	1.307	1.243
21. Public transport	1.157	1.177	1.135	1.327	1.177	0.876
22. PTT charges	1.387	1.238	1.128	1.185	1.238	1.434
23. Recreation	1.482	1.317	1.180	1.256	1.317	1.392
24. Public entertainment ..	0.904	0.780	0.743	0.529	0.780	1.002
25. Books and newspapers ..	0.706	0.788	0.872	0.603	0.788	0.990
26. Personal care	0.739	0.664	0.655	0.822	0.664	0.557
27. Misc. goods and services	2.233	1.780	1.414	1.435	1.780	2.218
28. Restaurants, hotels, etc.	1.696	1.388	1.198	1.164	1.388	1.789
I Food, beverages and tobacco	0.505	0.540	0.619	0.626	0.540	0.482
II Clothing and footwear .	1.189	1.022	0.951	0.949	1.022	1.082
III Housing, fuel and furniture	0.877	0.869	0.890	0.900	0.869	0.833
IV Travel and recreation .	1.590	1.396	1.226	1.289	1.396	1.527
V Other goods and services	1.302	1.215	1.130	1.048	1.215	1.417

Table B.9. Estimates of Engel elasticities for the model specification with age of the main income earner as the only background variable included.
A priori restrictions: $\beta=0$. $s_{in}=a'_{in}=b'_{in}=0$

Commodity group i	A=50			u=50 000		
	u= 30 000	u= 50 000	u= 100 000	A=30	A=50	A=70
1. Flour and bread	0.493	0.429	0.396	0.258	0.429	0.602
2. Meat and eggs	0.862	0.870	0.899	0.884	0.870	0.856
3. Fish	0.683	0.663	0.694	0.458	0.663	0.763
4. Canned meat and fish ..	0.561	0.499	0.482	0.864	0.499	-0.029
5. Dairy products	0.450	0.379	0.326	0.182	0.379	0.583
6. Butter and margarine ..	0.378	0.268	0.119	0.036	0.268	0.483
7. Potatoes and vegetables	0.559	0.607	0.693	0.604	0.607	0.610
8. Other food	0.580	0.564	0.597	0.570	0.564	0.558
9. Beverages	1.001	0.956	0.946	0.838	0.956	1.081
10. Tobacco	0.315	0.220	0.075	-0.135	0.220	0.783
11. Clothing	1.130	1.068	1.029	1.116	1.068	1.021
12. Footwear	1.338	1.198	1.102	1.225	1.198	1.165
13. Housing	0.819	0.840	0.881	0.900	0.840	0.764
14. Fuel and power	0.255	0.219	0.177	0.237	0.219	0.202
15. Furniture	1.513	1.327	1.182	1.516	1.327	1.139
16. Household equipment ...	0.962	1.060	1.082	1.071	1.060	1.050
17. Misc. household goods .	0.794	0.846	0.903	1.241	0.846	0.331
18. Medical care	1.194	1.167	1.113	1.737	1.167	0.892
19. Motorcars, bicycles ...	2.941	1.875	1.411	2.068	1.875	1.660
20. Running costs of vehicles	1.320	1.208	1.118	0.991	1.208	1.489
21. Public transport	1.126	1.069	1.031	0.548	1.069	1.281
22. PTT charges	1.150	1.086	1.043	0.455	1.086	1.213
23. Recreation	1.251	1.162	1.091	1.070	1.162	1.280
24. Public entertainment ..	0.815	0.804	0.832	0.567	0.804	0.949
25. Books and newspapers ..	0.864	0.844	0.862	1.109	0.844	0.632
26. Personal care	0.818	0.769	0.778	0.777	0.769	0.760
27. Misc. goods and services	2.696	1.828	1.400	2.258	1.828	1.336
28. Restaurants, hotels, etc.	1.152	1.108	1.065	0.808	1.108	1.359
I Food, beverages and tobacco	0.626	0.629	0.678	0.558	0.629	0.703
II Clothing and footwear .	1.161	1.090	1.042	1.146	1.090	1.030
III Housing, fuel and furniture	0.838	0.887	0.933	0.992	0.887	0.770
IV Travel and recreation .	1.482	1.320	1.183	1.311	1.320	1.329
V Other goods and services	1.240	1.165	1.097	1.232	1.165	1.106

Table B.10. Estimates of Engel elasticities for the model specification with age of the main income earner and family size as background variables. A priori restriction: $\beta=0$

Commodity group i	n=3, A=50			u=50 000, A=50			u=50 000, n=3		
	u= 30 000	u= 50 000	u= 100 000	n=1	n=3	n=5	A=30	A=50	A=70
1. Flour and bread	0.276	0.242	0.208	0.293	0.242	0.218	0.039	0.242	0.409
2. Meat and eggs ..	0.787	0.773	0.802	0.702	0.773	0.820	0.760	0.773	0.784
3. Fish	0.704	0.626	0.612	0.981	0.626	0.274	0.674	0.626	0.602
4. Canned meat and fish	0.353	0.435	0.566	0.507	0.435	0.380	0.824	0.435	-0.094
5. Dairy products .	0.279	0.133	-0.163	0.211	0.133	0.102	-0.054	0.133	0.284
6. Butter and margarine	0.217	0.019	-0.508	-0.374	0.019	0.181	-0.363	0.019	0.299
7. Potatoes and vegetables	0.441	0.471	0.545	0.555	0.471	0.421	0.509	0.471	0.434
8. Other food	0.427	0.429	0.472	0.540	0.429	0.361	0.472	0.429	0.387
9. Beverages	1.068	1.050	1.031	0.937	1.050	1.209	0.882	1.050	1.242
10. Tobacco	0.160	0.285	0.490	0.151	0.285	0.584	-0.267	0.285	1.116
11. Clothing	1.172	1.022	0.958	1.014	1.022	1.028	1.080	1.022	0.966
12. Footwear	1.507	1.120	0.963	0.942	1.120	1.254	1.117	1.120	1.123
13. Housing	0.904	0.864	0.870	0.873	0.864	0.854	0.938	0.864	0.768
14. Fuel and power .	0.249	0.145	-0.053	0.298	0.145	0.023	0.259	0.145	0.040
15. Furniture	1.694	1.414	1.221	1.507	1.414	1.247	1.668	1.414	1.113
16. Household equipment	1.002	1.036	1.040	1.042	1.036	1.030	1.058	1.036	1.016
17. Misc. household goods	0.785	0.807	0.853	0.946	0.807	0.680	1.265	0.807	0.218
18. Medical care ...	1.211	1.125	1.065	1.103	1.125	1.145	1.695	1.125	0.868
19. Motorcars, bicycles	3.271	1.960	1.440	1.652	1.960	2.389	2.028	1.960	1.878
20. Running costs of vehicles	1.254	1.195	1.124	1.099	1.195	1.276	0.918	1.195	1.536
21. Public transport	1.168	1.179	1.133	1.299	1.179	0.973	0.946	1.179	1.299
22. PTT charges	1.472	1.277	1.145	1.206	1.277	1.469	1.163	1.277	1.319
23. Recreation	1.388	1.236	1.140	1.097	1.236	1.418	1.088	1.236	1.440
24. Public entertainment	0.860	0.760	0.737	0.382	0.760	1.048	0.296	0.760	1.012
25. Books and newspapers	0.812	0.872	0.928	0.772	0.872	0.976	1.077	0.872	0.710
26. Personal care ..	0.750	0.666	0.650	0.848	0.666	0.538	0.756	0.666	0.561
27. Misc. goods and services	2.748	1.909	1.447	1.713	1.909	2.169	2.241	1.909	1.503
28. Restaurants, hotels, etc. ...	1.406	1.260	1.146	0.921	1.260	1.863	0.844	1.260	1.662

Table B.10 (cont.). Estimates of Engel elasticities for the model specification with age of the main income earner and family size as background variables.
A priori restriction: $\beta=0$

Commodity group i	n=3, A=50			u=50 000, A=50			u=50 000, n=3		
	u= 30 000	u= 50 000	u= 100 000	n=1	n=3	n=5	A=30	A=50	A=70
I Food, beverages and tobacco	0.506	0.532	0.603	0.585	0.532	0.497	0.467	0.532	0.594
II Clothing and footwear	1.221	1.041	0.963	0.998	1.041	1.076	1.098	1.041	0.983
III Housing, fuel and furniture .	0.913	0.909	0.925	1.004	0.909	0.798	1.067	0.909	0.724
IV Travel and recreation	1.544	1.375	1.218	1.245	1.375	1.535	1.311	1.375	1.437
V Other goods and services	1.306	1.219	1.132	1.050	1.219	1.418	1.209	1.219	1.228

Table B.11. Estimates of budget shares for the model specification with age of the main income earner and family size as background variables. A priori restriction: $\beta=0$

Commodity group i	n=3, A=50			u=50 000, A=50			u=50 000, n=3			Sample mean of budget shares
	u= 30 000	u= 50 000	u= 100 000	n=1	n=3	n=5	A=30	A=50	A=70	
1. Flour and bread	2.80	1.92	1.12	1.23	1.92	2.60	1.73	1.92	2.10	2.63
2. Meat and eggs	6.91	6.16	5.30	4.91	6.16	7.41	5.83	6.16	6.49	6.70
3. Fish	1.74	1.46	1.12	1.46	1.46	1.47	0.99	1.46	1.93	1.60
4. Canned meat and fish ..	0.63	0.46	0.33	0.40	0.46	0.52	0.54	0.46	0.39	0.61
5. Dairy products ..	4.15	2.77	1.39	1.56	2.77	3.97	2.48	2.77	3.06	3.73
6. Butter and margarine .	1.09	0.70	0.30	0.41	0.70	0.99	0.59	0.70	0.81	0.98
7. Potatoes and vegetables	5.53	4.19	2.97	3.14	4.19	5.23	4.14	4.19	4.23	5.13
8. Other food	4.50	3.36	2.29	2.54	3.36	4.17	3.29	3.36	3.42	4.29
9. Beverages .	2.47	2.55	2.62	2.98	2.55	2.12	2.72	2.55	2.38	2.35
10. Tobacco ...	2.04	1.37	0.90	1.44	1.37	1.30	1.68	1.37	1.06	1.74
11. Clothing ..	7.66	8.00	7.90	7.35	8.00	8.66	7.85	8.00	8.16	7.58
12. Footwear ..	1.60	1.85	1.88	1.59	1.85	2.11	1.96	1.85	1.74	1.72
13. Housing ...	11.33	10.65	9.69	11.40	10.65	9.90	12.10	10.65	9.20	11.04
14. Fuel and power	4.86	3.23	1.68	2.87	3.23	3.59	3.09	3.23	3.37	4.78
15. Furniture .	4.21	5.55	6.86	7.12	5.55	3.97	6.02	5.55	5.08	4.45
16. Household equipment .	2.81	2.84	2.92	2.77	2.84	2.92	2.64	2.84	3.04	2.93
17. Misc. household goods	2.44	2.20	1.95	2.09	2.20	2.31	2.47	2.20	1.93	2.60
18. Medical care	1.51	1.64	1.75	1.51	1.64	1.78	1.02	1.64	2.26	1.71
19. Motorcars, bicycles ..	3.25	6.86	10.81	7.99	6.86	5.74	7.49	6.86	6.24	4.95
20. Running costs of vehicles ..	7.26	8.14	9.08	7.45	8.14	8.84	8.98	8.14	7.30	6.91
21. Public transport .	2.74	3.01	4.43	3.79	3.01	2.23	2.05	3.01	3.97	2.54
22. PTT charges ...	1.48	1.78	2.05	2.60	1.78	0.96	0.95	1.78	2.61	1.47

Tabell B.11 (cont.). Estimates of budget shares for the model specification with age of the main income earner and family size as background variables.
A priori restriction: $\beta=0$

Commodity group i	n=3, A=50			u=50 000, A=50			u=50 000, n=3			Sample mean of budget shares
	u= 30 000	u= 50 000	u= 100 000	n=1	n=3	n=5	A=30	A=50	A=70	
23. Recreation	5.83	6.75	7.66	7.65	6.75	5.84	7.81	6.75	5.68	5.89
24. Public entertainment	3.27	2.96	2.47	2.56	2.96	3.35	2.08	2.96	3.83	3.10
25. Books and newspapers	2.20	2.03	1.90	2.07	2.03	2.00	1.80	2.03	2.27	2.26
26. Personal care	2.24	1.92	1.51	1.59	1.92	2.26	2.09	1.92	1.76	2.00
27. Misc. goods and services	0.97	1.85	2.89	2.11	1.85	1.58	2.03	1.85	1.66	1.49
28. Restaurants, hotels, etc.	2.84	3.36	3.85	4.30	3.36	2.42	3.30	3.36	3.41	2.82
I Food, beverages and tobacco ...	31.82	24.87	18.40	20.05	24.87	29.68	24.04	24.87	25.70	29.75
II Clothing and footwear	9.27	9.83	9.78	8.95	9.83	10.71	9.82	9.83	9.84	9.30
III Housing, fuel and furniture .	25.68	24.51	23.12	26.30	24.51	22.73	26.38	24.51	22.65	25.81
IV Travel and recreation	25.35	32.00	39.12	35.26	32.00	28.75	31.69	32.00	32.31	27.12
V Other goods and services	7.64	8.73	9.84	9.45	8.73	8.01	8.35	8.73	9.11	8.03

Table B.12. Estimates of the individual part of the disturbance variance (ρ) based on single equation estimates of different specifications of the model.a)
A priori restriction: $\beta=0$

Commodity group i	M	H_{An}	H^*_{An}	H_A	H_n	H_*
1. Flour and bread	0.491	0.462	0.458	0.433	0.418	0.383
2. Meat and eggs	0.229	0.225	0.224	0.217	0.225	0.214
3. Fish	0.455	0.453	0.454	0.450	0.439	0.437
4. Canned meat and fish	0.203	0.200	0.200	0.196	0.197	0.194
5. Dairy products	0.485	0.473	0.482	0.393	0.451	0.369
6. Butter and margarine	0.294	0.262	0.261	0.238	0.263	0.235
7. Potatoes and vegetables	0.323	0.334	0.332	0.333	0.343	0.336
8. Other food	0.323	0.284	0.283	0.264	0.267	0.251
9. Beverages	0.321	0.312	0.305	0.299	0.311	0.297
10. Tobacco	0.704	0.720	0.720	0.710	0.718	0.707
11. Clothing	0.297	0.287	0.288	0.293	0.285	0.293
12. Footwear	-0.002	-0.009	-0.008	-0.019	-0.018	-0.026
13. Housing	0.360	0.362	0.361	0.359	0.354	0.349
14. Fuel and power	0.313	0.108	0.122	0.103	0.100	0.075
15. Furniture	0.015	0.003	0.002	-0.009	-0.007	-0.024
16. Household equipment	0.047	0.043	0.043	0.046	0.039	0.046
17. Misc. household goods	0.276	0.269	0.274	0.267	0.255	0.255
18. Medical care	-0.002	0.007	0.010	0.015	-0.001	-0.003
19. Motorcars, bicycles	-0.105	-0.081	-0.083	-0.073	-0.070	-0.069
20. Running costs of vehicles	0.359	0.311	0.310	0.307	0.296	0.293
21. Public transport	0.060	0.058	0.055	0.049	0.050	0.042
22. PTT charges	0.176	0.175	0.176	0.170	0.167	0.163
23. Recreation	0.104	0.071	0.071	0.065	0.063	0.058
24. Public entertainment	0.436	0.424	0.427	0.419	0.426	0.422
25. Books and newspapers	0.422	0.420	0.423	0.413	0.415	0.410
26. Personal care	0.263	0.258	0.261	0.250	0.258	0.251
27. Misc. goods and services	0.055	0.035	0.036	0.031	0.024	0.023
28. Restaurants, hotels, etc.	0.324	0.314	0.313	0.291	0.315	0.291

a) The table head reads:

M : Marginal ρ, calculated directly from the observed budget shares

H_{An} : No background variables and $\beta=0$

H^*_{An} : No background variables and $\beta=1$

H_A : One background variable (number of persons in the household, n) and $\beta=0$

H_n : One background variable (age of household's head, A) and $\beta=0$

H_* : Two background variables (n and A) and $\beta=0$

Table B.12 (cont.). Estimates of the individual part of the disturbance variance (ρ) based on single equation estimates of different specifications of the model.[a)]
A priori restriction: $\beta=0$

Commodity group i	M	H_{An}	H^*_{An}	H_A	H_n	H_*
I Food, beverages and tobacco .	0.400	0.409	0.412	0.403	0.406	0.398
II Clothing and footwear	0.256	0.248	0.248	0.250	0.243	0.249
III Housing, fuel and furniture .	0.202	0.181	0.182	0.181	0.157	0.145
IV Travel and recreation	0.281	0.292	0.292	0.300	0.292	0.289
V Other goods and services	0.220	0.211	0.207	0.208	0.207	0.207

a) See note a, page 206.

Table B.13. Estimates of the total disturbance variance (σ_ε^2) for model variants with different specifications of background variables.
A priori restriction: $\beta=0$

Commodity group i	Case a) H_{An}	H_A	H_n	H_*
1. Flour and bread	$4.440\cdot10^{-4}$	$3.982\cdot10^{-4}$	$4.140\cdot10^{-4}$	$3.676\cdot10^{-4}$
2. Meat and eggs	$3.870\cdot10^{-3}$	$3.813\cdot10^{-3}$	$3.869\cdot10^{-3}$	$3.808\cdot10^{-3}$
3. Fish	$4.888\cdot10^{-4}$	$4.852\cdot10^{-4}$	$4.775\cdot10^{-4}$	$4.728\cdot10^{-4}$
4. Canned meat and fish	$7.898\cdot10^{-5}$	$7.769\cdot10^{-5}$	$7.822\cdot10^{-5}$	$7.689\cdot10^{-5}$
5. Dairy products	$5.892\cdot10^{-4}$	$4.664\cdot10^{-4}$	$5.564\cdot10^{-4}$	$4.417\cdot10^{-4}$
6. Butter and margarine	$9.921\cdot10^{-5}$	$9.447\cdot10^{-5}$	$9.865\cdot10^{-5}$	$9.383\cdot10^{-5}$
7. Potatoes and vegetables	$1.570\cdot10^{-3}$	$1.504\cdot10^{-3}$	$1.553\cdot10^{-3}$	$1.486\cdot10^{-3}$
8. Other food	$9.422\cdot10^{-4}$	$8.856\cdot10^{-4}$	$9.239\cdot10^{-4}$	$8.664\cdot10^{-4}$
9. Beverages	$1.203\cdot10^{-3}$	$1.193\cdot10^{-3}$	$1.200\cdot10^{-3}$	$1.186\cdot10^{-3}$
10. Tobacco	$7.043\cdot10^{-4}$	$6.849\cdot10^{-4}$	$6.686\cdot10^{-4}$	$6.431\cdot10^{-4}$
11. Clothing	$6.265\cdot10^{-3}$	$6.243\cdot10^{-3}$	$6.256\cdot10^{-3}$	$6.237\cdot10^{-3}$
12. Footwear	$1.574\cdot10^{-3}$	$1.558\cdot10^{-3}$	$1.569\cdot10^{-3}$	$1.555\cdot10^{-3}$
13. Housing	$1.025\cdot10^{-2}$	$1.020\cdot10^{-2}$	$1.017\cdot10^{-2}$	$1.009\cdot10^{-2}$
14. Fuel and power	$9.612\cdot10^{-4}$	$9.497\cdot10^{-4}$	$9.599\cdot10^{-4}$	$9.415\cdot10^{-4}$
15. Furniture	$5.351\cdot10^{-3}$	$5.261\cdot10^{-3}$	$5.330\cdot10^{-3}$	$5.220\cdot10^{-3}$
16. Household equipment	$3.143\cdot10^{-3}$	$3.143\cdot10^{-3}$	$3.129\cdot10^{-3}$	$3.128\cdot10^{-3}$
17. Misc. household goods	$1.304\cdot10^{-3}$	$1.302\cdot10^{-3}$	$1.284\cdot10^{-3}$	$1.281\cdot10^{-3}$
18. Medical care	$2.204\cdot10^{-3}$	$2.200\cdot10^{-3}$	$2.176\cdot10^{-3}$	$2.175\cdot10^{-3}$
19. Motorcars, bicycles	$1.433\cdot10^{-2}$	$1.418\cdot10^{-2}$	$1.426\cdot10^{-2}$	$1.416\cdot10^{-2}$
20. Running costs of vehicles	$8.489\cdot10^{-3}$	$8.441\cdot10^{-3}$	$8.284\cdot10^{-3}$	$8.247\cdot10^{-3}$
21. Public transport ...	$2.923\cdot10^{-3}$	$2.877\cdot10^{-3}$	$2.874\cdot10^{-3}$	$2.847\cdot10^{-3}$
22. PTT charges	$3.130\cdot10^{-3}$	$3.087\cdot10^{-3}$	$3.091\cdot10^{-3}$	$3.063\cdot10^{-3}$
23. Recreation	$7.088\cdot10^{-3}$	$7.062\cdot10^{-3}$	$7.037\cdot10^{-3}$	$6.983\cdot10^{-3}$
24. Public entertainment	$2.882\cdot10^{-3}$	$2.862\cdot10^{-3}$	$2.869\cdot10^{-3}$	$2.843\cdot10^{-3}$
25. Books and newspapers	$1.986\cdot10^{-3}$	$1.973\cdot10^{-3}$	$1.971\cdot10^{-3}$	$1.963\cdot10^{-3}$

a) The table head reads:

H_{An} : No background variables

H_A : One background variable (number of persons in the household, n)

H_n : One background variable (age of household's head, A)

H_* : Two background variables (n and A)

Table B.13 (cont.). Estimates of the total disturbance variance (σ_ε^2) for model variants with different specifications of background variables.[a)]
A priori restriction: $\beta=0$

Commodity group i	Case[a)] H_{An}	H_A	H_n	H_*
26. Personal care	$5.301\cdot10^{-4}$	$5.197\cdot10^{-4}$	$5.273\cdot10^{-4}$	$5.187\cdot10^{-4}$
27. Misc. goods and services	$2.301\cdot10^{-3}$	$2.289\cdot10^{-3}$	$2.283\cdot10^{-3}$	$2.278\cdot10^{-3}$
28. Restaurants, hotels, etc.	$2.968\cdot10^{-3}$	$2.892\cdot10^{-3}$	$2.952\cdot10^{-3}$	$2.858\cdot10^{-3}$
I Food, beverages and tobacco	$1.567\cdot10^{-2}$	$1.401\cdot10^{-2}$	$1.548\cdot10^{-2}$	$1.389\cdot10^{-2}$
II Clothing and footwear	$8.680\cdot10^{-3}$	$8.603\cdot10^{-3}$	$8.659\cdot10^{-3}$	$8.593\cdot10^{-3}$
III Housing, fuel and furniture	$1.936\cdot10^{-2}$	$1.921\cdot10^{-2}$	$1.913\cdot10^{-2}$	$1.881\cdot10^{-2}$
IV Travel and recreation	$2.802\cdot10^{-2}$	$2.750\cdot10^{-2}$	$2.798\cdot10^{-2}$	$2.734\cdot10^{-2}$
V Other goods and services	$7.581\cdot10^{-3}$	$7.496\cdot10^{-3}$	$7.553\cdot10^{-3}$	$7.491\cdot10^{-3}$

a) See note a, page 208.

Table B.14. Likelihood Ratio tests of significance of background variables. Test statistics and final result. Overall level of significance: approximately 4 per cent[a].
A priori restriction: $\beta=0$

Commodity group i	Test statistics for subtests[b)]				Final result[b)]
	H_* versus H_A	H_* versus H_n	H_A versus H_{An}	H_n versus H_{An}	
1. Flour and bread	66.85	99.38	*	*	H_*
2. Meat and eggs .	1.10	13.29	12.40	*	H_A
3. Fish	21.64	8.27	*	19.55	H_n
4. Canned meat and fish	8.65	14.33	13.77	*	H_A
5. Dairy products	45.49	193.0	*	*	H_*
6. Butter and margarine	5.68	41.88	40.93	*	H_A
7. Potatoes and vegetables	10.07	36.87	35.90	*	H_A
8. Other food	18.32	53.72	*	*	H_*
9. Beverages	4.92	9.81	6.98	2.09	H_{An}
10. Tobacco	52.65	32.51	*	*	H_*
11. Clothing	0.80	2.54	2.94	1.20	H_{An}
12. Footwear	1.61	7.49	8.54	2.66	H_{An}
13. Housing	9.06	6.60	4.09	6.55	H_{An}
14. Fuel and power	7.25	16.18	10.06	*	H_{An}
15. Furniture	6.54	17.43	14.18	*	H_A
16. Household equipment	4.00	0.27	0.08	3.73	H_{An}
17. Misc. household goods	13.59	1.96	*	12.92	H_n
18. Medical care ..	9.55	0.38	1.52	10.69	H_{An}
19. Motorcars, bicycles	1.18	5.88	8.80	4.09	H_{An}
20. Running costs of vehicles ...	19.44	3.74	*	20.44	H_n
21. Public transport	8.76	7.89	13.26	14.13	H_A or H_n
22. PTT charges ...	6.52	7.61	11.56	10.48	H_{An}

*) Not computed. Statistic unnecessary for performing the test.
a) I.e., 1 per cent level for each subtest. The critical value of the χ^2 distribution with 3 degrees of freedom is 11.34.
b) The following symbols are used:
H_{An} : No background variables.
H_A : One background variable: number of household members, n.
H_n : One background variable: age of household's head, A.
H_* : Two background variables: n and A.

Table B.14 (cont.). Likelihood Ratio tests of significance of background variables. Test statistics and final result. Overall level of significance: approximately 4 per cent[a]. A priori restriction: $\beta=0$

Commodity group i	Test statistics for subtests[b]				Final result[b]
	H_* versus H_A	H_* versus H_n	H_A versus H_{An}	H_n versus H_{An}	
23. Recreation	9.40	6.43	3.07	6.04	H_{An}
24. Public entertainment	5.57	7.61	5.82	3.78	H_{An}
25. Books and newspapers	4.25	3.40	5.49	6.34	H_{An}
26. Personal care .	1.61	13.75	16.56	*	H_A
27. Misc. goods and services	4.03	1.83	4.37	6.57	H_{An}
28. Restaurants, hotels, etc. ..	9.89	27.06	21.69	*	H_A
I Food, beverages and tobacco	7.19	90.61	93.61	*	H_A
II Clothing and footwear	0.97	6.40	7.45	2.03	H_{An}
III Housing, fuel and furniture .	17.59	14.10	*	*	H_*
IV Travel and recreation	4.88	19.34	15.66	*	H_A
V Other goods and services	0.56	6.89	9.42	3.09	H_{An}

a) See note a, page 210.
b) See note b, page 210.

Table B.15. Likelihood Ratio tests of significance of the individual part of the disturbance variance, ρ, based on preferred specification of background variables.[a)]
A priori restriction: $\beta=0$

Commodity group i	Test statistic[b)c)]
1. Flour and bread	129.8
2. Meat and eggs	40.06
3. Fish	177.9
4. Canned meat and fish	32.63
5. Dairy products	121.4
6. Butter and margarine	48.82
7. Potatoes and vegetables	97.27
8. Other food	53.72
9. Beverages	85.12
10. Tobacco	567.3
11. Clothing	71.91
12. Footwear	0.07
13. Housing	117.0
14. Fuel and power	9.61
15. Furniture	0.06
16. Household equipment	1.53
17. Misc. household goods	56.05
18. Medical care	0.04
19. Motorcars, bicycles	5.38
20. Running costs of vehicles	76.73
21. Public transport[d)]	2.01/2.05
22. PTT charges	25.84
23. Recreation	4.16
24. Public entertainment	164.7
25. Books and newspapers	161.9
26. Personal care	53.34
27. Misc. goods and services	1.02
28. Restaurants, hotels, etc.	73.61

a) See table B.14.
b) The test statistic is $2M \log\{\tilde{\sigma}^2/(\hat{\sigma}^2(1-\hat{\rho}^2))\}$, where $\tilde{\sigma}^2$ and $\hat{\sigma}^2$ are the estimates of the total disturbance variance when $\rho=0$ and $\rho\neq0$, respectively, and $\hat{\rho}$ is the estimate of ρ.
c) The critical χ^2 (1) value is 3.84 at the 5 per cent level and 6.64 at the 1 per cent level.
d) The Likelihood Ratio test of significance gave ambiguous result for this commodity group.

Table B.15 (cont.). Likelihood Ratio tests of significance of the individual part of the disturbance variance, ρ, based on preferred specification of background variables.[a]
A priori restriction: $\beta=0$

Commodity group i		Test statistic[b)c)]
I	Food, beverages and tobacco	146.3
II	Clothing and footwear	52.91
III	Housing, fuel and furniture	17.24
IV	Travel and recreation	76.74
V	Other goods and services	37.65

a) See note a, page 212.
b) See note b, page 212.
c) See note c, page 212.

Table B.16. Marginal skewness and kurtosis of the budget shares as compared with skewness and kurtosis of the residuals for the model with the preferred specification of background variables.
A priori restriction: $\beta=0$

Commodity group i	Marginal skewness	Marginal kurtosis	Preferred variant[a)]	Residual skewness	Residual kurtosis
1. Flour and bread ..	2.17	9.75	H_*	1.99	12.23
2. Meat and eggs	3.52	20.35	H_A	2.65	11.23
3. Fish	1.66	4.97	H_n	3.37	18.46
4. Canned meat and fish	1.13	1.36	H_A	2.55	10.39
5. Dairy products ...	1.52	3.70	H_*	1.03	6.23
6. Butter and margarine	1.27	2.15	H_A	2.96	20.18
7. Potatoes and vegetables	3.47	24.45	H_A	2.45	12.70
8. Other food	2.10	8.10	H_*	2.07	9.43
9. Beverages	2.37	10.51	H_{An}	2.77	10.31
10. Tobacco	1.32	3.31	H_*	2.39	11.93
11. Clothing	3.53	19.99	H_{An}	2.17	7.54
12. Footwear	2.18	7.65	H_{An}	4.19	27.97
13. Housing	3.38	17.41	H_{An}	2.24	7.88
14. Fuel and power ...	4.19	33.09	H_{An}	1.61	18.90
15. Furniture	4.33	27.29	H_A	2.90	12.25
16. Household equipment	5.84	51.68	H_{An}	4.39	28.53
17. Misc. household goods	3.75	23.89	H_n	4.28	25.23
18. Medical care	4.80	37.89	H_{An}	6.34	56.28
19. Motorcars, bicycles	4.34	20.30	H_{An}	2.66	7.65
20. Running costs of vehicles	3.58	19.08	H_n	2.31	6.73
21. Public transport[b)]	9.26	142.68	H_A or	6.31	64.08
			H_n	6.22	61.97
22. PTT charges	5.15	38.68	H_{An}	7.41	76.01

a) The symbols are defined as follows:
H_{An} : No background variables
H_A : One background variable (number of persons in the household, n)
H_n : One background variable (age of household's head, A)
H_* : Two background variables (n and A)

b) The Likelihood Ratio test of significance of background variables gave ambiguous result for this commodity.

Table B.16 (cont.). Marginal skewness and kurtosis of the budget shares as compared with skewness and kurtosis of the residuals for the model with the preferred specification of background variables.
A priori restriction: $\beta=0$

Commodity group i	Marginal skewness	Marginal kurtosis	Preferred variant[a)]	Residual skewness	Residual kurtosis
23. Recreation	3.77	19.39	H_{An}	2.56	8.29
24. Public entertainment	4.29	28.21	H_{An}	3.02	12.68
25. Books and newspapers	6.23	62.31	H_{An}	5.59	43.28
26. Personal care	1.91	7.20	H_{A}	2.36	8.35
27. Misc. goods and services	8.42	98.51	H_{An}	7.90	83.88
28. Restaurants, hotels, etc.	7.17	82.38	H_{A}	5.06	40.70
I Food, beverages and tobacco	1.07	0.44	H_{A}	0.64	0.86
II Clothing and footwear	2.88	11.88	H_{An}	1.86	5.01
III Housing, fuel and furniture	1.43	1.94	H_{*}	1.16	1.80
IV Travel and recreation	1.53	1.66	H_{A}	0.54	-0.12
V Other goods and services	4.15	24.10	H_{An}	2.89	12.41

a) See note a, page 214.
b) See note b, page 214.

Appendix A

DEMAND ELASTICITIES IN THE FOURGEAUD-NATAF MODEL

The demand function of commodity i, expressed in budget share form, is (cf.eq. (2.20))

$$(A.1)\qquad w_i = \frac{p_i x_i}{y} = \frac{v_i x_i}{u} = f_i(u,v_i) = s_i + (t_i v_i^{\beta} - s_i)\frac{C(u)}{u} \qquad (i=1, \ldots, N),$$

where the symbols are defined as in section 2.2. The purpose of this appendix is to derive and comment upon the corresponding expressions for the income and price elasticities.

The partial elasticities of w_i with respect to the total real expenditure (real income) u and the real price v_i are, respectively,

$$(A.2)\qquad E_{iu} = \frac{\partial f_i}{\partial u}\frac{u}{w_i} = \frac{(t_i v_i^{\beta}-s_i)(\check{C}(u)-1)C(u)}{s_i u+(t_i v_i^{\beta}-s_i)C(u)},$$

$$(A.3)\qquad E_{iv} = \frac{\partial f_i}{\partial v_i}\frac{v_i}{w_i} = \frac{\beta t_i v_i^{\beta} C(u)}{s_i u+(t_i v_i^{\beta}-s_i)C(u)},$$

where $\check{C}(u) = (\partial C(u)/\partial u)(u/C(u))$. From the inequality contraints (2.16), (2.17), and (2.19) it follows that the common denominator in (A.2) and (A.3) is always positive. We then have:

(i) $E_{iu} \equiv 0$, i.e. the budget shares are income independent, if either

$C(u) \equiv 0$,

or $\check{C}(u) \equiv 1 \Rightarrow C(u) = Au$ (A constant),

or $t_i = s_i$ and $\beta = 0$.

(ii) If $\check{C}(u) > 1$, then $E_{iu} \gtreqless 0$ according as $t_i v_i^{\beta} \gtreqless s_i$.

(iii) If $\check{C}(u) < 1$, then $E_{iu} \gtreqless 0$ according as $t_i v_i^{\beta} \lesseqgtr s_i$.

(iv) If t_i is strictly positive, the sign of E_{iv} will always be the same as the sign of β: An increase in the real price of commodity i will increase its budget share if $\beta > 0$, leave it unaffected if $\beta = 0$, and decrease it if $\beta < 0$.

The *Engel (expenditure) elasticity* of commodity i can be written as

$$(A.4) \qquad E_i = \frac{\partial x_i}{\partial y}\frac{y}{x_i} = \frac{\partial w_i}{\partial y}\frac{y}{w_i} + 1 = E_{iu} + 1 = \frac{s_i u + (t_i v_i^{\beta} - s_i)\check{C}(u)C(u)}{s_i u + (t_i v_i^{\beta} - s_i)C(u)} \,.$$

To find expressions for the price elasticities, we first notice that

$$(A.5) \qquad \frac{\partial w_i}{\partial p_j}\frac{p_j}{w_i} = \begin{cases} E_{iv} - (E_{iv} + E_{iu})\pi_i & \text{for } j=i \\ -(E_{iv} + E_{iu})\pi_j & \text{for } j\neq i, \end{cases}$$

where (cf.eq. (2.18))

$$(A.6) \qquad \pi_j = \frac{\partial P}{\partial p_j}\frac{p_j}{P} = t_j v_j^{\beta}.$$

The *Cournot elasticity* of commodity i with respect to the price of commodity j is definitionally equal to

$$e_{ij} = \frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i} = \begin{cases} \dfrac{\partial w_i}{\partial p_i}\dfrac{p_i}{w_i} - 1 & \text{for } j=i. \\[2ex] \dfrac{\partial w_i}{\partial p_j}\dfrac{p_j}{w_i} & \text{for } j\neq i. \end{cases}$$

Inserting from (A.5) and rearranging terms, we get

$$(A.7)\quad e_{ij} = \frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i} = \delta_{ij}(E_{iv}-1) - (E_{iv}+E_{iu})\pi_j$$

$$= (E_{iv}-1)(\delta_{ij}-\pi_j) - (E_{iu}+1)\pi_j \qquad (i=1, \ldots, N; j=1, \ldots, N).$$

where $\delta_{ij}=1$ for $j=i$, and 0 otherwise.

From (A.4) and (A.7) it now follows that the *Slutsky elasticity* - i.e. the income compensated price elasticity - of commodity i with respect to the price of commodity j can be written as

$$(A.8)\quad S_{ij} = e_{ij}+w_jE_i = (E_{iv}-1)(\delta_{ij}-\pi_j)+(E_{iu}+1)(w_j-\pi_j) \qquad (i=1, \ldots, N; j=1, \ldots, N).$$

It is convenient to define

$$(A.9)\quad S^*_{ii} = E_{iv} - 1 = \frac{(\beta-1)t_iv_i^\beta C(u) - s_i(u-C(u))}{s_iu + (t_iv_i^\beta - s_i)C(u)} \qquad (i=1, \ldots, N).$$

From (A.8) and (A.9) we see that $S_{ii} \approx S^*_{ii}$ if w_j and π_j are small, which they will normally be when a detailed commodity classification is used. Then S^*_{ii} serves as a first order approximation to the direct Slutsky elasticity (as well as the direct Cournot elasticity) of commodity i. Utilizing (A.4) and (A.7)-(A.9), we can now express all price elasticities in terms of E_i, S^*_{ii}, w_j, and π_j:

$$(A.10)\quad e_{ij} = S^*_{ii}(\delta_{ij}-\pi_j) - E_i\pi_j,$$

$$(A.11)\quad S_{ij} = S^*_{ii}(\delta_{ij}-\pi_j) + E_i(w_j-\pi_j) \qquad (i=1, \ldots, N; j=1, \ldots, N).$$

With the particular parametrization of the C function used in the present study,

$$(A.12)\quad C(u) = a \log u + b,$$

which implies

$$\text{(A.13)} \quad \check{C}(u) = \frac{a}{a \log u + b},$$

the expressions for E_i and S^*_{ii} take the form

$$\text{(A.14)} \quad E_i = \frac{s_i u + (t_i v_i^{\beta} - s_i) a}{s_i u + (t_i v_i^{\beta} - s_i)(a \log u + b)},$$

$$\text{(A.15)} \quad S^*_{ii} = \frac{(\beta-1) t_i v_i^{\beta} (a \log u + b) - s_i(u - a \log u - b)}{s_i u + (t_i v_i^{\beta} - s_i)(a \log u + b)} \qquad (i=1, \ldots, N).$$

Inserting these formulae, together with (A.1) and (A.6), in (A.10) and (A.11), we get the explicit expressions for the price elasticities. As they are fairly complex, we refrain from writing them out.

Let us then turn to the income flexibility (or more precisely, the flexibility of the marginal utility of income) which is another interesting demand elasticity in the Fourgeaud-Nataf model. In section 2.2, we showed, by differentiating an indirect utility indicator consistent with the model, (2.11)[1], that the marginal utility of (nominal) income can be written as

$$\text{(A.16)} \quad \frac{\partial V}{\partial y} = V_y = \frac{1}{P\{u - C(u)\}}$$

for an arbitrary choice of the C function (cf.eq. (2.14)). The corresponding expression for the *income flexibility* is

$$\text{(A.17)} \quad \omega = \frac{\partial V_y}{\partial y}\frac{y}{V_y} = -\frac{u\{1 - C'(u)\}}{u - C(u)} = -\frac{u - \check{C}(u)\,C(u)}{u - C(u)}.$$

1) Recall that the marginal utility of income, and concepts derived from it, are constructs whose interpretation is confined to a particular choice of utility indicator, in this case (2.11). They are not ordinal concepts, in the sense that they are not invariant with respect to arbitrary monotonic tranformations of the utility indicator used to represent the consumer's preferences. The income flexibility, however, is invariant with respect to *linear* transformations.

Obviously, a necessary and sufficient condition for ω to be negative is that $C'(u) < 1$, i.e. that C increases at a slower pace than u. The income flexibility is identically equal to -1 in two particular cases: (i) $\check{C}(u) \equiv 1$, i.e. C and u are proportional, and (ii) $C \equiv 0$.[2] In general, the departure of ω from -1 is determined by the elasticity of the C function: $|\omega| \gtreqless 1$ according as $\check{C} \lesseqgtr 1$.

Inserting (A.12) and (A.13) in (A.17), we find that the income flexibility function takes the following specific form:

$$\text{(A.18)} \quad \omega = - \frac{u-a}{u-a \log u-b} .$$

Finally, let us refer briefly some *asymptotic properties* of the demand elasticities. Since (A.12) and (A.13) imply $\lim_{u\to\infty} C(u)/u = 0$ and $\lim_{u\to\infty} \check{C}(u) = 0$, it follows from (A.1)-(A.3) that

$$\lim_{u\to\infty} w_i = s_i ,$$

$$\lim_{u\to\infty} E_{iu} = \lim_{u\to\infty} E_{iv} = 0 \qquad (i=1, \ldots, N).$$

Combining these with (A.4), (A.9)-(A.11), we obtain

$$\lim_{u\to\infty} E_i = 1 ,$$

$$\lim_{u\to\infty} S^{*}_{ii} = -1 ,$$

$$\lim_{u\to\infty} e_{ii} = -1 ,$$

$$\lim_{u\to\infty} e_{ij} = 0 \qquad \text{for } j \neq i ,$$

$$\lim_{u\to\infty} S_{ii} = s_i - 1 ,$$

$$\lim_{u\to\infty} S_{ij} = s_j \qquad \text{for } j \neq i \qquad (i,j=1, \ldots, N).$$

---

2) In these cases, the model also implies unitary Engel elasticities for all commodities, cf. eq.(A.4).

Thus, the asymptotic properties of the price elasticities depend on the s coefficients only. The asymptotic value of the income flexibility is, not unexpectedly,

$$\lim_{u\to\infty} \omega = -1.$$

Appendix B

GENERAL RESTRICTIONS ON THE DISTURBANCES OF A COMPLETE SYSTEM OF CONSUMER DEMAND FUNCTIONS. WHY DO WE TRANSFORM OUR DEMAND FUNCTIONS TO BUDGET SHARES?[1)]

1. In this appendix, we elaborate our arguments, briefly mentioned in section 3.2, for working with demand functions expressed in terms of budget shares. We shall, however, first state some general restrictions which the probability distribution of the disturbances must satisfy to ensure that the equation system is consistent with the adding-up and the homogeneity constraints of the static theory of consumer demand.

Consider the following general system of demand functions:

$$\text{(B.1)} \qquad x_{iht} = f_i(y_{ht}, p_{1t}, \ldots, p_{Nt}) + \bar{\varepsilon}_{iht},$$

where x_{iht} and p_{it} denote the quantity demanded by household h and the price respectively of commodity i in period t (i=1, ..., N; h=1, ..., H; t=1, ..., T), y_{ht} is total consumption expenditure of household h in period t, definitionally equal to the sum of the values of the N commodities:

$$\text{(B.2)} \qquad y_{ht} = \sum_{i=1}^{N} p_{it}x_{iht} \qquad\qquad h=1, \ldots, H, \quad t=1, \ldots, T,$$

and $\bar{\varepsilon}_{iht}$ is a stochastic disturbance.[2)] The unspecified functions $f_1, \ldots, f_N$, representing the deterministic part of the model, are assumed to be compatible with utility maximizing behaviour. This implies, *inter alia*, that the 'adding-up constraint'

$$\text{(B.3)} \qquad \sum_{i=1}^{N} p_{it}f_i(y_{ht},p_{1t},\ldots,p_{Nt}) = y_{ht}$$

and the 'homogeneity conditions'

---

1) This appendix is a revised version of Biørn (1977, pp. 1-8). The symbols used deviate in some respects from the general notation in this report.
2) The interpretation of the subscripts h and t used here deviates slightly from their interpretation in chapters 3 and 4 of the main text. This should not bring confusion.

$$(B.4)\quad f_i(y_{ht},p_{1t},\ldots,p_{Nt}) = f_i(ky_{ht},kp_{1t},\ldots,kp_{Nt}) \qquad i=1,\ldots,N,$$

hold identically in $y_{ht},p_{1t},\ldots,p_{Nt}$, and k.

2. Equations (B.1)-(B.3) involve the following restrictions on the disturbances:

$$(B.5)\quad \sum_i p_{it}\bar{\varepsilon}_{iht} = 0 \qquad h=1,\ldots,H,\ t=1,\ldots,T.$$

In the sequel, we shall consider the p's and the y's as non-stochastic variables and assume that all disturbances have zero expectations:

$$(B.6)\quad E(\bar{\varepsilon}_{iht}) = 0 \qquad i=1,\ldots,N,\ h=1,\ldots,H,\ t=1,\ldots,T,$$

and that all disturbances relating to different households are uncorrelated:

$$(B.7)\quad E(\bar{\varepsilon}_{iht}\bar{\varepsilon}_{jks}) = 0 \qquad i,j=1,\ldots,N,\ h,k=1,\ldots,H;k\neq h,\ t,s=1,\ldots,T.$$

A consequence of our disturbance component specification (cf. section 3.2 above) is, however, that we allow for correlation between all disturbances which relate to the same household, irrespective of the time period.

Multiplying (B.5) by $\bar{\varepsilon}_{jhs}$ and taking expectations, we obtain the following set of restrictions on the second order moments of the disturbances:

$$(B.8)\quad \sum_i p_{it}E(\bar{\varepsilon}_{iht}\bar{\varepsilon}_{jhs}) = 0 \qquad j=1,\ldots,N,\ h=1,\ldots,H,\ t,s=1,\ldots,T.$$

From this we deduce

Proposition 1. The variances/covariances of the disturbances of the demand functions expressed in quantity form, (B.1), cannot be independent of the period number (i.e. we cannot impose

$E(\bar{\varepsilon}_{iht}\bar{\varepsilon}_{jht}) = \tau_{ij}$ and $E(\bar{\varepsilon}_{iht}\bar{\varepsilon}_{jhs}) = \tau^{\mu}_{ij}$ for all h,t, and s≠t) unless all prices change proportionally.[3)]

3. Multiplying through the demand system (B.1) by p_{it}, we obtain the corresponding system of 'expenditure functions'. Their disturbances,

$$\text{(B.9)} \quad \overset{*}{\varepsilon}_{iht} = p_{it}\bar{\varepsilon}_{iht}, \qquad \begin{array}{l} i=1,\dots,N, \\ h=1,\dots,H, \\ t=1,\dots,T, \end{array}$$

will, in view of (B.5), be subject to the restrictions

$$\text{(B.10)} \quad \sum_i \overset{*}{\varepsilon}_{iht} = 0 \qquad \begin{array}{l} h=1,\dots,H, \\ t=1,\dots,T. \end{array}$$

From this we conclude that the following specification of the second order moments is admissible:

$$\text{(B.11)} \quad E(\overset{*}{\varepsilon}_{iht}\overset{*}{\varepsilon}_{jks}) = \begin{cases} \tau_{ij} & \text{for } k=h, s=t \\ \tau^{\mu}_{ij} & \text{for } k=h, s\neq t \\ 0 & \text{otherwise}, \quad i,j=1,\dots,N. \end{cases}$$

provided that

$$\text{(B.12)} \quad \sum_i \tau_{ij} = \sum_i \tau^{\mu}_{ij} = 0, \qquad j=1,\dots,N.$$

The corresponding second order moments of the quantity form of the model read

$$\text{(B.13)} \quad E(\bar{\varepsilon}_{iht}\bar{\varepsilon}_{jks}) = \begin{cases} \dfrac{\tau_{ij}}{p_{it}p_{jt}} & \text{for } k=h, s=t \\ \dfrac{\tau^{\mu}_{ij}}{p_{it}p_{js}} & \text{for } k=h, s\neq t \\ 0 & \text{otherwise}, \quad i,j=1,\dots,N. \end{cases}$$

---

3) The specification $E(\bar{\varepsilon}_{iht}\bar{\varepsilon}_{jhs}) = \tau_{ijts}$, subject to the constraints $\Sigma_i p_{it}\tau_{ijts} = 0$ for all j,h,t, and s, is, however, admissible. I.e., the second order moments of $\bar{\varepsilon}_{iht}$ may be allowed to be independent of the household number.

Thus, we can state

Proposition 2. *(i) The variances/covariances of the disturbances of the demand functions expressed in expenditure (value) form can be specified to take the same values for all households in all the periods of observation.*

(ii) The model's adding-up constraint implies that the covariance matrices (τ_{ij}) *and* (τ^{μ}_{ij}) *are both singular; i.e. an assumption that all disturbances in different equations are uncorrelated would not be admissible.*[4)]

4. The homogeneity conditions (B.4) restrict the admissible set of demand functions. It seems reasonable to place similar restrictions on the random components of demand, i.e. to require that a proportional change of all prices and total expenditure leave the distribution of the disturbances of the demand functions expressed in quantity form - or at least its second order moments - unaffected. Would this requirement be compatible with the expenditure version of the model, (B.11)-(B.12)? Obviously, the answer is no. If all the τ's are constants, multiplication of all prices and total expenditure by a factor k would reduce the disturbance variances/covariances of the quantity version of the model, (B.1), by a factor of $1/k^2$, as is easily seen from (B.13). The specification (B.11)-(B.12) thus imposes a rather implausible kind of heteroscedasiticity on the model.

5. This motivates considering the following, more general, problem: Assume that we multiply through eq. (B.1) by a non-stochastic weight b_{iht}, its value depending on the commodity group as well as on the household and the period of observation. Which restrictions should be imposed on these weights to ensure that both the adding-up conditions and the homogeneity conditions are satisfied?

The transformed disturbances are

$$\text{(B.14)} \quad \varepsilon_{iht} = b_{iht}\bar{\varepsilon}_{iht} \qquad i=1,\ldots,N,\; h=1,\ldots,H,\; t=1,\ldots,T,$$

with second order moments equal to

4) Confer also Pollak and Wales (1969, p.615).

$$\text{(B.15)}\quad E(\varepsilon_{iht}\varepsilon_{jks}) = \begin{cases} b_{iht}b_{jht}E(\bar{\varepsilon}_{iht}\bar{\varepsilon}_{jht}) & \text{for } k=h, s=t, \\ b_{iht}b_{jhs}E(\bar{\varepsilon}_{iht}\bar{\varepsilon}_{jhs}) & \text{for } k=h, s\neq t, \\ 0 & \text{otherwise, } i,j=1,\ldots,N. \end{cases}$$

Combining (B.15) and (B.8), we find that the adding-up constraint of the model implies the following restrictions:

$$\text{(B.16)}\quad \sum_i \frac{p_{it}}{b_{iht}} E(\varepsilon_{iht}\varepsilon_{jhs}) = 0 \qquad \begin{array}{l} j=1,\ldots,N, \\ h=1,\ldots,H, \\ t,s=1,\ldots,T. \end{array}$$

We now restrict the second order moments of the ε_{iht}'s to take the same value for all the households in all the periods, i.e.

$$\text{(B.17)}\quad E(\varepsilon_{iht}\varepsilon_{jks}) = \begin{cases} \sigma_{ij} & \text{for } k=h, s=t, \\ \sigma^{\mu}_{ij} & \text{for } k=h, s\neq t, \\ 0 & \text{otherwise, } i,j=1,\ldots,N. \end{cases}$$

For (B.16) to be satisfied identically in the p_{it}'s when the latter conditions are imposed, the weights must be of the form

$$\text{(B.18)}\quad b_{iht} = \frac{p_{it}}{c_i d_{ht}} \qquad \begin{array}{l} i=1,\ldots,N, \\ h=1,\ldots,H, \\ t=1,\ldots,T, \end{array}$$

where c_i and d_{ht} are constants, the c_i's being subject to the restrictions

$$\text{(B.19)}\quad \sum_i c_i\sigma_{ij} = \sum_i c_i\sigma^{\mu}_{ij} = 0 \qquad j=1,\ldots,N.$$

The d_{ht}'s, however, can be chosen freely.

Adopting this specification, the disturbance variances/covariances in the untransformed quantity version of the model, (B.1), can be written in the following way:

$$\text{(B.20)}\quad E(\bar{\varepsilon}_{iht}\bar{\varepsilon}_{jks}) = \begin{cases} \dfrac{c_i c_j d_{ht}^2}{p_{it}p_{jt}}\sigma_{ij} & \text{for } k=h, s=t \\[2ex] \dfrac{c_i c_j d_{ht} d_{hs}}{p_{it}p_{js}}\sigma_{ij}^{\mu} & \text{for } k=h, s\neq t \\[2ex] 0 & \text{otherwise, } i,j=1,\ldots,N. \end{cases}$$

What about the homogeneity conditions in this case? From (B.20) we see that it is perfectly possible to satisfy the requirement that all second order moments of the quantity disturbances $\bar{\varepsilon}_{iht}$ be unaffected when all prices and total expenditure change proportionally - even if σ_{ij} and σ_{ij}^{μ} are constants. We only have to let d_{ht} be a function homogeneous of the first degree in prices and total expenditure. More generally, since the d's are free parameters in our problem, they are "tools" by which we can impose the pattern of heteroscedasticity that we would like our $\bar{\varepsilon}_{iht}$'s to possess. The only claim is that their values are common to all the N commodity groups.

Summing up, we have:

Proposition 3. (i) *If we multiply through the demand functions expressed in quantity form by the weight b_{iht}, while constraining the transformed disturbances to have identical second order moments (σ_{ij}, σ_{ij}^{μ}) for all households in all periods, then the weighting system must be of the form $b_{iht} = p_{it}/(c_i d_{ht})$ for all i, h, and t, where $\Sigma_i c_i \sigma_{ij} = \Sigma_i c_i \sigma_{ij}^{\mu}$ for all j. This implies that both (σ_{ij}) and (σ_{ij}^{μ}) are singular matrices, with ranks at most equal to N-1.*

(ii) If we impose the additional constraint of linear homogeneity, i.e. that a proportional change of all prices and total expenditure leave the second order moments of the quantity disturbances $\bar{\varepsilon}_{iht}$ *unaffected, then* d_{ht} *must be homogeneous of degree one in* $p_{1t},\ldots,p_{Nt}$, *and* y_{ht}. *I.e., the weights must be of the form* $b_{iht} = p_{it}/\{c_i\lambda(p_{1t},\ldots,p_{Nt}, y_{ht})\}$ *for all i, h, and t, where* $\lambda(.)$ *is a linear homogeneous function.*

6. It is often asserted that the scope for variation in consumption habits is larger for rich households than for poorer ones. This would suggest that the disturbance variances/covariances in the quantity version of the demand functions ($E(\bar{\varepsilon}_{iht}\bar{\varepsilon}_{jhs})$) are increasing functions of the income, or expenditure level. Let us take a closer look a this hypothesis in the light of Proposition 3.

One version of the heteroscedasticity hypothesis might be that the standard deviation of $\bar{\varepsilon}_{iht}$ is proportional to total expenditure y_{ht}. Another version might be that the standard deviation of $\bar{\varepsilon}_{iht}$ is proportional to expected consumption $E(x_{iht}) = f_i(y_{ht},p_{1t},\ldots,p_{Nt})$ for all i, h, and t. This corresponds to fixing the weights b_{iht} equal to $1/y_{ht}$ and $1/E(x_{iht})$, respectively, while restricting the resulting disturbances ε_{iht} to be homoscedastic. Would this be compatible with the basic assumptions of our model? Generally, the answer is no in both cases, as is easily seen from (B.18). For b_{iht} to be equal to $1/y_{ht}$, we should have $p_{it}/c_i = d_{ht}/y_{ht}$. This equality would hold for all i, h, and t only if all prices change proportionally. On the other hand, the restriction $b_{iht} = 1/E(x_{iht})$ would imply $p_{it}f_i(\cdot) = c_i d_{ht}$. Combining this with the adding-up constraint (B.3), we find $f_i(\cdot) = (c_i/\Sigma_k c_k)y_{ht}/p_{it}$; i.e. a disturbance specification of that form would be admissible for demand functions giving constant budget shares only.

The transformation applied in the present study is $b_{iht} = p_{it}/y_{ht}$. This implies that we work with the following set of *'income normalized expenditure functions'*, or *'budget share functions'*:

$$\text{(B.21)}\quad \frac{p_{it}x_{iht}}{y_{ht}} = w_{iht} = \frac{p_{it}}{y_{ht}} f_i(y_{ht},p_{1t},\ldots,p_{Nt}) + \varepsilon_{iht},\qquad \begin{array}{l} i=1,\ldots,N,\\ h=1,\ldots,H,\\ t=1,\ldots,T, \end{array}$$

where $\varepsilon_{iht} = p_{it}\bar{\varepsilon}_{iht}/y_{ht}$ is homoscedastic. This specification satisfies both the adding-up and the homogeneity constraints, since it is the special case of eq. (B.18) where $c_i = 1$ for all i and $d_{ht} = y_{ht}$ for all h and t - the latter being obviously linear homogeneous in prices and total expenditure. Moreover, it pays regard to the idea of heteroscedasticity since the standard deviation of $\bar{\varepsilon}_{iht}$ will be proportional to y_{ht} when ε_{iht} is homoscedastic. The budget shares transformation is in fact the simplest transformation which satisfies these three restrictions simultaneously.

Appendix C

MAXIMUM LIKELIHOOD ESTIMATION IN GENERALIZED NON-LINEAR REGRESSION MODELS[1)]

C.1. Introduction

In chapter IV we propose an iterative procedure for maximizing the likelihood function with respect to the vector of structural parameters α and the unknown coefficients of the variance-covariance matrix Ω. The conditions for such a procedure to be convergent is discussed in Oberhofer and Kmenta (1974). They consider, in particular, its application to generalized *linear* regression models, and the purpose of this appendix is to modify their assumptions in order to make the results applicable for our non-linear model as well. In doing this, we shall follow the exposition in the Oberhofer-Kmenta article rather closely.

We start by quoting a fundamental lemma from Oberhofer and Kmenta (1974) (section C.2). In section C.3 we demonstrate its applicability to the generalized *non-linear* regression model. Finally, in section C.4 we discuss the relevance of the results to the estimation of the error component model in chapter IV.

C.2. The fundamental lemma[2)]

Let $f(a)$ be a function which is to be maximized with respect to a, and $a \in U$. Further, let a be partitioned as $a=(a_1,a_2)$ with $a_1 \in U_1$ and $a_2 \in U_2$, $(U=U_1 x U_2)$. The number of components in a_1 and a_2 is taken to be n and m, respectively, i.e., $U_1 \subseteq R^n$ and $U_2 \subseteq R^m$. It is assumed that $f(a)$ has the following properties:

(i) There exists a s such that the set

$$S = \{a \mid a \in U_1 x U_2,\ f(a) \geq s\} \text{ is non-empty and bounded;}$$

(ii) $f(a)$ is continuous in S; and

(iii) the parameter space U is closed, or U_2 is closed and $U_1 = R^n$.

1) The symbols used in this appendix deviate in some respects from the general notation in this report.
2) See Oberhofer and Kmenta (1974, p. 579). This lemma is also given in Sargan (1964).

From these assumptions it follows that S is compact. Now we define the following iteration procedure:

(i) Let $a_1^{(0)}$ be a vector of initial values of a_1 such that there exists an $a_2 \in U_2$ for which $f(a_1^{(0)}, a_2) \geq s$.

(ii) We maximize $f(a_1^{(0)}, a_2)$ in U_2. Because of the compactness property, the maximum will be reached at $a_2 = a_2^{(0)} \in U_2$.

(iii) We suppose inductively that we have obtained $(a_1^{(j)}, a_2^{(j-1)})$ for all $1 \leq j \leq k$.

Now we maximize $f(a_1^{(k)}, a_2)$ in U_2. The maximum will be reached at $a_2 = a_2^{(k)}$:

Then we maximize $f(a_1, a_2^{(k)})$ in U_1. The maximum will be attained at $a_1 = a_1^{(k+1)}$.

In this way, the iterative procedure is unequivocally defined.

Lemma: (i) The sequence $\{a^{(k)}\}$ *has at least one accumulation point* a^* *in S. (ii) If* a^* *and* a^+ *are two accumulation points of the sequence, then* $f(a^*) = f(a^+)$. *(iii) For every accumulation point* $a^* = (a_1^*, a_2^*)$,

$$\max_{a_1 \in U_1} f(a_1, a_2^*) = \max_{a_2 \in U_2} f(a_1^*, a_2) = f(a^*).$$

This lemma is given a proof in Oberhofer and Kmenta (1974), which is left out here.

C.3. Maximum likelihood estimation of the generalized non-linear regression model

Consider the model

$$y = F(X, \beta) + u, \tag{C.1}$$

where y is the T x 1 vector of dependent variables, X is a T x K_1 matrix of values of exogenous variables, β is a K_2 x 1 vector of structural parameters in the vector-valued function F, and u is a T x 1 disturbance vector. We make the following assumptions:

Assumption 1: u is normally distributed.

Assumption 2: $E(u) = 0$.

Assumption 3: $E(uu') = \Omega$, $|\Omega| \neq 0$.

Assumption 4: X is a fixed non-stochastic matrix.

Assumption 5: The parameters in β are independent of those in Ω.

Except for the irrelevant constant, the logaritmic likelihood function of y is then

$$(C.2) \qquad L(\beta,\omega) = \tfrac{1}{2} \log |\Omega^{-1}| - \tfrac{1}{2}(y-F(X,\beta))' \, \Omega^{-1}(y-F(X,\beta)),$$

where ω represents a vector containing all the parameters in Ω. The elements of β and ω are in general restricted *a priori*: $\beta \in V_1$ and $\omega \in V_2$. In most cases, β and ω can be expressed as functions of free parameters ξ and γ: $\beta = \beta(\xi)$ and $\omega = \omega(\gamma)$. Our problem is to determine those values of β and ω which maximize $L(\beta,\omega)$. When we maximize $L(\beta,\omega)$ with respect to ω, while considering β as given, we obtain the solution $\hat{\omega}(\beta)$. Conversely, when maximizing $L(\beta,\omega)$ with respect to β, with ω considered as given, we obtain $\hat{\beta}(\omega)$. We assume that $\hat{\omega}(\beta)$ and $\hat{\beta}(\omega)$ exist and are uniquely determined.

In order to develop an iterative procedure for finding the maximum of $L(\beta,\omega)$, whose convergence can be demonstrated by reference to the general lemma above, we need the following three definitions:

$$(C.3) \qquad U_2 = \{\omega \,|\, |\Omega| \geq \lambda > 0,\ \omega'\omega \leq M,\ \Omega \text{ non-negative definite}\} \cap V_2,$$

where λ and M are some arbitrarily chosen positive numbers.

$$(C.4) \qquad U_1 = R^{K_2} \cap V_1,$$

and

$$(C.5) \qquad S = \{\begin{pmatrix}\beta\\ \omega\end{pmatrix} \,|\, \beta \in U_1,\ \omega \in U_2,\ L(\beta,\omega) > s\},$$

where s is arbitrary.

The iterative zig-zag procedure consists in maximizing $L(\beta,\omega)$ with respect to $\beta \in U_1$ for given ω, and with respect to $\omega \in U_2$ for given β. We intend, however, to determine $\hat{\omega}(\beta)$ by maximization of $L(\beta,\omega)$ with respect to ω in V_2 instead of confining ω to the subset U_2 (cf. (C.3)), as the lemma presumes. If we denote the maximizing value of ω in U_2 by $\tilde{\omega}(\beta)$, we then have to provide a guarantee that $\tilde{\omega}(\beta) = \hat{\omega}(\beta)$ for all admissible values of s. For this purpose we introduce

<u>Assumption 6</u>: If for a given $\beta^\Delta \in U_1$ and a given $\omega^\Delta \in U_2$, we have $L(\beta^\Delta,\omega^\Delta) \geq s$,

then $\hat{\omega}(\beta^\Delta) \in U_2$.

In this case it follows that

$$L(\beta^\Delta,\tilde{\omega}(\beta^\Delta)) = \max_{\omega \in U_2} L(\beta^\Delta,\omega) = \max_{\omega \in V_2} L(\beta^\Delta,\omega) = L(\beta^\Delta,\hat{\omega}(\beta^\Delta)).$$

That means that $\hat{\omega}(\beta^\Delta) = \tilde{\omega}(\beta^\Delta)$.

Since S represents the space in which we search for the maximum of $L(\beta,\omega)$ we have to satisfy ourselves that at least one neighbourhood of the true β and ω belongs to S. But this is guaranteed, because of our assumptions about Ω, whenever we choose λ and s sufficiently small and M sufficiently large. The class of admissible matrices Ω has the additional property that there exist $\ell_1 > \ell_2 > 0$ such that all eigenvalues of every Ω with $\omega \in U_2$ lie in the interval $[\ell_2,\ell_1]$, that is,

$$\ell_1 \geq \lambda(\Omega) \geq \ell_2 > 0, \tag{C.6}$$

where $\lambda(\Omega)$ stands for any eigenvalue of Ω. The inequality (C.6) can be demonstrated as follows. The eigenvalues of Ω are always positive. The largest eigenvalue cannot be arbitrarily large because the trace of Ω (which is equal to the sum of all its eigenvalues) is restricted by the condition that $\omega'\omega \leq M$. Furthermore , the smallest eigenvalue cannot be arbitrarily small, because, by assumption, $|\Omega| \geq \lambda > 0$ and $|\Omega|$ is equal to the product of the eigenvalues of Ω. Moreover, we know that the function to be maximized, $L(\beta,\omega)$, is continuous and that U_2 is closed. The only thing that remains to be shown to invoke the general lemma for solving our problem is that S is bounded. For this purpose we introduce our last assumption:

Assumption 7:

The set

(C.7) $A = \{\beta | \beta \in V_1, \ F(X,\beta)'\Omega^{-1}F(X,\beta) \leq G\}$
is bounded for any positive constant G.

We can now demonstrate that S, as defined in (C.5), is bounded. Let us assume the converse, i.e. let there be a sequence

$$\begin{pmatrix} \beta^{(\nu)} \\ \omega^{(\nu)} \end{pmatrix}$$

in S such that

$$\text{(C.8)} \quad \lim_{\nu\to\infty} [(\beta^{(\nu)})' (\beta^{(\nu)}) + (\omega^{(\nu)})' (\omega^{(\nu)})] = \infty.$$

Since $\omega^{(\nu)} \in U_2$, which, by assumption, is closed, it follows from (C.8) that

$$\text{(C.9)} \quad \lim_{\nu\to\infty} [(\beta^{(\nu)})' (\beta^{(\nu)})] = \infty.$$

Since $\beta^{(\nu)} \in V_1$, then $\beta^{(\nu)}$ cannot belong to the bounded set A, in view of assumption 7. This implies

$$\text{(C.10)} \quad \lim_{\nu\to\infty} [F(X_1\beta^{(\nu)})' \ \Omega^{-1} \ F(X_1\beta^{(\nu)})] = \infty.$$

Therefore, since $|\Omega| \geq \lambda > 0$, (C.2) and (C.10) lead to the conclusion that

$$\text{(C.11)} \quad \lim_{\nu\to\infty} L(\beta^{(\nu)},\omega^{(\nu)}) = -\infty.$$

But equation (C.11) contradicts the fact that

$$\begin{pmatrix} \beta^{(\nu)} \\ \omega^{(\nu)} \end{pmatrix} \in S.$$

Consequently, S must be bounded.

We have now demonstrated that all premises for the general lemma in section C.2 are fulfilled.

The iterative method proceeds as follows. We choose some starting vector

$$\begin{pmatrix} \hat{\beta}^{(0)} \\ \hat{\omega}^{(0)} \end{pmatrix} .$$

For instance, $\hat{\beta}^{(0)}$ can represent the ordinary non-linear least squares estimate of β. and $\hat{\omega}^{(0)}$ can be defined as $\hat{\omega}(\hat{\beta}^{(0)})$. If we choose $s = L(\hat{\beta}^{(0)}, \hat{\omega}^{(0)})$, then it is guaranteed that for $\hat{\omega}^{(0)} \in U_2$ we have

$$\begin{pmatrix} \hat{\beta}^{(0)} \\ \hat{\omega}^{(0)} \end{pmatrix} \in S.$$

From $\hat{\omega}^{(0)}$ we construct $\hat{\beta}^{(1)} = \hat{\beta}(\hat{\omega}^{(0)})$, which leads to $\hat{\omega}^{(1)}$, and so on. Because of Assumption 6 and the fact that, in the process of iteration, $L(\beta,\omega)$ will always increase from one step to the next, the parameters are always confined to space S.

We are now in the position to formulate the equivalent of Theorem 1 of Oberhofer and Kmenta (1974) for the case of a non-linear model:

Theorem: (i) The sequence

$$\begin{pmatrix} \beta^{(\nu)} \\ \omega^{(\nu)} \end{pmatrix}$$

has at least one accumulation point.

(ii) If

$$\begin{pmatrix} \beta^{*} \\ \omega^{*} \end{pmatrix}$$

is an accumulation point, then $L(\beta,\omega^*)$, taken as a function of β, has its absolute maximum in S at $\beta=\beta^*$. Correspondingly, $L(\beta^*,\omega)$, considered as a function of ω, has its absolute maximum in S at $\omega=\omega^*$. If $\beta \in V_1$ is given by a differentiable (vector) function $\beta(\xi)$ and $\omega \in V_2$ by a differentiable function $\omega(\gamma)$, where ξ and γ are vectors of free parameters, then it follows that

$$\left.\frac{\partial L}{\partial \xi}\right|_{\substack{\beta=\beta^* \\ \omega=\omega^*}} = 0 \text{ and } \left.\frac{\partial L}{\partial \gamma}\right|_{\substack{\beta=\beta^* \\ \omega=\omega^*}} = 0 ,$$

providing only that

$$\begin{pmatrix} \beta^* \\ \omega^* \end{pmatrix}$$

is not a corner solution.

(iii) In all accumulation points, $L(\beta,\omega)$ takes on the same value.

The proof follows immediately from the lemma in section C.2.

The implication of Theorem is that the iterative procedure always converges to a solution of the first-order maximizing conditions (which, however, may or may not correspond to the absolute maximum of the likelihood function).

C.4. Application to a system of non-linear demand functions with error components

It is straightforward to verify that our basic model, (4.1), are of the form (C.1):

(C.12) $w = F(Z,\alpha) + \varepsilon,$

where w denote the 2NM x 1 vector of budget shares stacked as described by equations (3.9), (3.11) and (3.13), Z is the 2MN x K_1 matrix of values of the exogenous variables, α is the K_2 x 1 vector of demand coefficients, F is the vector-valued function representing the demand functions written in terms of budget shares, and ε is the 2MN x 1 disturbance vector defined by (3.13). The assumptions made in sections 3.2 and 4.1 of the main text imply that Assumptions 1 through 5 above are satisfied.

The logarithmic likelihood function, given by equation (C.2), is of course equivalent to $-g$ where g is defined as in equation (4.5). The maximum likelihood estimates of α and Ω are given as the solution to the nonlinear generalized least squares problem

$$\text{(C.13)} \quad \underset{\alpha \in V_2}{\text{Max}} \; \{- \varepsilon' \Omega^{-1} \varepsilon\} ,$$

which we assume has a unique solution, and

$$\text{(C.14)} \quad \hat{\Omega}(\alpha) = I_M \otimes \hat{\Sigma}_\nu(\alpha) \otimes I_2 + I_M \otimes \hat{\Sigma}_\mu(\alpha) \otimes E_2,$$

where $\hat{\Sigma}_\nu(\alpha)$ and $\hat{\Sigma}_\mu(\alpha)$ are given by equations (4.11) and (4.12).

The concentrated log-likelihood function is

$$\text{(C.15)} \quad L(\alpha,\hat{\omega}(\alpha)) = -\frac{1}{2} \log \left| \hat{\Omega}(\alpha) \right| - \text{constant}.$$

From the premise of Assumption 6 (i.e. $L(\alpha,\hat{\omega}(\alpha)) \geq s$ for all α), it follows that $\left|\hat{\Omega}(\alpha)\right| \leq C$ where C is some arbitrarily chosen positive number. To satisfy Assumption 6 we need[3)]

$$\text{(C.16)} \quad \inf \sigma^*_{ii} \geq \lambda_0 > 0$$

and

$$\text{(C.17)} \quad \left|\hat{R}\right| \geq \lambda_1 > 0 \text{ for all } \alpha,$$

where σ^*_{ij} is the typical element of Ω_* (recall that $\Omega = I_M \otimes \Omega_*$), $\hat{R}$ is a $2N \times 2N$ matrix with a typical element $\hat{\rho}_{ij} = \hat{\sigma}^*_{ij} / \sqrt{\sigma^*_{ii}\sigma^*_{jj}}$, and λ_0 and λ_1 are some arbitrarily chosen positive numbers. We then have

$$\text{(C.18)} \quad C \geq \left|\Omega\right| = \left(\prod_{i=1}^{2N} \hat{\sigma}^*_{ii}\right)^M \left|\hat{R}\right|^M \geq \prod_{i=1}^{2N} \hat{\sigma}^{*M}_{ii} \lambda_1^M \geq \lambda_1^M \lambda_0^{2MN}.$$

---

3) Confer Application 3 in Oberhofer and Kmenta (1974).

Hence,

$$(C.19) \quad C/\lambda_1^M \geq \prod_{i=1}^{2N} \hat{\sigma}_{ii}^{*M},$$

and

$$(C.20) \quad \left|\hat{\Omega}\right| \geq \lambda_1^M \lambda_0^{2MN}.$$

Eq. (C.19) implies that all σ_{ii}^* and therefore all σ_{ij}^* are bounded. Taking into account (C.19) and (C.20), we have shown that Assumption 6 is satisfied, since $\hat{\omega}(\alpha) = \max_{\omega \in V_2} L(\alpha,\omega) \in U_2$. However, we have not been able to demonstrate rigorously that Assumption 7 is necessarily satisfied.

Assumption 7 is a way of stating that $F(Z,\alpha)$ increases indefinitely with α for all admissible α. In general, this is a relatively restrictive assumption, but for the current case it is not implausible. Indeed, if we define V_1 to be the set of *economically plausible* values of α[4], V_1 is itself a bounded set. In this case, assumption 7 holds true, but it is abundant in the proof of the theorem in section C.3, since it now follows immediately that S in (C.5) is a bounded set.[5]

---

4) That is: $0 \leq s_i \leq 1$ $(\forall i)$, $0 \leq t_i \leq 1$ $(\forall i)$, $u > C(u) \geq 0$, and β finite.

5) Assumption 7 is closely related to one of four sufficient conditions for (C.13) to yield consistent estimates of α, as stated by Malinvaud (1970). Malinvaud assumes that the equivalent of assumption 7 holds uniformly in M, at least for all $M \geq M^0$, where M^0 is some positive number.

Appendix D

THE FIRST ORDER CONDITIONS FOR MAXIMIZATION OF THE LOG-LIKELIHOOD FUNCTION WITH RESPECT TO Σ_μ AND Σ_ν [1)]

In this appendix, we prove that the conditional estimators of the covariance matrices Σ_μ and Σ_ν used in sections 4.2 and 4.3 for solving subproblem (ii) of the iterative algorithm, are in accordance with the first order conditions for FIML estimation.

The log-likelihood function for our problem is

$$(D.1) \qquad l = -\frac{2MN}{2}\log(2\pi) - \tfrac{1}{2}g,$$

where

$$(D.2) \qquad g = M\log|\Omega_*| + \sum_{h=1}^{M} \varepsilon_h'\Omega_*^{-1}\varepsilon_h.$$

Here N is the number of commodity groups minus one, M is the number of households, each of which is observed twice, ε_h is the 2N x 1 vector of disturbances from household h, and

$$(D.3) \qquad \Omega_* = E(\varepsilon_h\varepsilon_h') = \Sigma_\nu \otimes I_2 + \Sigma_\mu \otimes E_2$$

$$= \Sigma_\nu \otimes (I_2 - \frac{E_2}{2}) + (2\Sigma_\mu + \Sigma_\nu) \otimes \frac{E_2}{2}\ .$$

By utilizing the fact that the matrices $(I_2 - \frac{E_2}{2})$ and $\frac{E_2}{2}$ are idempotent and orthogonal, we directly find that Ω_* has the following inverse:

$$(D.4) \qquad \Omega_*^{-1} = \Sigma_\nu^{-1} \otimes (I_2 - \frac{E_2}{2}) + (2\Sigma_\mu + \Sigma_\nu)^{-1} \otimes \frac{E_2}{2}$$

(cf. also Baltagi (1980, p. 1548)).

It is convenient to arrange the disturbances from household h in the following 2 x N matrix:

---

1) The preparation of this appendix has benefited greatly from Balestra (1975, section 6.2). Confer also Chamberlain and Griliches (1975, appendix). A generalization is discussed in Biørn (1981b, section 4.2).

$$\text{(D.5)} \quad \tilde{\varepsilon}_h = \begin{pmatrix} \varepsilon_{1h1} & \cdots\cdots & \varepsilon_{Nh1} \\ \\ \varepsilon_{2h1} & \cdots\cdots & \varepsilon_{Nh2} \end{pmatrix} \qquad (h=1,\ldots,M),$$

so that ε_h can be written as

$$\text{(D.6)} \quad \varepsilon_h = \text{vec}(\tilde{\varepsilon}_h),$$

where 'vec' is the vectorization operator. (A formal definition is given in eq. (F.9) in appendix F below.)

In order to obtain the first order conditions for maximizing l - or equivalently, minimizing g - with respect to Σ_ν and Σ_μ, we need expressions for the first derivatives of $\log|\Omega_*|$ and of the quadratic form

$$\text{(D.7)} \quad Q = \sum_{h=1}^{M} \varepsilon_h' \Omega_*^{-1} \varepsilon_h .$$

By making use of the formulae for matrix derivatives given in part III of appendix F, this is a fairly straightforward algebraic exercise.

From (D.3) and (F.17) it follows that

$$\frac{\partial \log|\Omega_*|}{\partial \Sigma_\nu} = [I_N \otimes (\text{vec}I_2)'][\Omega_*^{-1} \otimes I_2][I_N \otimes (\text{vec } I_2)],$$

$$\frac{\partial \log|\Omega_*|}{\partial \Sigma_\mu} = [I_N \otimes (\text{vec}I_2)'][\Omega_*^{-1} \otimes E_2][I_N \otimes (\text{vec}I_2)].$$

By inserting (D.4), while noting that

$$(\text{vec}I_2)'(I_2 \otimes I_2)(\text{vec}I_2) = 2,$$

$$(\text{vec}I_2)'(E_2 \otimes I_2)(\text{vec}I_2) = (\text{vec}I_2)'(I_2 \otimes E_2)(\text{vec}I_2) = 2,$$

$$(\text{vec}I_2)'(E_2 \otimes E_2)(\text{vec}I_2) = 4,$$

these expressions can be simplified to

(D.8) $$\frac{\partial \log|\Omega_*|}{\partial \Sigma_\nu} = \Sigma_\nu^{-1} + (2\Sigma_\mu + \Sigma_\nu)^{-1},$$

(D.9) $$\frac{\partial \log|\Omega_*|}{\partial \Sigma_\mu} = 2(2\Sigma_\mu + \Sigma_\nu)^{-1}.$$

We then turn to the quadratic form Q. From (D.4) and (D.6) it follows that the part of Q which relates to household h can be written as

$$\varepsilon_h'\Omega_*^{-1}\varepsilon_h = (\text{vec}\tilde{\varepsilon}_h)'\{\Sigma_\nu^{-1} \otimes (I_2 - \frac{E_2}{2})\}(\text{vec}\tilde{\varepsilon}_h)$$

$$+ (\text{vec}\tilde{\varepsilon}_h)'\{(2\Sigma_\mu + \Sigma_\nu)^{-1} \otimes \frac{E_2}{2}\}(\text{vec}\tilde{\varepsilon}_h).$$

Since both terms on the right hand side of this equation have the same form as the right hand side of eq. (F.15), it can be reformulated as

(D.10) $$\varepsilon_h'\Omega_*^{-1}\varepsilon_h = \text{tr}[\tilde{\varepsilon}_h'(I_2 - \frac{E_2}{2})\tilde{\varepsilon}_h\Sigma_\nu^{-1}] + \text{tr}[\tilde{\varepsilon}_h' \frac{E_2}{2} \tilde{\varepsilon}_h(2\Sigma_\mu + \Sigma_\nu)^{-1}],$$

'tr' denoting the trace operation. Inserting (D.10) in (D.7), while recalling (F.13), we obtain

(D.11) $$Q = \text{tr}[(C-\bar{C})\Sigma_\nu^{-1}] + \text{tr}[\bar{C}(2\Sigma_\mu + \Sigma_\nu)^{-1}],$$

where C and $\bar{C}$ are N x N matrices defined as

(D.12) $$C = \sum_{h=1}^{M} \tilde{\varepsilon}_h'\tilde{\varepsilon}_h,$$

(D.13) $$\bar{C} = \tfrac{1}{2}\sum_{h=1}^{M} \tilde{\varepsilon}_h' E_2\tilde{\varepsilon}_h.$$

From (D.11), while utilizing (F.18) and the fact that all matrices involved are symmetric, we directly find the following expressions for the derivatives of Q:

(D.14) $$\frac{\partial Q}{\partial \Sigma_\nu} = - \Sigma_\nu^{-1} (C-\bar{C})\Sigma_\nu^{-1} - (2\Sigma_\mu + \Sigma_\nu)^{-1} \bar{C}(2\Sigma_\mu + \Sigma_\nu)^{-1},$$

(D.15) $$\frac{\partial Q}{\partial \Sigma_\mu} = -2(2\Sigma_\mu + \Sigma_\nu)^{-1}\bar{C}(2\Sigma_\mu + \Sigma_\nu)^{-1}.$$

By combining (D.2), (D.7) - (D.9), (D.14), and (D.15), we can now state the first order conditions for maximization of l with respect to Σ_ν and Σ_μ as follows:

(D.16) $$\frac{\partial g}{\partial \Sigma_\nu} = M\frac{\partial \log|\Omega_*|}{\partial \Sigma_\nu} + \frac{\partial Q}{\partial \Sigma_\nu}$$

$$= M[\hat{\Sigma}_\nu^{-1} + (2\hat{\Sigma}_\mu + \hat{\Sigma}_\nu)^{-1}] - \hat{\Sigma}_\nu^{-1}(\hat{C}-\hat{\bar{C}})\hat{\Sigma}_\nu^{-1}$$

$$- (2\hat{\Sigma}_\mu + \hat{\Sigma}_\nu)^{-1}\hat{\bar{C}}(2\hat{\Sigma}_\mu + \hat{\Sigma}_\nu)^{-1} = 0,$$

(D.17) $$\frac{\partial g}{\partial \Sigma_\mu} = M\frac{\partial \log|\Omega_*|}{\partial \Sigma_\mu} + \frac{\partial Q}{\partial \Sigma_\mu}$$

$$= 2M(2\hat{\Sigma}_\mu + \hat{\Sigma}_\nu)^{-1} - 2(2\hat{\Sigma}_\mu + \hat{\Sigma}_\nu)^{-1}\hat{\bar{C}}(2\hat{\Sigma}_\mu + \hat{\Sigma}_\nu)^{-1} = 0,$$

the 'hats' denoting the maximizing values. From (D.17) we directly obtain

(D.18) $$2\hat{\Sigma}_\mu + \hat{\Sigma}_\nu = \frac{\hat{\bar{C}}}{M}.$$

By inserting this into (D.16), we get

$$M\hat{\Sigma}_\nu^{-1} - \hat{\Sigma}_\nu^{-1}(\hat{C} - \hat{\bar{C}})\hat{\Sigma}_\nu^{-1} = 0.$$

Hence, the FIML estimator of Σ_ν becomes

(D.19) $$\hat{\Sigma}_\nu = \frac{\hat{C} - \hat{\bar{C}}}{M}.$$

Inserting this into (D.18) and solving for $\hat{\Sigma}_\mu$, we get the FIML estimator

(D.20) $$\hat{\Sigma}_\mu = \frac{2\hat{\bar{C}} - \hat{C}}{2M}.$$

The corresponding estimator of the 'total' covariance matrix Σ is

(D.21) $$\hat{\Sigma} = \hat{\Sigma}_{\mu} + \hat{\Sigma}_{\nu} = \frac{\hat{C}}{2M} .$$

It is illuminating to write these expressions out in scalar notation. Elements (i,j) of C and $\bar{C}$ are, respectively,

$$c_{ij} = \sum_{h=1}^{M} (\varepsilon_{ih1}\varepsilon_{jh1} + \varepsilon_{ih2}\varepsilon_{jh2}),$$

$$\bar{c}_{ij} = \tfrac{1}{2} \sum_{h=1}^{M} (\varepsilon_{ih1} + \varepsilon_{ih2})(\varepsilon_{jh1} + \varepsilon_{jh2}) \qquad (i,j=1,\ldots,N),$$

i.e.

$$c_{ij} - \bar{c}_{ij} = \tfrac{1}{2} \sum_{h=1}^{M} (\varepsilon_{ih2} - \varepsilon_{ih1})(\varepsilon_{jh2} - \varepsilon_{jh1}),$$

$$2\bar{c}_{ij} - c_{ij} = \sum_{h=1}^{M} (\varepsilon_{ih1}\varepsilon_{jh2} + \varepsilon_{ih2}\varepsilon_{jh1}).$$

Thus, (D.19) - (D.21) are equivalent to

(D.22) $$\hat{\sigma}^{\nu}_{ij} = \frac{1}{2M} \sum_{h=1}^{M} (\hat{\varepsilon}_{ih2} - \hat{\varepsilon}_{ih1})(\hat{\varepsilon}_{jh2} - \hat{\varepsilon}_{jh1}),$$

(D.23) $$\hat{\sigma}^{\mu}_{ij} = \frac{1}{2M} \sum_{h=1}^{M} (\hat{\varepsilon}_{ih1}\hat{\varepsilon}_{jh2} + \hat{\varepsilon}_{ih2}\hat{\varepsilon}_{jh1}),$$

(D.24) $$\hat{\sigma}_{ij} = \hat{\sigma}^{\mu}_{ij} + \hat{\sigma}^{\nu}_{ij} = \frac{1}{2M} \sum_{h=1}^{M} (\hat{\varepsilon}_{ih1}\hat{\varepsilon}_{jh1} + \hat{\varepsilon}_{ih2}\hat{\varepsilon}_{jh2}) \qquad (i,j=1,\ldots,N).$$

This completes our proof.

Appendix E

LIKELIHOOD RATIO TESTS OF SIGNIFICANCE OF THE INDIVIDUAL DISTURBANCE COMPONENTS

E.1. General properties

The Likelihood Ratio testing principle is well-known from the literature, see e.g. Kendall and Stuart (1973) or Madansky (1976, pp. 208-209 and 212-214). In this appendix, we shall sketch the test in general terms and derive the statistics for testing hypotheses on the second order moments of the individual disturbance components used in sections 6.1 and 7.5.

The log-likelihood function of our model can in general be written (see (4.2), (4.5), and (3.12))

(E.1) $$\log \Lambda = \text{constant} - \tfrac{1}{2} g,$$

where

(E.2) $$g = M \log |\Omega_*| + \sum_{h=1}^{M} \varepsilon_h' \Omega_*^{-1} \varepsilon_h,$$

(E.3) $$\Omega_* = \Sigma_\nu \otimes I_2 + \Sigma_\mu \otimes E_2.$$

Consider two hypotheses, H_1 and H_0, where the more restrictive hypothesis H_0 is formed by imposing r restrictions on the parameters under the specification H_1. Let $\tilde{\Omega}_*$ and $\hat{\Omega}_*$ be the values of the covariance matrix Ω_*, and $\tilde{\varepsilon}_h$ and $\hat{\varepsilon}_h$ the values of the disturbance vector ε_h calculated from FIML estimates of the parameters under the hypotheses H_0 and H_1, respectively. The maximal value of log Λ under H_0 is thus

(E.4) $$\log \Lambda_{HO} = \text{constant} - \tfrac{1}{2} g_{HO},$$

where

(E.5) $$g_{HO} = M \log |\tilde{\Omega}_*| + \sum_{h=1}^{M} \tilde{\varepsilon}_h' \tilde{\Omega}_*^{-1} \tilde{\varepsilon}_h,$$

and the maximal value of $\log \Lambda$ under H_1 is

$$\text{(E.6)} \qquad \log \Lambda_{H1} = \text{constant} - \tfrac{1}{2}\, g_{H1},$$

where

$$\text{(E.7)} \qquad g_{H1} = M \log|\hat{\Omega}_*| + \sum_{h=1}^{M} \hat{\varepsilon}_h{}' \hat{\Omega}_*^{-1} \hat{\varepsilon}_h.$$

Obviously, $\log \Lambda_{H0} \leqq \log \Lambda_{H1}$ and $g_{H0} \geqq g_{H1}$.

In general terms, the Likelihood Ratio test principle is based on the fact that minus 2 times the (natural) logarithm of the likelihood ratio, defined as $\lambda = \Lambda_{H0}/\Lambda_{H1}$, i.e.

$$-2 \log \lambda = -2[\log \Lambda_{H0} - \log \Lambda_{H1}]$$

is approximately distributed as $\chi^2(r)$ under the null hypothesis H_0. From (E.4)-(E.7), while utilizing the fact that

$$\sum_{h=1}^{M} \tilde{\varepsilon}_h{}' \tilde{\Omega}_*^{-1} \tilde{\varepsilon}_h = \sum_{h=1}^{M} \hat{\varepsilon}_h{}' \hat{\Omega}_*^{-1} \hat{\varepsilon}_h = 2MN,$$

which is an inherent property of the FIML method (see e.g. Madansky (1976,p. 96, Theorem 19)), it follows that the Likelihood Ratio test statistic can be written as

$$\text{(E.8)} \qquad -2 \log \lambda = g_{H0} - g_{H1} = M\{\log|\tilde{\Omega}_*| - \log|\hat{\Omega}_*|\}.$$

E.2. Testing for individual disturbance components in the multi-equation case

In this case, our problem is to test

$$H_0: \; \Sigma_\mu = (\sigma^\mu_{ij}) = 0$$

against

$$H_1: \Sigma_\mu = (\sigma^\mu_{ij}) \neq 0.$$

Let $\tilde{\Sigma}_\nu$ and $\hat{\Sigma}_\nu$ denote the FIML estimate of Σ_ν under H_0 and H_1, respectively, and let $\hat{\Sigma}_\mu$ be the FIML estimate of Σ_μ under H_1. Then,

$$\tilde{\Omega}_* = \tilde{\Sigma}_\nu \otimes I_2,$$

$$\hat{\Omega}_* = \hat{\Sigma}_\nu \otimes I_2 + \hat{\Sigma}_\mu \otimes E_2.$$

Inserting these expressions in (E.8), the test statistic becomes

$$\text{(E.9)} \qquad -2 \log \lambda = M\{\log|\tilde{\Sigma}_\nu \otimes I_2| - \log|\hat{\Sigma}_\nu \otimes I_2 + \hat{\Sigma}_\mu \otimes E_2|\}.$$

Since Σ_μ is symmetric, the number of restrictions in H_0 is $r = N(N+1)/2$, where N is the number of commodity groups minus one.

E.3. Testing for individual disturbance components in the single equation case

When the number of commodities $N = 1$, we have

$$\text{(E.10)} \qquad \Omega_* = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

where $\sigma^2 = \sigma^2_\nu + \sigma^2_\mu$ is the total disturbance variance, and $\rho = \sigma^2_\mu/\sigma^2$ is the part of this total which is due to individual variations. (See section 4.5.) In this case, our problem is equivalent to testing

$$H_0: \rho = 0$$

against

$$H_1: \rho > 0,$$

recalling that ρ is non-negative.[1)] Letting $\tilde{\sigma}^2$ and $\hat{\sigma}^2$ be the FIML estimate of σ^2 under H_0 and H_1, respectively, and $\hat{\rho}$ the estimate of ρ under H_1, it follows from (E.8) and (E.10) that

$$\text{(E.11)} \quad -2 \log \lambda = 2M\{\log \tilde{\sigma}^2 - \log \hat{\sigma}^2 - \log (1-\hat{\rho}^2)\}.$$

Since $r = 1$ in this case, this statistic will be approximately distributed as $\chi^2(1)$.

---

1) Note, however, the possible generalization discussed in section 3.2, in particular eq. (3.6').

Appendix F

SOME USEFUL PROPERTIES OF KRONECKER PRODUCTS, 'VEC' OPERATIONS, AND MATRIX DERIVATIVES[1)]

In this appendix, we have collected and refer, mostly without proofs, some properties of Kronecker product and vectorization operations, and some results on matrix derivatives which we have found useful in discussing the econometric formulation of the model and its estimation procedure in chapters III and IV of the main text.

I

Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices of dimension $M \times m$ and $N \times n$, respectively. The *Kronecker product* of A and B is, by definition, the $MN \times mn$ matrix

(F.1) $$A \otimes B = \begin{pmatrix} a_{11}B & \cdots\cdots & a_{1m}B \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{M1}B & \cdots\cdots & a_{Mm}B \end{pmatrix},$$

where $\otimes$ is the Kronecker product operator. If C and D are conformable matrices, the following rules for matrix operations are valid:[2)]

Multiplication and addition.

(F.2) $(A + C) \otimes B = A \otimes B + C \otimes B,$

(F.3) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD),$

(F.4) $A \otimes (B \otimes C) = (A \otimes B) \otimes C.$

Transposition.

(F.5) $(A \otimes B)' = A' \otimes B'.$

Inversion. If A and B are non-singular ($m=M, n=N$), then

(F.6) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$

---

1) The symbols used in this appendix deviate from the general notation in the main text.

2) Proofs of (F.2)-(F.8) can be found in several textbooks in econometrics, e.g. Theil (1971, pp. 303-306).

Rank.

(F.7) $\quad \text{rank}(A \otimes B) = \text{rank}(A) \cdot \text{rank}(B).$

Determinant value. If A and B are quadratic, i.e. $m=M$ and $n=N$, then

(F.8) $\quad |A \otimes B| = |A|^N |B|^M.$

II

Let X be the M x N matrix

$$X = \begin{pmatrix} x_{11} & \cdots\cdots & x_{1N} \\ \vdots & & \vdots \\ x_{M1} & \cdots\cdots & x_{MN} \end{pmatrix},$$

and define

$$x_r = \begin{pmatrix} x_{1r} \\ \vdots \\ x_{Mr} \end{pmatrix} = \text{the } r\text{'th column of } X \qquad (r=1,\ldots,N),$$

and

$$\bar{x}_i = \begin{pmatrix} x_{i1} \\ \vdots \\ x_{iN} \end{pmatrix} = \text{the } i\text{'th column of } X'$$

$$(\text{i.e., } \bar{x}_i' = \text{the } i\text{'th row of } X) \qquad (i=1,\ldots,M).$$

The *vectorization operator* 'vec' is the operator which stacks all columns of a matrix along one vector, i.e. by definition we have

(F.9) $$\text{vec}(X) = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = x.$$

Similarly,

$$\text{(F.10)} \quad \mathrm{vec}(X') = \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \vdots \\ \bar{x}_M \end{pmatrix} = \bar{x}.$$

Provided that A and B are quadratic matrices of dimension M x M and N x N, respectively, we can now state the following important result on *quadratic forms:*

$$\text{(F.11)} \quad Q = x'(B \otimes A)x = \bar{x}'(A \otimes B)\bar{x} = \sum_{r=1}^{N} \sum_{s=1}^{N} \sum_{i=1}^{M} \sum_{j=1}^{M} x_{ir} a_{ij} b_{rs} x_{js}.$$

Proof of (F.11):

From the definitions of A, B, and x, we have

$$(*) \qquad Q = (x_1' \ \ldots\ldots x_N') \begin{pmatrix} b_{11}A & \ldots & b_{1N}A \\ \vdots & & \vdots \\ b_{N1}A & \ldots & b_{NN}A \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \sum_{r=1}^{N} \sum_{s=1}^{N} b_{rs} x_r' A x_s.$$

Expanding the bilinear form $x_r'Ax_s$, we get

$$(**) \qquad x_r'Ax_s = \sum_{i=1}^{M} \sum_{j=1}^{M} x_{ir} a_{ij} x_{js} \qquad (r,s=1,\ldots,N).$$

In a similar way, we find

$$(\Delta) \qquad \bar{x}'(A \otimes B)\bar{x} = \sum_{j=1}^{M} \sum_{i=1}^{M} a_{ij} \bar{x}_i' B \bar{x}_j,$$

and

$$(\Delta\Delta) \qquad \bar{x}_i' B \bar{x}_j = \sum_{r=1}^{N} \sum_{s=1}^{N} x_{ir} b_{rs} x_{js} \qquad (i,j=1,\ldots,M).$$

Eq. (F.11) now follows by inserting (**) in (*) and inserting (ΔΔ) in (Δ).
Q.E.D.

This equation may be readily generalized to bilinear forms. Let us consider four important applications of (F.11).

i) $B = I_N$, i.e. $b_{rr} = 1, b_{rs} = 0$ for $s \neq r$:

$$x'(I_N \otimes A)x = \sum_{r=1}^{N} \sum_{i=1}^{M} \sum_{j=1}^{M} x_{ir} a_{ij} x_{jr}.$$

ii) $A = I_M$, i.e. $a_{ii} = 1, a_{ij} = 0$ for $j \neq i$:

$$x'(B \otimes I_M)x = \sum_{i=1}^{M} \sum_{r=1}^{N} \sum_{s=1}^{N} x_{ir} b_{rs} x_{is}.$$

iii) $B = I_N$, $A = E_M$; i.e. $b_{rr} = 1$, $b_{rs} = 0$ for $s \neq r$; $a_{ij} = 1$ for all i and j:

$$x'(I_N \otimes E_M)x = \sum_{r=1}^{N} \sum_{i=1}^{M} \sum_{j=1}^{M} x_{ir} x_{jr} = \sum_{r=1}^{N} \left(\sum_{i=1}^{M} x_{ir}\right)^2.$$

iv) $A = I_M$, $B = E_N$; i.e. $a_{ii} = 1$, $a_{ij} = 0$ for $j \neq i$; $b_{rs} = 1$ for all r and s:

$$x'(E_N \otimes I_M)x = \sum_{i=1}^{M} \sum_{r=1}^{N} \sum_{s=1}^{N} x_{ir} x_{is} = \sum_{i=1}^{M} \left(\sum_{r=1}^{N} x_{ir}\right)^2.$$

III

The *trace* of a M x M matrix A is defined as the sum of the elements along its main diagonal:[3)]

$$\text{(F.12)} \quad trA = trA' = \sum_{i=1}^{M} a_{ii} = (vecI_M)'(vecA).$$

---

3) Regarding the last equality, see Balestra (1975, eq. (2.2.21)).

It is easy to prove that[4)]

(F.13) $\quad \mathrm{tr}(A+C) = \mathrm{tr}A + \mathrm{tr}C,$

(F.14) $\quad \mathrm{tr}(AC) = \mathrm{tr}(CA).$

One may also show that if A, B, C, and D are matrices such that the product ABCD is a quadratic matrix, then[5)]

(F.15) $\quad \mathrm{tr}(ABCD) = (\mathrm{vec}A')'(D' \otimes B)(\mathrm{vec}C).$

Let λ be a scalar which is a function of the elements of the M x m matrix $A = (a_{ij})$. The first derivative of λ with respect to A, $\partial\lambda/\partial A$, is, by definition, the M x m matrix with element (i,j) equal to $\partial\lambda/\partial a_{ij}$:

(F.16) $$\frac{\partial\lambda}{\partial A} = \left(\frac{\partial\lambda}{\partial a_{ij}}\right).$$

In Balestra (1975), proofs are given for the following two results on *matrix derivatives*, which are very useful when dealing with Maximum Likelihood estimation problems:

(i) Let B be a quadratic and positive definite matrix of the form

$$B = A \otimes C \pm P, \text{ where}$$

$$\dim(A) = M \times M,$$

$$\dim(C) = S \times S,$$

$$\dim(P) = MS \times MS.$$

Then,[6)]

(F.17) $$\frac{\partial \log|B|}{\partial A} = [I_M \otimes \mathrm{vec}I_S]'[B^{-1} \otimes C][I_M \otimes \mathrm{vec}I_S]$$

(where $B = A \otimes C + P$).

(ii) Let B and C be quadratic matrices of the same dimension. If B is non-singular, then[7)]

4) See e.g. Theil (1971, p. 15)
5) Cf. Balestra (1975, eq. (2.2.24)).
6) Balestra (1975, eq. (5.3.18)).
7) Balestra (1975, eq. (5.2.19)).

$$(\text{F.18}) \quad \frac{\partial \operatorname{tr}(CB^{-1})}{\partial B} = -(B^{-1})' C' (B^{-1})' .$$

Finally, let μ be a scalar function of the elements of the $M \times 1$ vector $\alpha = (\alpha_1 \ldots \alpha_M)'$. According to (F.16), the first derivative of μ with respect to α is the $M \times 1$ vector with i'th element equal to $\partial\mu/\partial\alpha_i$. The second derivative of μ with respect to α, often denoted as the *Hessian* of μ, is defined as the $M \times M$ matrix with element (i,j) equal to $\partial^2\mu/\partial\alpha_i\partial\alpha_j$:

$$(\text{F.19}) \quad \frac{\partial^2 \mu}{\partial\alpha\partial\alpha'} = \left[\frac{\partial^2 \mu}{\partial\alpha_i\partial\alpha_j}\right] .$$

Appendix G

ON THE STANDARD ERRORS OF THE MAXIMUM LIKELIHOOD ESTIMATES[1]

1. In this appendix, we present and discuss the formulae applied for calculating the standard errors of the FIML estimates in the simultaneous demand model in chapter VI. We first show that these formulae, under certain conditions, correspond to the Cramer-Rao bound on the variance-covariance matrix. Second, we demonstrate that they can alternatively be interpreted as the result of a non-linear least squares estimation, after having transformed the model into the least squares regression format. Finally, we consider two alternative numerical approximations to the Hessian matrix of the quadratic form in the log-likelihood function, which is a crucial element in the standard error formulae.

2. Consider a general expression for the likelihood function

$$\text{(G.1)} \qquad \Lambda = \Lambda(w|\theta),$$

where w is the vector of endogenous variables, the budget shares in our case, and θ denotes the vector of structural parameters. The minimum variance bound on a set of estimators for θ, known as *the Cramer-Rao bound* (CRB), is given by

$$\text{(G.2)} \qquad \mathrm{CRB}(\theta) = -\left[E\left(\frac{\partial^2 \log \Lambda}{\partial\theta\partial\theta'}\right)\right]^{-1},$$

see e.g. Kendall and Stuart (1973, chapter 17). The matrix $\partial^2 \log \Lambda/\partial\theta\partial\theta'$ is the Hessian of log Λ with respect to θ (cf. eq. (F.19) in appendix F). It is well-known[2] that the maximum likelihood estimators of the elements of θ are asymptotically normally distributed with mean equal to the unknown value θ_0 and covariance matrix equal to the Cramer-Rao bound. If the joint density function of w admits a set of *minimal sufficient statistics* for θ - which the joint normal distribution does - then the maximum likelihood estimators are such minimal sufficient statistics. A vector of statistics, T(w), is said to be sufficient if and only if the likelihood function can be written in the following separable form

1) The symbols used in this appendix deviate in some respects from the general notation in the main text.
2) Confer op.cit. or Goldfeld and Quandt (1972,p. 61)

$$\text{(G.3)} \quad \Lambda(w|\theta) = g(T|\theta)h(w).$$

The vector $T(w)$ is a set of minimal sufficient statistics if it has the smallest number of elements among all sets of sufficient statistics.[3] Eq. (G.3) implies that the maximization of the likelihood function with respect to θ is equivalent to maximization of the density function g of T.

3. In our particular problem, we have $\theta = (\alpha, \omega)$, where α is the vector of structural coefficients in the demand system and ω is the vector of unknown elements in the disturbance variance-covariance matrix Ω_*. Let now $(T_1(w), T_2(w))$ denote a set of jointly minimal sufficient statistics for (α, ω), and suppose that:

Assumption 1: The distribution of the statistics $T_2(w)$ does not depend on the coefficient vector α.

Assumption 2: $T_2(w)$ is a set of sufficient statistics for ω when α is known.

Then, the likelihood function of w can be factorized as follows:

$$\text{(G.4)} \quad \Lambda(w|\alpha,\omega) = g_1(T_1|T_2,\alpha)\ g_2(T_2|\omega)\ h(w)$$

and the conditional density function of T_1 with respect to T_2, g_1, is all we need to consider in making inferences on α.[4]

4. Subproblem (i) of the iterative procedure for maximizing our particular log-likelihood function (4.2) consists in maximizing - conditionally on Ω_* -

$$\text{(G.5)} \quad \log \Lambda = \text{constant} - \tfrac{1}{2}Q$$

with respect to α, where

$$\text{(G.6)} \quad Q = \sum_{h=1}^{M} \varepsilon_h' \Omega_*^{-1} \varepsilon_h.$$

---

3) See Kendall and Stuart (1973, par. 23.1?.
4) This is called the Conditionality Principle by Kendall and Stuart (1973, par. 23.37).

(Cf. section 4.2.) The Hessian of log Λ - conditionally on Ω_* - can thus be written as

$$(G.7) \qquad \frac{\partial^2 \log \Lambda}{\partial\alpha\partial\alpha'} = -\tfrac{1}{2}\left(\frac{\partial^2 Q}{\partial\alpha\partial\alpha'}\right),$$

where $(\partial^2 Q/\partial\alpha\partial\alpha')$ is the Hessian of Q. If, however, Assumptions 1 and 2 hold true for the maximum likelihood estimators of α and Ω_* - corresponding to, respectively, T_1 and T_2 above - then (G.7) is valid not only conditionally, but also marginally (i.e., for the global maximum of log Λ). Hence, we conclude, with reference to (G.4), that the Hessian of log Λ is block diagonal:

$$(G.8) \qquad \frac{\partial^2 \log \Lambda}{\partial\theta\partial\theta'} = \begin{bmatrix} \frac{\partial^2 \log \Lambda}{\partial\alpha\partial\alpha'} & 0 \\ 0 & \frac{\partial^2 \log \Lambda}{\partial\omega\partial\omega'} \end{bmatrix}.$$

We can now employ the result[5] that when the maximum likelihood estimator is sufficient, then

$$(G.9) \qquad E\left(\frac{\partial^2 \log \Lambda}{\partial\theta\partial\theta'}\right) = \frac{\partial^2 \log \Lambda}{\partial\theta\partial\theta'}$$

for θ equal to the estimated parameter values $\hat{\theta} = (\hat{\alpha},\hat{\omega})$. From (G.2) and (G.7) - (G.9) it then follows that the covariance matrix estimator

$$(G.10) \qquad \text{cov}(\alpha) = -\left[\frac{\partial^2 \log \Lambda}{\partial\alpha\partial\alpha'}\right]^{-1} = 2\left[\frac{\partial^2 Q}{\partial\alpha\partial\alpha'}\right]^{-1}$$

attains the Cramer-Rao bound.[6]

5. The formula actually used for calculating the estimated covariance matrix of α in the present study is

$$(G.11) \qquad \text{cov}(\alpha) = \frac{2Q}{2MN-K}\left[\frac{\partial^2 Q}{\partial\alpha\partial\alpha'}\right]^{-1},$$

where K is the number of unknown coefficients in the vector α. Since

---

5) See Goldfeld and Quandt (1972, p.63).

6) On the other hand, if the sufficiency assumptions 1 and 2 above are not fulfilled for our particular model we may still consider (G.10) as an estimator of the covariance matrix of α, but then it is an estimator which is conditional on the estimated covariance matrix Ω_*.

it is easy to show[7] that Q=2MN at the global maximum of the likelihood function, (G.11) gives asymptotically the same result as (G.10). Our application of (G.11) relies on the interpretation of the minimization of the quadratic form Q with respect to α as a non-linear least squares regression problem, that is, we interpret Q as an expression of the form

$$(G.12) \quad Q(\alpha) = \sum_{j=1}^{2MN} (R_j(\alpha))^2,$$

where $R_j(\alpha)$ has *the same functional form in* α *for all* j. Then (G.11) provides unbiased estimates for the covariance matrix of α (conditionally on Ω_*), see Wolberg (1967, p.60). Our next task is therefore to show how Q can be written in this sum of squares format.

6. Let us recapitulate the model in scalar notation:

$$(G.13) \quad \varepsilon_{iht} = w_{iht} - f_i(z_{iht};\phi_i,\gamma) \qquad \begin{array}{l} i=1,\dots,N, \\ h=1,\dots,M, \\ t=1,2, \end{array}$$

where the budget share demand equation for commodity i (h denoting, as before, the household number and t the number of the report),

$$f_i(z_{iht};\phi_i,\gamma) = s_i+(t_i v_{iht}^{\beta}-s_i)\,\frac{a \log u_{ht}+b}{u_{ht}} \quad ,$$

is defined in terms of the arguments:[8]

$z_{iht} = (v_{iht}, u_{ht})$, i.e. the explanatory variables in the demand equation for commodity i,

$\phi_i = (s_i, t_i)$, i.e. the coefficients which are specific to commodity i, and

$\gamma = (a,b,\beta)$, i.e. the coefficients which are common to all commodity groups.

If $\Omega_* = I_{2N}$, we immediately see that (G.6) is a sum of squares in the ε disturbances:

$$Q = \sum_{h=1}^{M} \sum_{t=1}^{2} \sum_{i=1}^{N} \varepsilon_{iht}^2,$$

7) See e.g. Klein (1974, p.147).
8) Omitting, for simplicity, the background variables.

but this is not an expression of the form (G.12), since the expression for ε_{iht} contains the commodity specific coefficients ϕ_i. In our disturbance components model, where Ω_* is not an identity matrix, we factorise Ω_*^{-1} into an expression of the form U'U (cf. eqs. (4.8) and (4.14)), which transforms Q into a sum of squares:

$$\text{(G.14)} \quad Q = \sum_{h=1}^{M} \eta_h'\eta_h = \sum_{h=1}^{M}\sum_{t=1}^{2}\sum_{i=1}^{N} \eta_{iht}^2,$$

where

$$\text{(G.15)} \quad \eta_h = U\,\varepsilon_h \qquad h=1,\dots,M,$$

or equivalently,

$$\begin{bmatrix} \eta_{1h1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \eta_{Nh2} \end{bmatrix} = U \begin{bmatrix} \varepsilon_{1h1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_{Nh2} \end{bmatrix} \qquad h=1,\dots,M,$$

i.e.,

$$\text{(G.16)} \quad \eta_{iht} = \sum_{j=1}^{N}\sum_{\tau=1}^{2} U_{ij\tau}\,\varepsilon_{jh\tau} \qquad i=1,\dots,N,\ h=1,\dots,M,\ t=1,2,$$

where $(U_{i11},\dots,U_{iN2})$ is the i'th row of U. But this sum of squares expression is certainly not of the form (G.12).[9)]

To circumvent this problem, we define

$$\text{(G.17)} \quad \mu_{iht} = \sum_{j=1}^{N} \delta_{ij}\,\eta_{iht} \qquad i=1,\dots,N,\ h=1,\dots,M,\ t=1,2,$$

where δ_{ij} is the Kronecker delta. Obviously,

$$\text{(G.18)} \quad Q = \sum_{h=1}^{M}\sum_{t=1}^{2}\sum_{i=1}^{N} \mu_{iht}^2.$$

---

9) This is particularly easy to see in the case of a Cholesky factorisation of Ω_*^{-1}, when U is a triangular matrix.

Now, combining (G.13), (G.16), and (G.17) we have

$$(G.19)\quad \mu_{iht} = \sum_{j=1}^{N} \delta_{ij} \sum_{k=1}^{N} \sum_{\tau=1}^{2} U_{jk\tau}\, \varepsilon_{kh\tau}$$

$$= \overset{*}{w}_{iht} - \Phi(z_{1h1},\ldots,z_{Nh2},\delta_{i1},\ldots,\delta_{iN};\phi_1,\ldots,\phi_N,\gamma,U_{111},\ldots,U_{NN2}),$$

where

$$\overset{*}{w}_{iht} = \sum_{k=1}^{N} \sum_{\tau=1}^{2} U_{ik\tau}\, w_{kh\tau},$$

$$\Phi(\cdot) = \sum_{j=1}^{N} \delta_{ij} \sum_{k=1}^{N} \sum_{\tau=1}^{2} U_{jk\tau}\, f_k(z_{kh\tau};\phi_k,\gamma).$$

From this we see that Q, expressed as a sum of squares in the transformed disturbances μ_{iht}, is of the form (G.12) when we consider $\delta_{ij}(i,j=1,\ldots,N)$ as variables in the function Φ and include $U_{jk\tau}$ $(j,k=1,\ldots,N,$ and $\tau=1,2)$ among its parameters (represented by α in the general expression (G.12)).

7. In this concluding paragraph, we consider two different approximations to the Hessian matrix of Q with respect to α.

Method 1 exploits the result from the preceding paragraph that Q can be written in the form (G.12). The Hessian matrix of $Q(\alpha)$ in (G.12) can be written as

$$(G.20)\quad \frac{\partial Q}{\partial\alpha\partial\alpha'} = 2\left[J'J + \sum_{j=1}^{2MN} R_j H_j\right],$$

where J is the (2MN x K) Jacobian matrix of $R_j (j=1,\ldots,2MN)$ and H_j is the (K x K) Hessian matrix of R_j. In the neighbourhood of the solution, $||Q(\alpha)||$ is small as compared with $||J'J||$ and thus 2 J'J provides an adequate approximation to the Hessian of Q with respect to α. In this way, we avoid the need to compute or approximate second derivatives of R_j.

Method 2 consists in using a finite difference approximation to the Hessian matrix of Q. Such an approximation is available since we employ a Quasi-Newton method to minimize Q with respect to α, for the simultaneous estimation of the complete model. In practice, a Quasi-Newton approximation $H^{(k)}$ to the Hessian of Q is maintained and updated at every iteration (k), using the recurrence relation (H.6).[10)]

In table G.1 we have compared the standard errors of the estimates from the complete model specification (without background variables), using these two alternative approximations to the Hessian matrix in equation (G.11). We observe that the difference between the resulting standard errors are within 10 per cent of the smallest estimated standard error. Throughout this work we have chosen to report the standard errors which emerge from the use of method 1 only.

Table G.1. Approximate standard errors of the coefficient estimates of the complete demand model for the case with $\beta=1$ and no background variables, based on two different approximations to the Hessian matrix $\partial Q/\partial\alpha\partial\alpha'$ in equation (G.11)

Coefficient	Estimated value	Standard error	
		Method 1 *)	Method 2 *)
s_1	0.137	0.014825	0.015056
s_2	0.118	0.008715	0.008710
s_3	0.188	0.013182	0.013288
s_4	0.449	0.018393	0.019165
a	0.779	0.937256	0.852372
b	1.287	1.526763	1.388188
t_1	0.361	0.275147	0.250333
t_2	0.086	0.044053	0.040111
t_3	0.295	0.127224	0.115823
t_4	0.194	0.311675	0.283613

*) Method 1 means that $\partial^2 Q/\partial\alpha\partial\alpha'$ is estimated by 2 J'J, while method 2 employs a finite difference approximation to this matrix.

10) See appendix H for a further discussion of the computer routines.

Appendix H

THE COMPUTER PROGRAMS[1)]

H.1. The scope of the appendix

Needless to say, this project has raised several computer (and computer related) problems, and a substantial part of the resources spent has been costs in terms of computer run-time and programming man-hours. This motivates a closer look at the computer programs which we have developed during this work.

In this final appendix, we shall be particularly concerned with the implementation of the iterative procedure for FIML estimation, described algebraically in chapter IV. First, in section H.2, we present two alternative algorithms for solving the problem stated as subproblem (i) in section 4.2: Minimize

$$(H.1) \qquad Q = \sum_{h=1}^{M} \varepsilon_h' \, \Omega_*^{-1} \, \varepsilon_h$$

with respect to the K dimensional vector of unknown structural coefficients, α, conditionally on the 2N x 2N covariance matrix Ω_*, where ε_h is the 2N x 1 vector of disturbances from household h (cf. eq. (3.11)). We discuss the choice between these algorithms for the simultaneous estimation of the complete demand system as well as for the single equation estimation problem. In section H.3, we outline the structure of our computer programs, with reference to an example program, reproduced as a separate annex to this study. In the final section H.4, we summarize some experiences from the programming, *inter alia* by recording computational statistics, which characterize the estimation of different variants of the complete demand model.

H.2. Two alternative numerical methods for solving the minimization problem in the iterative procedure for FIML estimation (subproblem (i))

We have approached the problem of minimizing the quadratic form Q with respect to α in two different ways. The first approach is to apply an algorithm which solves a non-linear least squares problem, while the second one makes use of a subroutine for minimization of a general non-linear function. In both cases, we have employed subroutines from the NAG Library - subroutines *E04GAF* and *E04KBF*, respectively (see NAG (1978)).

---

1) The symbols used in this appendix deviate from the general notation in the main text.

The first subroutine *(E04GAF)* finds a solution to the following least squares problem: Minimize

(H.2) $$F(\alpha) = \sum_{i=1}^{T} R_i(\alpha)^2 = \sum_{i=1}^{T} R^*(Z_i,\alpha)^2$$

with respect to α, where R^* is a specified function of the observation vector Z_i $(i=1,\dots,T)$, α is an unknown K dimensional parameter vector, and T is the number of observation sets. The method used is due to Marquandt (see Marquandt (1963)). The algorithm solves this problem iteratively: Suppose - at the r'th iteration - we have reached the point $\alpha^{(r)}$. The correction vector

(H.3) $$\delta^{(r)} = \alpha^{(r+1)} - \alpha^{(r)},$$

required to give an improved estimate of the minimum, is obtained by solving for $\delta^{(r)}$ the following normal equations

(H.4) $$(J^{(r)'}J^{(r)} + \lambda^{(r)}D)\ \delta^{(r)} = -\ J^{(r)'}R,$$

where J is the $T \times K$ Jacobian matrix of the functions R_i, R is a $T \times 1$ vector-valued function with R_i as its i'th element, D is a diagonal $K \times K$ scaling matrix with non-negative elements on the diagonal, and $\lambda^{(r)}$ is a scalar to be explained below. When $\lambda^{(r)} = 0$, these normal equations are identical to those in the Gauss-Newton method, for which convergence is quadratic, but in some cases it may diverge. The effect of including $\lambda^{(r)}$ is - whenever the method appears to be diverging - to introduce an adjustable bias towards the steepest descent vector of $F(\alpha)$, i.e. $2J'R$, where progress is assured (but may be slow). Thus, if the sum of squares $F(\alpha^{(r)}+\delta^{(r)})$ is less than $F(\alpha^{(r)})$, then $(\alpha^{(r)}+\delta^{(r)})$ will be taken as the starting point for the next iteration, otherwise $\lambda^{(r)}$ will be increased and the process repeated. The iterative procedure continues until convergence of $F(\alpha)$ with respect to α is obtained.

The second subroutine *(E04KBF)* is based on a quasi-Newton iterative algorithm for finding the minimum of a general function $F(\alpha)$.[2)] The essential feature of all Newton-type methods is that the Hessian matrix of F - or an approximation to it - is used to define the search

---

2) This subroutine has an option for constrained minimization, i.e. minimization where some of the parameters in the α vector are restricted to a priori given intervals. We have in fact exploited this option at several stages of the estimation work (see sections H.3 and H.4 below), but in this section we shall, for simplicity, consider the unconstrained case only.

direction. Let $H^{(r)}$ define this approximation at the r'th iteration. Then the equations

$$(H.5) \quad H^{(r)} \kappa^{(r)} = -g^{(r)}$$

are solved to give the search direction $\kappa^{(r)}$, where $g^{(r)}$ is the gradient of $F(\alpha)$ at the current point $\alpha^{(r)}$.[3] Second, a parameter γ is found such that $F(\alpha^{(r)}+\gamma\kappa^{(r)})$ approximately attains a minimum with respect to γ. Third, the matrix $H^{(r)}$ is updated so as to be consistent with the change produced in the gradient by the step $\gamma\kappa^{(r)}$. This updating is performed using the recurrence relation

$$(H.6) \quad H^{(r+1)} = H^{(r)} + \frac{d^{(r)} d^{(r)'}}{\gamma d^{(r)'} \kappa^{(r)}} + \frac{g^{(r)} g^{(r)'}}{g^{(r)'} \kappa^{(r)}},$$

where $d^{(r)} = g^{(r+1)} - g^{(r)}$. The iterative procedure proceeds until convergence of $F(\alpha)$ with respect to α.

There are notable similiarities between the two approaches. First, both algorithms require that the user supplies analytical expressions for the first derivations of the function to be minimized. Second, the expression on the right hand side of equation (H.4) is proportional to the gradient $g^{(r)}$ in equation (H.5). Third, as stated in appendix G, $2J'J$ provides an estimate of the Hessian of $F(\alpha)$ for the final parameter values, and we see that $J'J$ plays a similar role in equation (H.4) as $H^{(r)}$ does in equation (H.5), viz. to determine the correction vector for the unknown parameter vector α. The essential difference is thus that the quasi-Newton algorithm is gradually building up an approximation to the Hessian matrix with the purpose of minimizing $F(\alpha)$ as a general non-linear function, while the Marquandt algorithm exploits the specific structure of $F(\alpha)$ - i.e. that it is a non-linear sum of squares, and

---

3) A unique solution to (H.5) exists only if $H^{(r)}$ is positive definite. In order to assure this property of the Hessian, the approximation (H.6) below is modified. A Cholesky factorisation is computed to satisfy

$$L^{(r)} D^{(r)} L^{(r)'} = H^{(r)} + E^{(r)}$$

where $L^{(r)}$ is a lower-triangular matrix with unitary diagonal elements, $D^{(r)}$ is a diagonal matrix with strictly positive diagonal elements (which is a necessary and sufficient condition for $L^{(r)} D^{(r)} L^{(r)'}$ to be positive definite, see Lau (1978, p.429)), and $E^{(r)}$ is a diagonal matrix with non-negative elements. If $H^{(r)}$ is sufficiently positive definite, $E^{(r)}$ will be a zero matrix (see Gill and Murray (1977)). In the NAG-subroutine *E04KBF*, the equation (H.5) is calculated with the Cholesky factorisation above substituted for $H^{(r)}$.

solves the first order conditions for minimization of this function accordingly.

We observe from eq. (H.2) that the subroutine *E04GAF* requires that the minimization problem is in the single equation (non-linear) regression format since the function R^* is the same for all observation sets. This is obviously true for a single equation model, and this subroutine is thus the only one which we have tried for estimating the single equation version of the demand model (cf. chapter VII).[4)]

In the case of simultaneous estimation of the complete model, we have tried both algorithms for minimizing (H.1). This permitted a direct comparison of the computing efficiency of the two methods. As noted in paragraph 6 of appendix G, the general problem of minimizing Q with respect to α is not a problem in the single equation regression format, but it can be transformed to that format. In order to make subroutine *E04GAF* applicable to our problem, we define

$$\text{(H.7)} \qquad \mu_{ht} = \left\{ \sum_{i=1}^{N} \eta_{iht}^2 \right\}^{\frac{1}{2}} ,$$

where η_{iht} is the transformed residual corresponding to commodity i, household h, and report t (see eq. (G.16) in appendix G). The minimization of Q, as defined in (H.1), with respect to α is now equivalent to the minimization of

$$\text{(H.8)} \qquad Q = \sum_{h=1}^{M} \sum_{t=1}^{2} \mu_{ht}^2$$

with respect to α (cf. eqs. (G.17) and (G.18) in appendix G). This equation is in the same format as (H.2), since μ_{ht} is a function of the form $R^*(Z_i,\alpha)$ for all values of h and t.

We have compared the efficiency of the two algorithms for the version of the complete demand model with no background variables included and with $\beta=0$ *a priori*.[5)] We found that the program using the quasi-Newton

4) Moreover, when the demand functions are linear in α, it is easy to show that, provided $\lambda=0$, (H.4) is identical to the normal equations for the ordinary least squares estimation problem. Since $\lambda=0$ is the starting value of λ (by default), OLS estimates for α are obtained directly (without iteration) in such cases.
5) There is, however, one difference between the two programs which is worth mentioning: *E04KBF* implements the algorithm described in section 4.3 (with some minor deviations), while *E04GAF* is used to implement the alternative algorithm described in section 4.4.

method (subroutine *E04KBF*) converged to a final maximum likelihood solution after 0.37 CPU-hours[6], whereas the program based on the Marquandt algorithm (subroutine *E04GAF*) failed to reach this maximum after 1.6 CPU-hours.[7] The programs based on the NAG-subroutine *E04KBF* were thus preferred for the estimation of all other variants of the complete demand model.

H.3. The structure of the computer programs

Let us then sketch the structure of the computer programs used for the simultaneous estimation of the complete demand model. The programs are written in standard FORTRAN and consist of a main program and a set of subroutines, see figure H.1. The main program governs the iterative procedure for maximizing the log-likelihood function for our problem (confer chapter IV). From the main program we call the NAG-subroutine *E04KBF*, two supporting user-supplied subroutines (*FUNCT* and *MONIT*), and five auxiliary subroutines (*TRANSOME*, *OMEGINV*, *CHOLOME*, *DOMEGA*, and *SER*). Throughout the exposition below we shall refer to the example program for the case with no background variables and with $\beta=1$ *a priori* (and t_i unrestricted), which is reproduced as a separate annex to this report.

It is convenient to distinguish between three parts of the main program:

I. The introductory tasks

II. The iterative maximization of the log-likelihood function, cf. section 4.3

III. Miscellaneous editing tasks

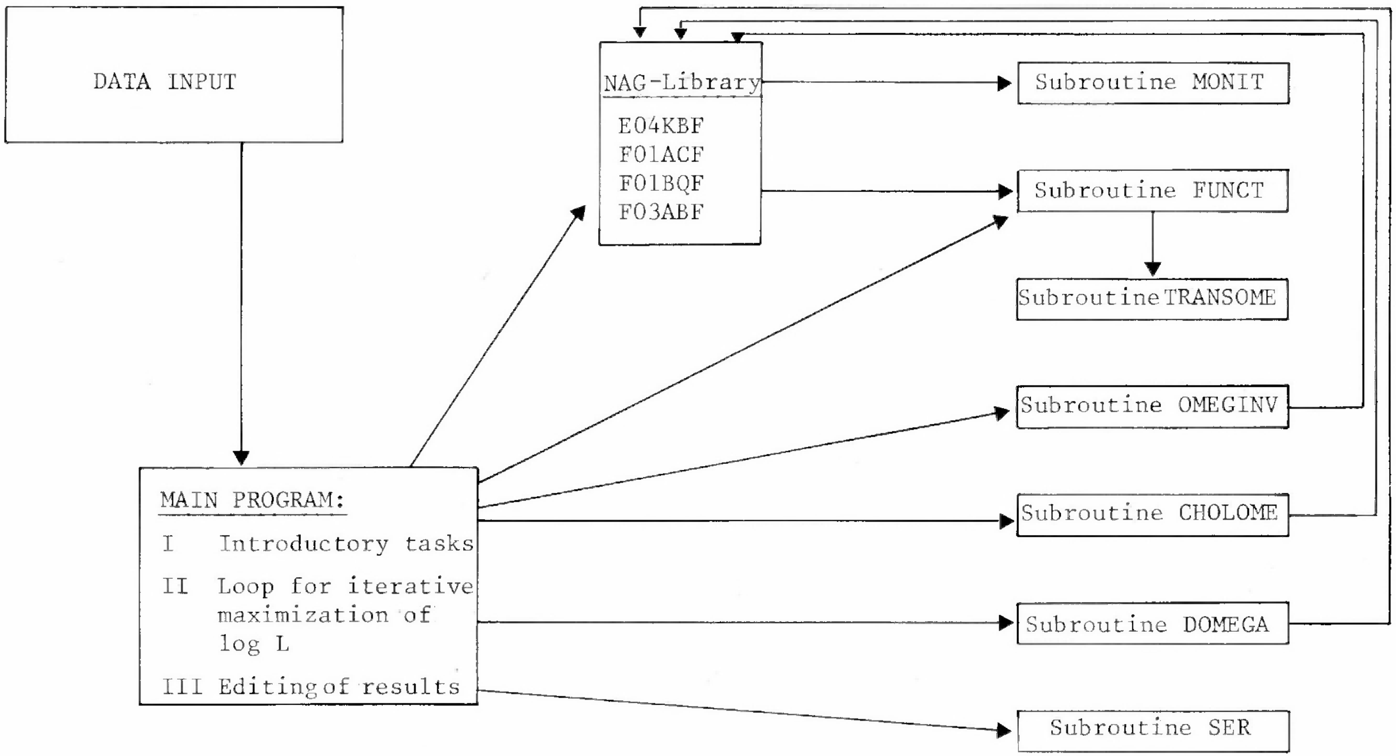
In part I, we define program variables and the data input is read from external files and subsequently rescaled. We also assign values to the input parameters of the NAG-routines, some of which will be commented upon below.

The iterative procedure for maximizing the log-likelihood function in our problem constitutes part II of our program, and it is implemented as a loop (labelled 2000), which is run twice - once for stage A (with $\Sigma_\mu=0$ *a priori*) and once for stage B (with Σ_μ unrestricted). As explained

6) This refers to the total CPU-time used on a Honeywell-Bull L60/6000 computer, cf. table H.1 below.

7) In the latter case, we reached the limit fixed on the number of iterations in stage A of the zig-zag FIML procedure after 1.3 CPU-hours. At that time the current estimate of α implied a value of the log-likelihood function equal to 2703.46, as compared with 2747.75 for the corresponding maximum found in one sixth of the time by means of the alternative routine.

Figure H.1. The structure of the computer programs



in section 4.3, the maximization procedure is completely analogous in the two stages. The logical variable *ASTAGE* is tested whenever differences between the two stages occur in the loop. The delimiting parameter of the loop, *NITER*, is set equal to 10. If the loop index *ITERA* equals 1, we go to label 590 in order to calculate $\hat{\Omega}_*$ (i.e. $\hat{\Sigma}$ in step A2 and $\hat{\Sigma}_\nu$ and $\hat{\Sigma}_\mu$ in step B2). Then we find its inverse, $\hat{\Omega}_*^{-1}$, by using a separate subroutine *(OMEGINV)*, which in turn calls the NAG-subroutine *F01ACE*. When inverting $\hat{\Omega}_*$, we do not exploit our a priori knowledge about the specific *structure* of Ω_* - given by equation (4.7) for stage A and equation (4.13) for stage B.[8] Further we proceed to calculate the Cholesky factorisation of Ω_* in the subroutine *CHOLOME*, which calls the NAG-subroutine *F01BQF*, to obtain the transformation matrix *U* (see Remark 2 of section 4.3). The matrix *U* is in turn vectorized into a transformation vector *BETA*. If the loop index *ITERA* equals 1, this completes steps A2 and B2, respectively, and we return to the top of the loop again.

Step A3 (or step B3) now consists in minimizing the quadratic form Q (eq. (H.1)) with respect to α, i.e. solving subproblem (i) of section 4.2. This is done by calling NAG-subroutine *E04KBF*. The algorithm is documented in NAG (1978) and given a brief description in section H.2 above. The user of NAG is required to supply two subroutines - *FUNCT* and *MONIT* - which are declared as EXTERNAL before calling *E04KBF*.

The first subroutine, *FUNCT*, is the most important one, since the value of Q and its first derivatives with respect to α are calculated here. A brief outline of this subroutine is in order: First, we calculate the residuals ε_{iht} for all i, h, and t. These are transformed into η_{iht} by means of the transformation vector *BETA* in a separate subroutine *TRANSOME* for all i, h, and t. We are now in the position to evaluate the function ($FC = Q(\alpha)$) for the current coefficient values. We proceed by calculating the derivatives $\partial\varepsilon_{iht}/\partial\alpha_k$ for all i, h, t and for all k, and transform the derivatives in the subroutine *TRANSOME* to obtain the derivatives of the transformed residuals.[9] Then we calculate the derivatives of Q as follows (confer eqs. (H.7) and (H.8)):

$$\text{(H.9)} \qquad \frac{\partial Q}{\partial \alpha_k} = 2 \sum_{i=1}^{N} \sum_{h=1}^{M} \sum_{t=1}^{2} \frac{\partial \eta_{iht}}{\partial \alpha_k}, \qquad k=1,\ldots,K.$$

---

8) The reason for this neglect of information is a purely practical one: The reduction in computer run-time to be gained from a reduction in the dimensions of the matrices to be inverted (from 2N x 2N to N x N) is negligible when N is as small as 4 and does not offset the extra programming efforts. With larger values of N, one might draw the opposite conclusion.
9) The simplicity of this procedure follows from the fact that the transformation is linear and dependent only on Ω_*, which is considered as fixed in this part of the program.

The second user-supplied subroutine, *MONIT*, is designed to monitor the progress (or lack of progress!) of the iterative minimization of Q in *E04KBF*.

Some input parameters must be assigned a value before calling the subroutine *E04KBF*. *MAXCAL* is the maximum number of times *FUNCT* is called by *E04KBF* and it is set equal to 2000.[10] *XTOL*, the parameter which specifies the desired accuracy in α, is set equal to $2.17\cdot 10^{-18}$ by default (i.e. 10 times the computer's accuracy in DOUBLE PRECISION). The input parameter *IBOUND* indicates whether the coefficients to be estimated are subject to inequality constraints (e.g. $0\leq t_i\leq 1$) or not, while the parameter vectors *ISTATE*, *BU*, and *BL* contain specific information on the nature of the constraints.

Provided the minimization is successful (or *MAXCAL* is reached), we return from *E04KBF* to the main program. There we proceed to reestimate Ω_* and to calculate a new transformation vector *BETA* as described above. When *ITERA*>1, however, we also calculate the changes in the estimated covariance matrices, $\hat{\Sigma}_\nu$ and $\hat{\Sigma}_\mu$, from the previous iteration[11] in order to test whether the norms of these changes satisfy the specified convergence criteria (given by equation (4.10) for stage A and equation (4.17) for stage B). The critical value *(KSI)* is set equal to 10^{-5}. When convergence is obtained, the program tests the logical variable *ASTAGE*. If this variable is "TRUE", we start all over again from label 410 (above the loop (labelled 2000)), otherwise part II of the program is terminated.

The final task (part III) of the main program is to edit the results. These routines are somewhat scattered around in the main program, and quite a few of them are, for practical reasons, performed in a separate program constructed for calculating the demand elasticities corresponding to the final estimates of α.[12] (See appendix A.) The log-likelihood function, however, is evaluated several times throughout the main program. The determinant of Ω_*, which is an element of this function,

10) One iteration with respect to α in *E04KBF* requires at least K(= *NVAR*) calls on *FUNCT*.

11) In stage A when $\Sigma_\mu=0$ *a priori* (and thus $\Sigma=\Sigma_\nu$), the sum of the changes in $\hat{\Sigma}_\mu$ equals 0 by assumption.

12) In this "ex post" program, we also compute the standard errors according to Method 1 of appendix G, which we consider the best approximation to the true standard errors of α. In the example program, which is reproduced in the program annex, the standard errors are calculated according to Method 2, which relies on the approximation to the Hessian matrix of Q which is built up in subroutine *E04KBF*. The inversion of this matrix, which is given on a Cholesky factorised form on exit from *E04KBF*, is carried out in the subroutine *SER* only in the cases where the Hessian is non-singular.

is calculated in a separate subroutine *DOMEGA*, which calls the NAG-subroutine *F03ABF*.

H.4. Experiences from the programming

The computer programs for the estimation of the complete demand models were designed with the explicit purpose to bring down the computer costs. The use of the NAG library at a Honeywell-Bull L60/6000 computer requires DOUBLE PRECISION to obtain a sufficient accuracy in the numerical calculations. Considering the size of the sample, this makes saving of storage space an important issue. Substantial gains were obtained by using COMMON fields to transfer the observations between the main program and the subroutine *FUNCT*. In addition, we obtained considerable reductions in storage space by a parsimonious use of working space vectors in *FUNCT*: When calculating the derivatives of the residuals ε_{iht} with respect to α_k (for all i, h, t, and k) in *FUNCT*, we exploited the fact that only the 2N residuals which relate to each of the M households are needed at the same time. The use of working space vectors of dimension 2MN, rather than the actual dimension 2N, would have increased the computer costs with a percentage ranging from 80 to 160.[13)]

The dominating part of the effective run-time on the computer is the time spent in the subroutine *FUNCT*, and we have therefore in several ways tried to increase the efficiency of this routine. Let us mention two:

- We have preferred use of vectors instead of matrices in *FUNCT*, since the computer finds a given position in a vector quicker than a given position in a matrix.[14)] It is possible, though not ascertained, that this causes a considerable gain, particularly in the calculation of the derivatives of the function to be minimized. On the other hand the costs in terms of programming man-hours by manipulating a lengthy and clumsy subroutine cast reasonable doubt on whether the efforts were worthwhile or not.

- Secondly, the use of a Cholesky factorisation of Ω_*^{-1} to establish the transformation vector *BETA* reduces the time spent in the supporting subroutine *TRANSOME* by approximately 40 per cent.[15)] This is important

13) The exact relative increase depends on the number of demand coefficients to be estimated (K).
14) The difference has an order of magnitude equal to the difference between the time the computer needs to multiply two numbers and the time needed for adding them.
15) This follows from the fact that the Cholesky factorisation (i.e. $U = \sqrt{D}'L'$, see Remark 2 in section 4.3) is triangular, and thus reduces the number of multiplications to K(K+1)/2 as against K^2 in the case with a transformation matrix with no zero elements.

because each call on *FUNCT* from *E04KBF* implies M(K+1) calls on *TRANSOME*.[16)]

Throughout the programming we have followed the principle of attacking the problems in an order of increasing complexity. We started with the single equation estimation (see chapter VII), which rendered valuable experience on the nature of the problems which were likely to emerge in FIML estimation of the complete model. Moreover, the testing of the various programs have, whenever possible, been performed on reduced samples (M=20). Finally, the parallel development of two alternative algorithms for simultaneous estimation served as a safeguard against programming errors, cf. section H.2 above.

Special mention should be made of the way in which we have created starting values for the iterative minimization of (H.1) with respect to α, see step A1 of section 4.3. As stated there, we have followed the principle of going from the most restricted cases to the more general formulations of the basic demand model. In doing this, we have utilized the estimation results from one variant as starting values for the estimation of the next one. For the first model variant to be estimated by simultaneous equation methods - i.e. the variant with no background variables and $\beta=1$ *a priori* - we chose starting values from a test run with a reduced sample (M=20).[17)]

In table H.1 we have recorded some computer statistics from the simultaneous estimation of the different demand model variants. The table contains the number of function evaluations (i.e. the number of calls on *FUNCT* made from *E04KBF*) in each iteration of stage A and stage B. The last two columns record two factors:

i) computer storage space required to run the model variant (in K = 1024 8-bits words),

ii) computer run-time measured in CPU-hours,

the product of which determines the computer costs.

---

16) Recall that M=418, $K\in(10,22)$ and that *MAXCAL* (i.e. the maximum number of calls on *FUNCT* from *E04KBF* in each iteration) is set to 2000. Cf. table H.1 below.

17) The initial values in the test run were, in turn, based on the results from the single equation estimation for the 5 aggregated commodities.

Table H.1. Number of iterations, function evaluations, and other computer statistics for the simultaneous estimation of the different variants of the complete model

Model variant		Number of function evaluations$^{\Delta)}$							Computer storage space (in K)	Total run-time in CPU-hours
		Stage A (Σ_μ=0)				Stage B (Σ_μ unrestricted)				
Background variables*)	Restrictions	Iteration				Iteration				
		1	2	3	4	1	2	3		
None	β=1, $0\leq t_i\leq 1$	389	136	28		134	40	23	75	0.37
None	β=1, t_i unrestr.	446	109	32		127	43	18	75	0.37
None	$0\leq t_i\leq 1$	105	22			71	28		84	0.29
None	None	164	15			123	49	31	84	0.42
A	β=1, $0\leq t_i\leq 1$	178	86	46	19	147	36	25	77	0.41
A	β=1, t_i unrestr.	245	47			154	39	27	77	0.34
n	β=1, $0\leq t_i\leq 1$	138	32	29		105	29	30	77	0.25
n	β=1, t_i unrestr.	298	59			166	48	30	77	0.40
n,A	β=1, $0\leq t_i\leq 1$	209	27	41		149	36	21	79	0.44
n,A	β=1, t_i unrestr.	357	75	33		164	32	36	79	0.56

Δ) I.e. the number of calls on *FUNCT*.

*) A = age of household head, n = number of household members.

The model variants are listed sequentially in accordance with the principle of increasing complexity. The statistics listed should be comparable between the variants, but for two exceptions:

- In the variants with no background variables and β=1 *a priori*, we did not calculate $\hat{\Sigma}$ from $\hat{\alpha}$ in the first iteration of stage A, but used an *a priori* estimate for $\hat{\Sigma}$ based on the test run from which $\hat{\alpha}$ was obtained.

- In the variant with age of household head as the only background variable and β=1 and $0\leq t_i\leq 1$ ($\forall$i) *a priori*, we used $\hat{\alpha}_k$ = 1.0 as starting values for the coefficients of the age variable[18], while the value zero would have been consistent with the other initial estimates applied in this variant.

18) Viz., s_{1A}, s_{2A}, s_{3A}, s_{4A}, a_A, and b_A.

```
C
C
C            MAIN PROGRAM
C            ============
C
C
C         BY EILEV S. JANSEN AND ANNE SAGSVEEN.
C
C
C         THIS PROGRAM IS DESIGNED TO FIND THE MINIMUM OF A GENERAL
C         FUNCTION BY MEANS OF A QUASI-NEWTON ITERATIVE MINIMIZATION
C         ALGORITHM. THE PROGRAM CALLS SEVERAL SUBROUTINES FROM THE NAG
C         LIBRARY (SEE NAG(1978)).
C
C         NOTE THAT THIS PARTICULAR VERSION OF THE PROGRAM IS MADE FOR
C         THE MODEL VARIANT WITHOUT BACKGROUND VARIABLES, WHERE THE
C         PARAMETER BETA IS SET EQUAL TO 1.0 A PRIORI.
C
C         PROGRAM TASKS:
C         **************
C         PART I     -  INTRODUCTORY TASKS
C         PART II    -  ITERATIVE MAXIMIZATION OF THE LOG-LIKELIHOOD
C                       FUNCTION DEFINED IN SECTION 4.3 OF THE MAIN TEXT.
C                       (THE PROGRAM MINIMIZES THE NEGATIVE OF THIS
C                       FUNCTION.)
C         PART III   -  MISCELLANEOUS EDITING TASKS.
C
C
C*********************************************************************
C
C          PART I    -  INTRODUCTORY TASKS
C
C*********************************************************************
C
C         VARIABLES WHICH ARE COMMON WITH OTHER SUBROUTINES.
C
C
      DOUBLE PRECISION  W1(836), W2(836), W3(836), W4(836),
     &      P1(836), P2(836), P3(836), P4(836), P5(836), Y(836),
     &      EPS1(836), EPS2(836), EPS3(836), EPS4(836), SP(836),
     &      ETA1(836), ETA2(836), ETA3(836), ETA4(836), N(836), A(836),
     &      S1(836), S2(836), S3(836), S4(836), AA(836), BB(836)
C
      LOGICAL  TRANSF
C
      INTEGER  K
C
      COMMON /BLK1/ TRANSF, K, W1, W2, W3, W4, N, A,
     &      P1, P2, P3, P4, P5, Y, SP,
     &      EPS1, EPS2, EPS3, EPS4, ETA1, ETA2, ETA3, ETA4,
     &      S1, S2, S3, S4, AA, BB
C
C
C  W1,..W4      -  BUDGET SHARES FOR COMMODITY GROUPS 1,...,4.
C  N            -  NO. OF PERSONS IN THE HOUSEHOLD.
C  A            -  AGE OF THE MAIN INCOME EARNER IN THE HOUSEHOLD.
C  P1,..P5      -  NOMINAL PRICES FOR THE 5 COMMODITY GROUPS.
C  Y            -  NOMINAL INCOME, I.E. TOTAL CONSUMPTION EXPENDITURE OF
C                  THE HOUSEHOLD.
C  SP           -  VALUE OF THE PRICE INDEX FUNCTION.
C  EPS1,..
C   ....,EPS4   -  UNTRANSFORMED RESIDUALS FOR COMMODITY GROUPS 1,...,4.
C  ETA1,..
C   ....,ETA4   -  TRANSFORMED RESIDUALS FOR COMMODITY GROUPS 1,...,4.
C  S1,..,S4     -  AUXILIARY VARIABLES
C  AA           -  AUXILIARY VARIABEL.
C  BB           -  AUXILIARY VARIABEL.
C  TRANSF       -  LOGICAL VARIABEL. IF TRUE, ALL RESIDUALS ARE TRANS-
C                  FORMED IN THE SUBROUTINE FUNCT.
C  K            -  NO. OF HOUSEHOLDS ( = NRESID/2).
C
```

```
C          BLOCK 2 OF COMMON VARIABLES.
C
      DOUBLE PRECISION  OMEGADET, BETA(36)
C
      COMMON /BLK2/  OMEGADET, BETA
C
C  OMEGADET    -   THE DETERMINANT OF OMEGA-STAR.
C  BETA        -   THE VECTORIZED CHOLESKY FACTORIZATION OF OMEGA-STAR.
C
C          LOCAL DOUBLE PRECISION VARIABLES.
C
      DOUBLE PRECISION  X(10), F, Q, XTOL, ETA, STEPMX, FEST, G(10),
     &        SNORM1, SNORM2, SIGMA(4,4), BL(10), BU(10), HESL(45),
     &        SIGMAI(4,4), SIGMAC(4,4), SIGMAO(4,4), SIGMAIO(4,4),
     &        DEL1(4,4), DEL2(4,4), SE(10),
     &        HESD(10), W(90), OMEGHJ(9,9), U(8,8),
     &        OMEGAST(8,8), MLLOG, XL(45), SD(10), TOL
C
      DATA  XTOL,  ETA, STEPMX, FEST,       BL,         BU
     &     /0.0D0,0.5D0, 1.0D3,0.0D0,6*-1.0D6,4*0.0D0,6*1.0D6,4*1.0D0/
C
      DATA  HESL, HESD, TOL  /45*0.0D0, 10*1.0D0, 1.0D-5/
C
C  X         -  VECTOR OF UNKNOWN STRUCTURAL COEFFICIENTS IN THE
C               DEMAND MODEL. THE VECTOR IS GIVEN A SET OF INITIAL
C               VALUES ON EXIT FROM E04KBF, X  CONTAINS THE ESTIMATES
C               OF THE COEFFICIENTS.
C  Q         -  VALUE OF THE QUADRATIC FORM DEFINED IN EQ. (4.4).
C  F         -  VALUE OF THE FUNCTION TO BE MINIMIZED IN SUBROUTINE
C            -  E04KBF. EQUAL TO F=Q*10**-2.
C  XTOL      -  ACCURACY (IN X) TO WHICH THE SOLUTION IS REQUIRED.
C  ETA       -  ACCURACY REQUIRED IN THE LINEAR SEARCH PROCEDURE. USED TO
C               DETERMINE THE OPTIMAL STEP-SIZE IN A PARTICULAR SEARCH
C               DIRECTION FOR EACH ITERATION IN SUBROUTINE E04KBF.
C  W         -  WORKSPACE IN SUBROUTINE E04KB.
C  G         -  VECTOR CONTAINING THE FIRST DERIVATIVES OF THE
C               MINIMAND F WITH RESPECT TO X. (I.E. THE GRADIENT.)
C  GT        -  ESTIMATES OF THE GRADIENT, CALCULATED IN SUBROUTINE
C               E04HBF.
C  SIGMA     -  THE TOTAL COVARIANCE MATRIX SIGMA.
C  SIGMAI    -  COVARIANCE MATRIX OF THE INDIVIDUAL COMPONENTS.
C  SIGMAC    -  COVARIANCE MATRIX OF THE REMAINDER COMPONENTS.
C  SIGMAO    -  VALUE OF THE LAST SIGMA MATRIX.
C  SIGMAIO   -  VALUE OF THE LAST SIGMAI MATRIX.
C  DEL1      -  DIFFERENCE BETWEEN SIGMA AND SIGMAO.
C  DEL2      -  DIFFERENCE BETWEEN SIGMAI AND SIGMAIO.
C  SNORM1    -  THE NORM OF THE DEL1 MATRIX.
C  SNORM2    -  THE NORM OF THE DEL2 MATRIX.
C  TOL       -  CONVERGENCE CRITERION FOR SNORM1 AND SNORM2.
C  U         -  CHOLESKY-FACTORIZASION OF OMEGA-STAR,
C               CALCULATED IN THE SUBROUTINE CHOLOME (F01BQF).
C  OMEGAST   -  THE COVARIANCE MATRIX OMEGA-STAR, DEFINED IN EQ (3.12)
C               OF THE MAIN TEXT.
C  OMEGAHJ   -  AUXILIARY MATRIX. USED TO STORE OMEGAST WHEN CALLING
C               SUBROUTINES CHOLOME AND OMEGINV.
C  MLLOG     -  VALUE OF THE LOG-LIKELIHOOD FUNCTION (EXCEPT FOR ITS
C               CONSTANT TERM ).
C  STEPMX    -  GUESTIMATE OF THE EUCLIDIAN DISTANCE BETWEEN THE FINAL
C               SOLUTION AND THE INITIAL VALUE OF F.
C  FEST      -  ESTIMATE OF THE FUNCTION VALUE AT THE MINIMUM.
C  BL        -  FIXED LOWER BOUNDS ON THE COEFFICIENT VECTOR X.
C  BU        -  FIXED UPPER BOUNDS ON THE COEFFICIENT VECTOR X.
C  HESD,
C     HESL   -  HESL AND HESD CONTAIN THE HESSIAN MATRIX OF F. (THE
C               SECOND DERIVATIVES WITH RESPECT TO X.) HESL IS A VECTOR
C               CONTAINING THE ELEMENTS BELOW THE DIAGONAL OF THE
C               HESSIAN, STACKED ROW BY ROW. HESD CONTAINS THE DIAGONAL
C               ELEMENTS. USED IN THE CALCULATION OF STANDARD ERRORS.
C  SE        -  VECTOR OF ESTIMATED STANDARD ERRORS.
C  XL        -  THE INVERSE OF THE STACKED MATRIX HESL, CALCULATED IN
C               SUBROUTINE SER.
C
```

```
C         LOCAL INTEGER VARIABLES.
C
C
      INTEGER  NRESID, NVAR, INTYPE, IPRINT, MAXCAL, IFAIL, NITER
      DATA     NRESID, NVAR, IPRINT, MAXCAL, IFAIL, NITER
     &        /   836,   10,   15  ,  2000 ,   0  ,   10  /
      INTEGER  NIN1, NOUT, NOUT2
      DATA     NIN1, NOUT, NOUT2
     &        /   1,    6,    8 /
      INTEGER   IBOUND, LH, ISTATE(10), IW(2), LIW, LW, IFLAG, NF
      DATA      IBOUND, LH, LIW, LW
     &        /      0, 45,   2, 90 /
      INTEGER  IAAR(836), ID(836)
C
C  NRESID  -  NO. OF RESIDUALS.
C  NVAR    -  NO. OF UNKNOWN COEFFICIENTS TO BE ESTIMATED.
C  INTYPE  -  INDICATES WHETHER THE USER PROVIDES INITIAL VALUES FOR
C             THE PROGRAM PARAMETERS F, G, HESL, HESD, AND ISTATE.
C  LW      -  DIMENSION OF W.
C  IPRINT  -  FREQUENCY WITH WHICH THE SUBROUTINE MONIT IS TO BE CALLED.
C  MAXCAL  -  MAX. NO. OF CALLS ON THE SUBROUTINE FUNCT TO  BE MADE
C             FROM THE SUBROUTINE E04KBF.
C  IFAIL   -  DIAGNOSTIC PARAMETER. UNLESS NAG-SUBROUTINES E04KBF,
C             F01ACF, F01BQF, OR F03ABF DETECT AN ERROR, IFAIL CONTAINS
C             0 ON EXIT.
C  NIN1    -  LOGICAL UNIT NUMBER FOR READING CONSUMER EXPENDITURE DATA.
C  NOUT    -  LOGICAL UNIT NUMBER FOR WRITING RESULTS.
C  NOUT2   -  LOGICAL UNIT NUMBER FOR WRITING RESULTS.
C  IAAR    -  YEAR OF BOOK-KEEPING.
C  ID      -  HOUSEHOLD IDENTIFICATION NUMBER.
C  NITER   -  MAX. NO. OF ITERATIONS IN STAGE A AND STAGE B OF THE
C             ITERATIVE MINIMIZATION IN PART II OF THIS PROGRAM.
C  IBOUND  -  SPECIFIES WHETHER THE MINIMIZATION PROBLEM  IS
C             CONSTRAINED OR NOT.
C  LH      -  DIMENSION OF HESL.
C  ISTATE  -  CONTAINS INFORMATION ABOUT WHICH VARIABLES HAVE CURRENTLY
C             THEIR BOUNDARY VALUES AND WHICH ARE FREE.
C  IW      -  WORKSPACE. VECTOR USED IN SUBROUTINE E04KBF.
C  LIW     -  DIMENSION OF IW.
C  IFLAG   -  INDICATES WHETHER AN EVALUATION OF THE GRADIENT IN
C             SUBROUTINE FUNCT IS REQUIRED.
C  NF      -  NO. OF CALLS ON SUBROUTINE FUNCT ACTUALLY MADE FROM
C             SUBROUTINE E04KBF.
C
C
C         LOCAL LOGICAL VARIABLES.
C
      LOGICAL  ASTAGE, LOCSCH
      DATA     LOCSCH
     &        /  .TRUE.  /
C
C
C  ASTAGE  - SPECIFIES WHETHER WE ARE IN STAGE A OR IN STAGE B
C            OF THE ITERATIVE MINIMIZATION . (SEE PART II.)
C  LOCSCH  - SPECIFIES WHETHER OR NOT WE WANT A LOCAL SEARCH TO BE
C            PERFORMED WHEN A POINT IS FOUND WHICH IS THOUGHT TO BE A
C            CONSTRAINED MINIMUM.
C
C
C         EXTERNAL SUBROUTINES.
C
      EXTERNAL  FUNCT, MONIT, E04JBQ
C
C  FUNCT   -  SUBROUTINE FOR CALCULATING THE FUNCTION VALUE AND ITS
C             DERIVATIVES AT A GIVEN POINT. NOTE ALSO THAT
C             THE RESIDUALS ARE COMPUTED IN THIS SUBROUTINE.
C  MONIT   -  SUBROUTINE MONITORING THE MINIMIZATION PROGRESS.
C  E04JBQ  -  SUBROUTINE USED IN NAG-SUBROUTINE E04KBF.
C
      K = NRESID/2
```

```
C
C
C          INITIAL VALUES FOR X.
C
      X(1) = 1.09117D-01
      X(2) = 1.40033D-01
      X(3) = 1.27203D-01
      X(4) = 5.60948D-01
      X(5) = 8.74848D 00
      X(6) = 1.67930D 01
      X(7) = 1.21819D-01
      X(8) = 1.38852D-01
      X(9) = 1.40332D-01
      X(10) = 5.38272D-01
C
C          READ, TRANSFORM, AND RESCALE THE DATA.
C
      READ(NIN1,99999) (N(I), A(I), W1(I), W2(I), W3(I),
     &        W4(I), Y(I), P1(I), P2(I), P3(I),
     &        P4(I), P5(I), I=1,836 )
      REWIND 1
      READ(NIN1,11000) (IAAR(I), ID(I), I= 1, 836 )
C
      DO 200 I = 1, 836
         W1(I) = W1(I) / Y(I)
         W2(I) = W2(I) / Y(I)
         W3(I) = W3(I) / Y(I)
         W4(I) = W4(I) / Y(I)
         N(I) = N(I) * 1.0D-2
         A(I) = A(I) * 1.0D-2
         Y(I) = Y(I) * 1.0D-4
 200  CONTINUE
C
      TRANSF = .FALSE.
      ASTAGE = .TRUE.
C
 410  CONTINUE
C
C*********************************************************************
C
C        PART II    -    ITERATIVE MAXIMIZATION OF THE LOG-LIKELIHOOD
C                        FUNCTION DEFINED IN SECTION 4.3 IN THE MAIN TEXT.
C                        (THE PROGRAM MINIMIZES THE NEGATIVE OF THIS
C                        FUNCTION.)
C
C*********************************************************************
C
C        THE LOOP LABELLED 2000 IS RUN TWICE. FIRST, IT IS RUN WITH THE
C        COVARIANCE MATRIX OF THE INDIVIDUAL COMPONENT OF THE RESIDUAL
C        (SIGMA1) EQUAL TO ZERO A PRIORI (STAGE A). THEN IT IS RUN WITH
C        NO RESTRICTIONS ON THE COVARIANCE MATRICES (STAGE B). THE LOGI-
C        CAL VARIABEL ASTAGE IS USED TO TEST WHETHER WE ARE IN STAGE A
C        OR IN STAGE B. SEE CHAPTER IV OF THE MAIN TEXT.
C
C
      DO 2000  ITERA = 1, NITER
         ITER = ITERA - 1
C
C        BEFORE CALLING THE NAG-SUBROUTINE E04KBF, WE NEED INITIAL ESTI-
C        MATES FOR THE COVARIANCE MATRICES. WE THEREFORE TEST ITERA.
C
         IF (ITERA .EQ. 1)  GO TO 590
            WRITE(NOUT,998) ITER, SNORM1, SNORM2
            INTYPE = 2
            IF (ITERA .EQ. 2)  INTYPE = 1
C
C         IFAIL IS SET EQUAL TO 1 BEFORE CALLING SUBROUTINE E04KBF.
C
            IFAIL = 1
```

```
C
         CALL E04KBF(NVAR,FUNCT,MONIT,IPRINT,LOCSCH,INTYPE,E04JBQ,
     &      MAXCAL,ETA,XTOL,STEPMX,FEST,IBOUND,BL,BU,X,HESL,LH,
     &      HESD,ISTATE,F,G,IW,LIW,W,LW,IFAIL)
C
C       THE FOLLOWING VARIABLES ARE CHANGED AFTER THIS CALL:
C       X, HESL, HESD, ISTATE, F, G, IFAIL.
C
C       WE TEST IFAIL FOR FAILURES.
C
         IF (IFAIL .EQ. 0)  GO TO 500
            WRITE(NOUT,97) IFAIL
            IF (IFAIL .EQ. 2)  GO TO 501
               GO TO 500
 501        CONTINUE
            WRITE(NOUT,996) MAXCAL
 500     CONTINUE
         Q = F * 1.0D2
C
C       PRELIMINARY RESULTS : THESE VALUES ARE USED AS INITIAL
C       ESTIMATES FOR F, X, AND G AT THE NEXT CALL ON SUBROUTINE
C       E04KBF (PROVIDED INTYPE IS SET EQUAL TO 2).
C
C
         WRITE(NOUT,9951) ITER, F, (J, X(J), G(J), J = 1, NVAR)
         WRITE(NOUT,96) Q
C
         MLLOG = - NRESID * 2 * ( DLOG(2.0 * 3.14159) )
     &      - 0.5 * K * DLOG(OMEGADET) - 0.5*Q - 0.5*K*DLOG(1.0D-16)
C
         WRITE(NOUT,988) MLLOG
C
 590    CONTINUE
        DO 600 I = 1, 4
           DO 601  J = 1, I
              SIGMAO(I,J) = SIGMA(I,J)
              SIGMAIO(I,J) = SIGMAI(I,J)
              SIGMA(I,J) = 0.0D0
              SIGMAI(I,J) = 0.0D0
              SIGMAC(I,J) = 0.0D0
 601       CONTINUE
 600    CONTINUE
C
C       WE MAKE AN EXTRA CALL ON SUBROUTINE FUNCT IN ORDER TO
C       COMPUTE THE RESIDUALS CORRESPONDING TO THE CURRENT
C       VALUES OF X. (STRICTLY, THIS IS ONLY NEEDED IN THE CASES
C       WHERE ITERA = 1.)
C
        IFLAG = 0
        CALL FUNCT(IFLAG,NVAR,X,F,G,IW,LIW,W,LW)
C
C
        DO 602 I = 1, NRESID
           SIGMA(1, 1) = EPS1(I) * EPS1(I) + SIGMA(1. 1)
           SIGMA(2, 2) = EPS2(I) * EPS2(I) + SIGMA(2, 2)
           SIGMA(3, 3) = EPS3(I) * EPS3(I) + SIGMA(3, 3)
           SIGMA(4, 4) = EPS4(I) * EPS4(I) + SIGMA(4, 4)
           SIGMA(2, 1) = EPS2(I) * EPS1(I) + SIGMA(2, 1)
           SIGMA(3, 1) = EPS3(I) * EPS1(I) + SIGMA(3, 1)
           SIGMA(3, 2) = EPS3(I) * EPS2(I) + SIGMA(3, 2)
           SIGMA(4, 1) = EPS4(I) * EPS1(I) + SIGMA(4, 1)
           SIGMA(4, 2) = EPS4(I) * EPS2(I) + SIGMA(4, 2)
           SIGMA(4, 3) = EPS4(I) * EPS3(I) + SIGMA(4, 3)
 602    CONTINUE
C
        IF ( ASTAGE )  GO TO 633
```

```
C
            DO 603 I = 1, K
               I1 = (I - 1) * 2 + 1
               I2 = I1 + 1
               SIGMAI(1, 1) = 2 * EPS1(I1) * EPS1(I2) + SIGMAI(1, 1)
               SIGMAI(2, 2) = 2 * EPS2(I1) * EPS2(I2) + SIGMAI(2, 2)
               SIGMAI(3, 3) = 2 * EPS3(I1) * EPS3(I2) + SIGMAI(3, 3)
               SIGMAI(4, 4) = 2 * EPS4(I1) * EPS4(I2) + SIGMAI(4, 4)
               SIGMAI(2, 1) = EPS2(I1) * EPS1(I2) + EPS2(I2) * EPS1(I1)
     &              +   SIGMAI(2, 1)
               SIGMAI(3, 1) = EPS3(I1) * EPS1(I2) + EPS3(I2) * EPS1(I1)
     &              +   SIGMAI(3, 1)
               SIGMAI(3, 2) = EPS3(I1) * EPS2(I2) + EPS3(I2) * EPS2(I1)
     &              +   SIGMAI(3, 2)
               SIGMAI(4, 1) = EPS4(I1) * EPS1(I2) + EPS4(I2) * EPS1(I1)
     &              +   SIGMAI(4, 1)
               SIGMAI(4, 2) = EPS4(I1) * EPS2(I2) + EPS4(I2) * EPS2(I1)
     &              +   SIGMAI(4, 2)
               SIGMAI(4, 3) = EPS4(I1) * EPS3(I2) + EPS4(I2) * EPS3(I1)
     &              +   SIGMAI(4, 3)
 603        CONTINUE
 633     CONTINUE
C
         DO 604 I = 1, 4
            DO 605 J = 1, I
               SIGMA(I,J) = SIGMA(I,J) / NRESID
               SIGMAI(I,J) = SIGMAI(I,J) / NRESID
               SIGMAC(I,J) = SIGMA(I,J) - SIGMAI(I,J)
               DEL1(I,J) = SIGMA(I,J) - SIGMAO(I,J)
               DEL2(I,J) = SIGMAI(I,J) - SIGMAIO(I,J)
 605        CONTINUE
 604     CONTINUE
C
C        PRELIMINARY RESULTS ARE PRINTED OUT.
C
         WRITE(NOUT,994) ITER
         DO 607 I = 1, 4
            DO 608 J = 1, I
               WRITE(NOUT,993) I, J, SIGMA(I,J), SIGMAI(I,J),
     &               SIGMAC(I,J), DEL1(I,J), DEL2(I,J)
 608        CONTINUE
 607     CONTINUE
C
C        COMPUTATION OF THE NORM OF DELRO.
C
         SNORM1 = 0.0D0
         SNORM2 = 0.0D0
         DO 700 I = 1, 4
         DO 700 J = 1, I
            SNORM1 = SNORM1 + DEL1(I,J) * DEL1(I,J)
            SNORM2 = SNORM2 + DEL2(I,J) * DEL2(I,J)
  700    CONTINUE
         SNORM1 = DSQRT(SNORM1)
         SNORM2 = DSQRT(SNORM2)
C
C        THE FOLLOWING TRANSFORMATIN IS BASED ON THE COVARIANCE
C        MATRIX OMEGA-STAR. FIRST, WE COMPUTE OMEGA-STAR.
C
         DO 800 I = 1, 4
         DO 800 J = 1, I
            SIGMA(J,I) = SIGMA(I,J)
            SIGMAI(J,I) = SIGMAI(I,J)
 800     CONTINUE
C
         DO 900 I = 1, 4
         DO 900 J = 1, 4
            I1 = 2 * (I - 1) + 1
            I2 = I1 + 1
            J1 = 2 * (J - 1) + 1
            J2 = J1 + 1
            OMEGAST(I1,J1) = SIGMA(I,J) * 1.0D02
            OMEGAST(I2,J2) = SIGMA(I,J) * 1.0D02
            OMEGAST(I1,J2) = SIGMAI(I,J) * 1.0D02
            OMEGAST(I2,J1) = SIGMAI(I,J) * 1.0D02
```

```
C
C         THEN, WE DEFINE AUXILIARY VARIABLES TO BE USED IN THE
C         SUBROUTINE OMEGINV.
C
             OMEGHJ(I1,J1) = OMEGAST(I1,J1)
             OMEGHJ(I2,J2) = OMEGAST(I2,J2)
             OMEGHJ(I1,J2) = OMEGAST(I1,J2)
             OMEGHJ(I2,J1) = OMEGAST(I2,J1)
 900      CONTINUE
C
          WRITE(NOUT,199)
          DO 1000 I = 1, 8
             WRITE(NOUT,198) (OMEGAST(I,J), J = 1, 8)
 1000     CONTINUE
C
C         NOTE: OMEGA-STAR IS A REAL, SYMMETRIC, AND POSITIVE
C               DEFINITE MATRIX.
C
          CALL OMEGINV(OMEGHJ)
C
          CALL CHOLOME(OMEGHJ,U)
C
C         WE STORE THE CHOLESKY FACTORIZATION OF OMEGHJ IN BETA.
C
          DO 1050 I = 1, 8
          DO 1050 J = 1, I
             IND = I * (I - 1) / 2 + J
             BETA(IND) = U(I,J)
 1050     CONTINUE
C
          WRITE(NOUT,98)
          WRITE(NOUT,99) (BETA(J), J = 1, 36)
C
          CALL DOMEGA(OMEGAST,OMEGADET)
C
          IF (ITERA .GT. 1) GO TO 1150
C
C         THE GRADIENT IS TESTED BY CALLING NAG-SUBROUTINE E04HCF.
C         E04HCF PROVIDES INITIAL VALUES FOR THE NEXT CALL ON SUBROUTINE
C         E04KBF, WHICH IS USED IF INTYPE=1.
C
             TRANSF = .TRUE.
             IFAIL = 1
             CALL E04HCF(NVAR,FUNCT,X,F,G,IW,LIW,W,LW,IFAIL)
             IF (IFAIL .EQ. 0) GO TO 1140
                WRITE(NOUT,997) IFAIL
 1140        CONTINUE
C
             WRITE(NOUT,194) F
             Q = F * 1.0D2
             WRITE(NOUT,96) Q
             WRITE(NOUT,193)
             DO 1130 J = 1, NVAR
                WRITE(NOUT,1193) J, X(J), G(J)
 1130        CONTINUE
C
             MLLOG = - NRESID * 2 * (DLOG(2.0 * 3.14159))
     &          - 0.5 *K* DLOG(OMEGADET) - 0.5*Q - 0.5*K*DLOG(1.0D-16)
             WRITE(NOUT,988) MLLOG
C
             GO TO 2000
 1150     CONTINUE
C
C         THIS POINT IN THE PROGRAM IS REACHED ONLY IF ITERA IS GREATER
C         THAN 1. ITS TASK IS THEN TO COMPUTE STANDARD ERRORS OF THE
C         COEFFICIENT ESTIMATES.
C
C         FIRST, WE TEST WHETHER ALL COEFFICIENTS ARE FREE OR NOT.
C         IF AT LEAST ONE IS AT ITS BOUNDARY, METHOD 2 (SEE APPENDIX G)
C         CANNOT BE USED, AND THIS TASK IS SKIPPED.
C
          DO 1175 J = 1, NVAR
             IF (ISTATE(J) .NE. J)  GO TO 1300
 1175     CONTINUE
             CALL SER(NVAR, LH, HESL, HESD, XL, SE)
C
             DO 1200 I = 1, NVAR
                SE(I) = DSQRT(2.0D0 * SE(I)) * 1.0D-1
 1200        CONTINUE
```

```
C
C         PRINT THE RESULTS.
C
             WRITE(NOUT,191)
             DO 1250 I = 1, NVAR
                WRITE(NOUT,1193) I, X(I), SE(I)
 1250        CONTINUE
C
 1300     CONTINUE
C
          IFLAG = 2
          CALL FUNCT(IFLAG, NVAR, X, F, G, IW, LIW, W, LW)
          Q = F * 1.0D2
          MLLOG = - NRESID * 2 * (DLOG(2.0 * 3.14159))
     &       -0.5 * K * DLOG(OMEGADET)-0.5 * K * DLOG(1.0D-16)-0.5 * Q
          WRITE(NOUT,988) MLLOG
C
C         TESTS OF CONVERGENCE WITH RESPECT TO SIGMA AND SIGMAI.
C
          IF (SNORM1.LT.TOL .AND. SNORM2.LT.TOL)  GO TO 3000
C
 2000  CONTINUE
```

```
C
C**********************************************************************
C
C          PART III  -   MISCELLANEOUS EDITING TASKS.THE BULK OF THESE
C                        TASKS ARE PERFORMED IN A SEPARATE PROGRAM,
C                        WHICH IS NOT INCLUDED IN THIS ANNEX. THERE WE
C                        COMPUTE STANDARD ERRORS ACCORDING TO METHOD 1,
C                        AND ESTIMATED BUDGET SHARES AND DEMAND ELASTI-
C                        CITIES FOR CERTAIN VALUES OF THE EXOGENOUS
C                        VARIABLES (REAL INCOME, REAL PRICES, AND
C                        (IF INCLUDED) BACKGROUND VARIABLES.)
C
C**********************************************************************
C
      WRITE(NOUT,189)
 3000 WRITE(NOUT,188) ITER, F
      Q = F * 1.0D2
      WRITE(NOUT,96) Q
C
      WRITE(NOUT,9952) (J, HESD(J), ISTATE(J), J = 1, NVAR)
      WRITE(NOUT,9953) (HESL(J), J = 1, LH)
C
      IF ( .NOT. ASTAGE)  GO TO 5000
         ASTAGE = .FALSE.
         GO TO 410
 5000 CONTINUE
      STOP
99999 FORMAT(26X,2F2.0,T463,4F9.2,T508,F9.2,T775,5F8.6)
11000 FORMAT(I2,T10,I4)
  998 FORMAT(1H1,5X,"CONDITIONAL MINIMIZATION - GIVEN OMEGA)  -  ",
     &        "  ITERATION = ",I2,"  SNORM1  ="D12.4,
     &        " SNORM2 =",D12.4)
  997 FORMAT(//,15X,"IFAIL=",I3,5X,"ERROR FOUND IN E04HCF")
  996 FORMAT(/,15X,"MAX. NO. OF CALLS ON FUNCT=",I5)
 9951 FORMAT(1H0,"AFTER",I3," ITERATIONS THE MINIMUM OF THE",
     &        " FUNCTION IS  :", F12.5,/1H0,6HINDEX ,3X,
     &        13HCOEFFICIENT  ,3X,"  GRADIENT      ",
     &        /(1X,I3,5X,1PD13.5,7X,1PD13.5))
 9952 FORMAT(1H0,6HINDEX ,9X,7HHESD(J),3X,"ISTATE(J)",
     &        /(1X,I3,5X,1PD13.5,3X,I3))
 9953 FORMAT(1H0,10X,"HESL(J):",/(5(3X,1PD13.5)))
12000 FORMAT(1H0,15X,"RESID.  AFTER ",I5,"ITERATIONS :",/,
     &        10X," EPS1",10X," ETA1",10X," EPS2",10X," ETA2",
     &        10X," EPS3",10X," ETA3",10X," EPS4",10X," ETA4")
12001 FORMAT(1H0,8(3X,E12.4))
  988 FORMAT(//," VALUE OF THE LOG-LIKELIHOOD FUNCTION =", D14.6)
  994 FORMAT(1H1,5X,"AFTER",I3," ITERATIONS THE FOLLOWING ESTIMATES",
     &        " FOR THE COVARIANCE MATRICES ARE OBTAINED :",/,
     &        5X,"COM. I",4X,"COM. J",4X,"    SIGMA",5X,
     &    "     SIGMAI",5X,"     SIGMAC",5X,"       DEL1",5X,"      DEL2")
  993 FORMAT(5X,I5,5X,I5,5(3X,D12.4))
  199 FORMAT(/,15X," OMEGA-STAR(8,8) ,MULTIPLIED BY  10**2",/)
  198 FORMAT(1H0, 8(3X, D12.4))
  194 FORMAT(1H0,"RESULTS FROM SUBROUTINE FUNCT:",/,5X,
     &        "THE FUNCTION VALUE =",1PD20.10)
  193 FORMAT(5X,"INDEX ",5X,"COEFFICIENT  ",8X,"GRADIENT",/)
 1193 FORMAT(8X,I3,1PD13.5,5X,1PD20.10)
  191 FORMAT(/,"THE OPTIMAL VALUES ANDS THEIR STANDARD ERRORS",
     &        //,10X,"INDEX ",5X,"COEFFICIENT  ",5X,"STANDARD ERROR")
  189 FORMAT(//," MAXIMUM NO. OF ITERATIONS (MINIMIZATIONS)",
     &        " IS REACHED")
  188 FORMAT(//,"FINAL MINIMUM OF THE FUNCTION",/,
     &        "AFTER",I5," MINIZATIONS EQUALS =", D12.4)
   99 FORMAT(6(3X,D13.5))
   98 FORMAT(//,3X,"BETA-VECTOR EQUALS:",/)
   97 FORMAT(//,15X,"ERROR DETECTED IN E04KBF, IFAIL=",I3)
   96 FORMAT(/,"THE SUM OF SQUARES Q = F*10**2 =",1PD20.10,//)
      END
```

```
C
C
C         SUBROUTINE OMEGINV
C         ==================
C
C         PURPOSE:
C         ========
C         TO FIND THE INVERSE OF A REAL, SYMMETRIC MATRIX,WHICH IS
C         POSITIVE DEFINITE. OMEGINV CALLS NAG-SUBROUTINE F01ACF.
C         SEE NAG(1978).
C
      SUBROUTINE OMEGINV(A)
C
      DOUBLE PRECISION  A(9,9), B(9,9), Z(8), X02AAF
C
C  A      -    MATRIX TO BE INVERTED.
C  B      -    WORKING SPACE (MATRIX).
C  Z      -    DITTO.
C  X02AAF-     SMALLEST NUMBER WHICH CAN BE DISTINGUISHED BY A HONEYWELL-
C              BULL L60/6000 COMPUTER. CF. SUBROUTINE X02AAF, SEE
C              NAG(1978).
C
      INTEGER  NOUT, N, IA, IB, L, IFAIL, IT
C
C  NOUT   -    LOGICAL UNIT NUMBER FOR PRINTER.
C  N      -    THE ORDER OF THE MATRIX A.
C  IA     -    THE FIRST DIMENSION OF MATRIX A.
C  IB     -    THE FIRST DIMENSION OF MATRIX B.
C  L      -    NUMBER OF CORRECTIONS. EVALUATED IN SUBROUTINE F01ACF.
C  IFAIL  -    DIAGNOSTIC VARIABEL. USED IN SUBROUTINE F01ACF.
C  IT     -    "DUMMY" VARIABEL USED IN SUBROUTINE X02AAF.
C
      DATA  NOUT /6/
      N = 8
      IA = 9
      IB = 9
      IFAIL = 1
      CALL F01ACF(N, X02AAF(IT), A, IA, B, IB, Z, L, IFAIL )
      WRITE(NOUT,99999)
      IF ( IFAIL .EQ. 0 ) GO TO 20
      WRITE(NOUT,99998) IFAIL
      STOP
C
C
   20 WRITE(NOUT,99997)
      DO 40 I = 1, N
         WRITE(NOUT,99996) (A(I+1,J), J= 1, I)
   40 CONTINUE
      WRITE(NOUT,99995) L
      DO 50 I = 1, N
      DO 50 J = 1, I
         A(I,J) = A(I+1,J)
   50 CONTINUE
99999 FORMAT(///,15X,"REPORT FROM SUBROUTINE OMEGINV:")
99998 FORMAT(/,20X,"ERROR IN F01ACF   IFAIL=",I2)
99997 FORMAT(20X,"LOWER TRIANGLE OF THE INVERSE:")
99996 FORMAT(20X,8D12.4)
99995 FORMAT(20X,"RESULT FOUND AFTER  ",I2,"ITERASIONS.")
      RETURN
      END
C
C
C          SUBROUTINE CHOLOME
C          ==================
C
C          PURPOSE:
C          =======
C          TO FIND THE CHOLESKY-FACTORIZATION OF A REAL,
C          SYMMETRIC, AND POSITIVE DEFINITE MATRIX.
C          CALLS NAG-SUBROUTINE F01BQF.
```

```
C
      SUBROUTINE CHOLOME(A, B)
C
C        LOCAL DOUBLE PRECISION VARIABLES.
C
      DOUBLE PRECISION A(9,9), B(8,8), EPS, X02AAF, RL(28), D(3)
C
C  A      -    CONTAINS THE MATRIX TO BE FACTORIZED.
C  B      -    THE CHOLESKY-FACTORIZED MATRIX. (EQUALS RL'SQRT(D).)
C  EPS    -    THE SMALLEST POSITIVE NUMBER TO BE DISTINGUISHED BY
C              THE HONEYWELL COMPUTER.(FROM  NAG-SUBROUTINE X02AAF.)
C  RL     -    ON ENTRY TO SUBROUTINE F01BQF, RL CONTAINS THE LOWER
C              TRIANGLE OF A. ON EXIT RL CONTAINS THE LOWER TRIANGLE
C              OF B.
C  D      -    ON ENTRY TO SUBROUTINE F01BQF, D CONTAINS THE DIAGONAL
C              OF A. ON EXIT, D CONTAINS THE SQUARED DIAGONAL ELEMENTS
C              OF B.
C
C        LOCAL INTEGER VARIABLES.
C
      INTEGER IFAIL, M, N, NOUT
C
C  IFAIL -    DIAGNOSTICS VARIABEL.
C  N     -    DIMENSION OF MATRIX B.
C  M     -    DIMENSION OF VECTOR RL.
C  NOUT  -    LOGICAL UNIT NUMBER FOR PRINTER.
C
      DATA NOUT / 6 /
      N = 3
      M = N * (N - 1) / 2
      IFAIL = 1
      EPS = X02AAF(EPS)
C
C        CREATE RL AND D FROM A.
C
      I1 = 0
      D(1) = A(1,1)
      DO 100 I = 2, N
         I2 = I - 1
         D(I) = A(I,I)
         DO 101 J = 1, I2
            I1 = I1 + 1
            RL(I1) = A(I,J)
 101     CONTINUE
 100  CONTINUE
C
      CALL F01BQF( N, EPS, RL, M, D, IFAIL )
C
C        EDIT OUTPUT.
C
      WRITE(NOUT,99999)
      IF ( IFAIL .EQ. 0 )  GO TO 200
         WRITE(NOUT,99998) IFAIL
         STOP
C
 200  WRITE(NOUT,99997) ( D(I), I = 1, N )
      WRITE(NOUT,99996)
      DO 250 I = 1, 7
         I1 = I * (I - 1) / 2 + 1
         I2 = I1 + I - 1
         WRITE(NOUT,99995) (RL(J), J = I1, I2)
 250  CONTINUE
C
C        CREATE MATRIX B. ( SEE DEFINITION ABOVE.)
C
      I1 = 0
      B(1,1) = DSQRT(D(1))
      DO 300 I = 2, N
         I2 = I - 1
         B(I,I) = DSQRT(D(I))
         DO 301 J = 1, I2
            I1 = I1 + 1
            B(I,J) = RL(I1) * B(J,J)
 301     CONTINUE
```

```
 300    CONTINUE
99999   FORMAT(///,15X," REPORT FROM SUBROUTINE CHOLOME : ")
99998   FORMAT(20X,"ERROR IN SUBROUTINE F01BQF     IFAIL=",I2)
99997   FORMAT(20X,"THE DIAGONAL D (I.E. D**2) : ",/,
     &        (20X,4(D12.4,2X) ) )
99996   FORMAT(20X,"THE LOWER TRIANGLE (I.E.  RL ) :",/)
99995   FORMAT(20X,8(D12.4))
        RETURN
        END
C
C
C           SUBROUTINE DOMEGA
C           =================
C
C           PURPOSE:
C           ========
C           COMPUTATION OF THE DETERMINANT OF OMEGA-STAR.
C           CALLS NAG-SUBROUTINE F03ABF.
C
        SUBROUTINE DOMEGA(A, DET)
C
C           LOCAL DOUBLE PRECISION VARIABLES.
C
        DOUBLE PRECISION A(8,8), DET, WKSPCE(8)
C
C   A       -     OMEGA-STAR.
C   DET     -     DETERMINANT OF OMEGA-STAR.
C   WKSPCE-       WORKING SPACE USED IN SUBROUTINE F03ABF.
C
C           LOCAL INTEGER VARIABLES.
C
        INTEGER IA, N, IFAIL, NOUT
C
C   IA      -     NO. OF ROWS IN THE MATRIX A.
C   N       -     NO. OF COLUMNS IN THE MATRIX A.
C   IFAIL   -     DIAGNOSTIC VARIABLE.
C   NOUT    -     LOGICAL UNIT NUMBER FOR PRINTER.
C
        DATA NOUT /6/
        N = 8
        IA = 8
        IFAIL = 1
C
        CALL F03ABF(A, IA, N, DET, WKSPCE, IFAIL)
C
        WRITE(NOUT,99999)
        IF(IFAIL .EQ. 0)   GO TO 100
           WRITE(NOUT,99998) IFAIL
           STOP
 100    WRITE(NOUT,99997) DET
C
99999   FORMAT(///,15X,"REPORT FROM SUBROUTINE  DOMEGA:")
99998   FORMAT(20X,"ERROR IN SUBROUTINE F03ABF     IFAIL=",I2)
99997   FORMAT(20X,"DETERMINANT OF OMEGA-STAR=",D14.6,"*1.0D-16")
        RETURN
        END
C
C
C           SUBROUTINE TRANSOME
C           ===================
C
C           PURPOSE:
C           ========
C           TRANSFORMATION OF THE RESIDUALS EPSILON INTO THE
C           TRANSFORMED RESIDUALS ETA. THIS SUBROUTINE IS ALSO
C           USED FOR TRANSFORMING THE DERIVATIVES OF THE RESIDUALS.
```

```
C
      SUBROUTINE TRANSOME(A, X, TX)
      DOUBLE PRECISION A(36), X(8), TX(8)
C
C  A      -    VECTOR OF COEFFICIENTS USED FOR THE TRANSFORMATION.
C  X      -    VECTOR TO BE TRANSFORMED.
C  TX     -    TRANSFORMED VECTOR OF RESIDUALS.
C
      TX(1) = A(1) * X(1) + A(2) * X(2) + A(4) * X(3) + A(7) * X(4)
     &        + A(11)* X(5) + A(16)* X(6) + A(22)* X(7) + A(29)* X(8)
      TX(2) =               A(3) * X(2) + A(5) * X(3) + A(8) * X(4)
     &        + A(12)* X(5) + A(17)* X(6) + A(23)* X(7) + A(30)* X(8)
      TX(3) =                             A(6) * X(3) + A(9) * X(4)
     &        + A(13)* X(5) + A(18)* X(6) + A(24) * X(7) + A(31)* X(8)
      TX(4) =                                           A(10)* X(4)
     &        + A(14)* X(5) + A(19)* X(6) + A(25) * X(7) + A(32)* X(8)
      TX(5) = A(15)* X(5) + A(20)* X(6) + A(26) * X(7) + A(33)* X(8)
      TX(6) =               A(21)* X(6) + A(27) * X(7) + A(34)* X(8)
      TX(7) =                             A(28) * X(7) + A(35)* X(8)
      TX(8) =                                           A(36)* X(8)
      RETURN
      END
C
C
C         SUBROUTINE FUNCT
C         ================
C
C         PURPOSE:
C         ========
C         COMPUTATION OF THE FUNCTION VALUE FC AND ITS FIRST
C         DERIVATIVES AT THE POINT XC. THE CONTROL PARAMETER IFLAG
C         IS SET EQUAL TO 0 OR 2 BY THE NAG-ROUTINES WHICH GOVERN
C         FUNCT.
C         IF IFLAG=2 BOTH FC AND GC ARE COMPUTED,
C         IF IFLAG=0 ONLY FC IS COMPUTED.
C         THOSE VARIABLES WHICH ARE NOT GIVEN A DEFINITION BELOW, ARE
C         DEFINED AS IN THE MAIN PROGRAM.
C
      SUBROUTINE FUNCT(IFLAG, NVAR, XC, FC, GC, IW, LIW, W, LW)
C
C         INTEGER VARIABLES (DEFINED IN THE MAIN PROGRAM).
C
      INTEGER
     &    IFLAG, NVAR, LIW, LW, IW(2), K, IANT, NRESID, NOUT
C
C         COMMON DOUBLE PRECISION VARIABLES
C
      DOUBLE PRECISION  W1(836), W2(836), W3(836), W4(836), SP(836),
     &        P1(836), P2(836), P3(836), P4(836), P5(836), Y(836),
     &        EPS1(836), EPS2(836), EPS3(836), EPS4(836),
     &        ETA1(836), ETA2(836), ETA3(836), ETA4(836), N(836), A(836),
     &        S1(836), S2(836), S3(836), S4(836), AA(836), BB(836)
C
C         COMMON LOGICAL VARIABLE.
C
      LOGICAL TRANSF
C
C         LOCAL DOUBLE PRECISION VARIABLES
C
      DOUBLE PRECISION
     &        XC(NVAR), FC, GC(NVAR), W(LW),
     &        SUMF, SUMG1, SUMG2, SUMG3, SUMG4, SUMG5, SUMG6, SUMG7,
     &        SUMG8, SUMG9, SUMG10,
     &        DER1(8), DER2(8), DER3(8),
     &        DER4(8), DER5(8), DER6(8), DER7(8), DER8(8),
     &        DER9(8), DER10(8), TDER1(8), TDER2(8),
     &        TDER3(8), TDER4(8), TDER5(8), TDER6(8), TDER7(8),
     &        TDER8(8),  TDER9(8), TDER10(8), ESJ(8), KEB(8),
     &        HJA1, HJA2, HJB1, HJB2, HJC1,
     &        HJC2, HJD1, HJD2, HJE1, HJE2,
     &        HJF1, HJF2,
     &        HELP
```

```
C
C   XC        -   CURRENT COEFFICIENT VALUES.
C   FC        -   FUNCTION VALUE AT XC.
C   GC        -   GRADIENT VALUE AT XC.
C   W         -   WORKING SPACE VECTOR (SEE MAIN PROGRAM).
C   SUMF,...
C   ..,SUMG10-    AUXILIARY SUMMATION VARIABLES.
C   DER1,...
C   ...,DER10-    VECTORS USED TO STORE THE DERIVATIVES OF THE
C                 RESIDUALS WITH RESPECT TO EACH OF THE UNKNOWN
C                 COEFFICIENTS.
C   TDER1,..
C   ..,TDER10-    VECTORS USED TO STORE THE TRANSFORMED VECTORS
C                 DER1,..........,DER10.
C   ESJ       -   VECTOR USED TO STORE THE RESIDUALS RELATING TO ONE
C                 PARTICULAR HOUSEHOLD.
C   KEB       -   THE VECTOR ESJ AFTER TRANSFORMATION.
C   HJA1,...
C   ...,HELP -    AUXILIARY INTERMEDIATE VARIABLES.
C
      COMMON /BLK1/ TRANSF,K,W1,W2,W3,W4,N,A,
     &         P1, P2, P3, P4, P5, Y, SP,
     &         EPS1, EPS2, EPS3, EPS4, ETA1, ETA2, ETA3, ETA4,
     &         S1, S2, S3, S4, AA, BB
C
      DATA NRESID,NOUT
     &      /  836,    6/
C
C        COMMON DOUBLE PRECISION VARIABLES (BLOCK 2)
C
      DOUBLE PRECISION OMEGADET,BETA(36)
C
C
      COMMON /BLK2/ OMEGADET, BETA
C
C        HELP IS USED TO ENSURE THAT THE WEIGTHS OF THE PRICE INDEX
C        FUNCTION ADD TO UNITY.
C
      HELP = 1.0 - XC(7) -XC(8) -XC(9) - XC(10)
C
C        TEST POSITIVITY OF HELP
C
      IF (HELP .GT. 0.0D0)  GO TO 10
C
         WRITE(NOUT,99) HELP
         WRITE(NOUT,98) (XC(J), J = 7, 10)
C
 10   CONTINUE
      DO 100 I = 1, NRESID
         SP(I) = XC(7) * P1(I) + XC(8) * P2(I) + XC(9) * P3(I)
     &         + XC(10) * P4(I) + HELP * P5(I)
 100  CONTINUE
C
C        CREATE A SET OF AUXILIARY VARIABLES.
C
      DO 101 I = 1, NRESID
         S1(I) = XC(7) * ( P1(I) / SP(I) ) - XC(1)
         S2(I) = XC(8) * ( P2(I) / SP(I) ) - XC(2)
         S3(I) = XC(9) * ( P3(I) / SP(I) ) - XC(3)
         S4(I) = XC(10) * ( P4(I) / SP(I) ) - XC(4)
         BB(I) = Y(I) / SP(I)
         AA(I) = XC(5) * DLOG(BB(I)) + XC(6)
 101  CONTINUE
C
C        CREATE THE VECTORS OF RESIDUALS EPS1 THROUGH EPS4.
C
      DO 200  I = 1, NRESID
         EPS1(I) = W1(I) - XC(1) - S1(I) * AA(I) / BB(I)
         EPS2(I) = W2(I) - XC(2) - S2(I) * AA(I) / BB(I)
         EPS3(I) = W3(I) - XC(3) - S3(I) * AA(I) / BB(I)
         EPS4(I) = W4(I) - XC(4) - S4(I) * AA(I) / BB(I)
 200  CONTINUE
```

```
C
C         TRANSFORMATION OF THE RESIDUALS:
C         WE CALL SUBROUTINE TRANSOME FOR THE RESIDUALS WHICH CORRESPOND
C         TO THE SAME HOUSEHOLD.
C
      DO 600 I = 1, K
         I1 = 2 * (I - 1) + 1
         I2 = I1 + 1
         ESJ(1) = EPS1(I1)
         ESJ(2) = EPS1(I2)
         ESJ(3) = EPS2(I1)
         ESJ(4) = EPS2(I2)
         ESJ(5) = EPS3(I1)
         ESJ(6) = EPS3(I2)
         ESJ(7) = EPS4(I1)
         ESJ(8) = EPS4(I2)
         CALL TRANSOME(BETA, ESJ, KEB)
         ETA1(I1) = KEB(1)
         ETA1(I2) = KEB(2)
         ETA2(I1) = KEB(3)
         ETA2(I2) = KEB(4)
         ETA3(I1) = KEB(5)
         ETA3(I2) = KEB(6)
         ETA4(I1) = KEB(7)
         ETA4(I2) = KEB(8)
 600  CONTINUE
C
      SUMF = 0.0D0
      DO 700 I = 1, K
         I1 = 2 * (I - 1) + 1
         I2 = I1 + 1
         SUMF = SUMF + ETA1(I1)*ETA1(I1) + ETA1(I2)*ETA1(I2)
     &               + ETA2(I1)*ETA2(I1) + ETA2(I2)*ETA2(I2)
     &               + ETA3(I1)*ETA3(I1) + ETA3(I2)*ETA3(I2)
     &               + ETA4(I1)*ETA4(I1) + ETA4(I2)*ETA4(I2)
 700  CONTINUE
      FC = SUMF
C
C        CALCULATION OF DERIVATIVES.
C
      IF (IFLAG .EQ. 0)  RETURN
C
C
      SUMG1 = 0.0D0
      SUMG2 = 0.0D0
      SUMG3 = 0.0D0
      SUMG4 = 0.0D0
      SUMG5 = 0.0D0
      SUMG6 = 0.0D0
      SUMG7 = 0.0D0
      SUMG8 = 0.0D0
      SUMG9 = 0.0D0
      SUMG10 = 0.0D0
C
C        BEFORE ENTERING THE LOOP LABELLED 1000, WE SET THE DERIVATIVES
C        WHICH SHALL ALWAYS BE ZERO, EQUAL TO THIS VALUE.
C
      DO 50 I = 1, 8
         DER1(I) = 0.0D0
         DER2(I) = 0.0D0
         DER3(I) = 0.0D0
         DER4(I) = 0.0D0
 50   CONTINUE
```

```
C
      DO 1000 I = 1, K
         I1 = 2 * (I - 1) + 1
         I2 = I1 + 1
C
         HJA1 = AA(I1) / BB(I1)
         HJA2 = AA(I2) / BB(I2)
         HJB1 = DLOG(BB(I1))
         HJB2 = DLOG(BB(I2))
         HJC1 = HJA1 / SP(I1)
         HJC2 = HJA2 / SP(I2)
         HJD1 = HJC1 / SP(I1)
         HJD2 = HJC2 / SP(I2)
         HJE1 = XC(5) / ( SP(I1) * BB(I1))
         HJE2 = XC(5) / ( SP(I2) * BB(I2))
         HJF1 = AA(I1) / Y(I1)
         HJF2 = AA(I2) / Y(I2)
C
         DER1(1) = - 1.0D0 + HJA1
         DER1(2) = - 1.0D0 + HJA2
         DER2(3) = DER1(1)
         DER2(4) = DER1(2)
         DER3(5) = DER1(1)
         DER3(6) = DER1(2)
         DER4(7) = DER1(1)
         DER4(8) = DER1(2)
         DER6(1) = - S1(I1) / BB(I1)
         DER6(2) = - S1(I2) / BB(I2)
         DER6(3) = - S2(I1) / BB(I1)
         DER6(4) = - S2(I2) / BB(I2)
         DER6(5) = - S3(I1) / BB(I1)
         DER6(6) = - S3(I2) / BB(I2)
         DER6(7) = - S4(I1) / BB(I1)
         DER6(8) = - S4(I2) / BB(I2)
         DER5(1) = DER6(1) * HJB1
         DER5(2) = DER6(2) * HJB2
         DER5(3) = DER6(3) * HJB1
         DER5(4) = DER6(4) * HJB2
         DER5(5) = DER6(5) * HJB1
         DER5(6) = DER6(6) * HJB2
         DER5(7) = DER6(7) * HJB1
         DER5(8) = DER6(8) * HJB2
         DER7(1) = - P1(I1) * HJC1 + ( P1(I1) - P5(I1) )
     &      * ( P1(I1) * XC(7) * HJD1 + S1(I1) * ( HJE1 - HJF1 ) )
         DER7(2) = - P1(I2) * HJC2 + ( P1(I2) - P5(I2) ) *
     &        ( P1(I2) * XC(7) * HJD2 + S1(I2) * ( HJE2 - HJF2 ) )
         DER7(3) = ( P1(I1) - P5(I1) ) * ( P2(I1) * XC(8) * HJD1 +
     &        S2(I1) * ( HJE1 - HJF1 ) )
         DER7(4) = ( P1(I2) - P5(I2) ) * ( P2(I2) * XC(8) * HJD2 +
     &        S2(I2) * ( HJE2 - HJF2 ) )
         DER7(5) = ( P1(I1) - P5(I1) ) * ( P3(I1) * XC(9) * HJD1 +
     &        S3(I1) * ( HJE1 - HJF1 ) )
         DER7(6) = ( P1(I2) - P5(I2) ) * ( P3(I2) * XC(9) * HJD2 +
     &        S3(I2) * ( HJE2 - HJF2 ) )
         DER7(7) = ( P1(I1) - P5(I1) ) * ( P4(I1) * XC(10) * HJD1 +
     &        S4(I1) * ( HJE1 - HJF1 ) )
         DER7(8) = ( P1(I2) - P5(I2) ) * ( P4(I2) * XC(10) * HJD2 +
     &        S4(I2) * ( HJE2 - HJF2 ) )
         DER8(1) = ( P2(I1) - P5(I1) ) * ( P1(I1) * XC(7) * HJD1 +
     &        S1(I1) * ( HJE1 - HJF1 ) )
         DER8(2) = ( P2(I2) - P5(I2) ) * ( P1(I2) * XC(7) * HJD2 +
     &        S1(I2) * ( HJE2 - HJF2 ) )
         DER8(3) = - P2(I1) * HJC1  + ( P2(I1) - P5(I1) ) *
     &        ( P2(I1) * XC(8) * HJD1 + S2(I1) * ( HJE1 - HJF1 ) )
         DER8(4) = - P2(I2) * HJC2 + ( P2(I2) - P5(I2) ) *
     &        ( P2(I2) * XC(8) * HJD2 + S2(I2) * ( HJE2 - HJF2 ) )
         DER8(5) = ( P2(I1) - P5(I1) ) * ( P3(I1) * XC(9) * HJD1 +
     &        S3(I1) * ( HJE1 - HJF1 ) )
         DER8(6) = ( P2(I2) - P5(I2) ) * ( P3(I2) * XC(9) * HJD2 +
     &        S3(I2) * ( HJE2 - HJF2 ) )
         DER8(7) = ( P2(I1) - P5(I1) ) * ( P4(I1) * XC(10) * HJD1 +
     &        S4(I1) * ( HJE1 - HJF1 ) )
```

```
      DER8(8) = ( P2(I2) - P5(I2) ) * ( P4(I2) * XC(10) * HJD2 +
     &      S4(I2) * ( HJE2 - HJF2 ) )
      DER9(1) = ( P3(I1) - P5(I1) ) * ( P1(I1) * XC(7) * HJD1 +
     &      S1(I1) * ( HJE1 - HJF1 ) )
      DER9(2) = ( P3(I2) - P5(I2) ) * ( P1(I2) * XC(7) * HJD2 +
     &      S1(I2) * ( HJE2 - HJF2 ) )
      DER9(3) = ( P3(I1) - P5(I1) ) * ( P2(I1) * XC(8) * HJD1 +
     &      S2(I1) * ( HJE1 - HJF1 ) )
      DER9(4) = ( P3(I2) - P5(I2) ) * ( P2(I2) * XC(8) * HJD2 +
     &      S2(I2) * ( HJE2 - HJF2 ) )
      DER9(5) = - P3(I1) * HJC1 + ( P3(I1) - P5(I1) ) *
     &     ( P3(I1) * XC(9) * HJD1 + S3(I1) * ( HJE1 - HJF1 ) )
      DER9(6) = - P3(I2) * HJC2 + ( P3(I2) - P5(I2) ) *
     &     ( P3(I2) * XC(9) * HJD2 + S3(I2) * ( HJE2 - HJF2 ) )
      DER9(7) = ( P3(I1) - P5(I1) ) * ( P4(I1) * XC(10) * HJD1 +
     &      S4(I1) * ( HJE1 - HJF1 ) )
      DER9(8) = ( P3(I2) - P5(I2) ) * ( P4(I2) * XC(10) * HJD2 +
     &      S4(I2) * ( HJE2 - HJF2 ) )
      DER10(1) = ( P4(I1) - P5(I1) ) * ( P1(I1) * XC(7) * HJD1 +
     &       S1(I1) * ( HJE1 - HJF1 ) )
      DER10(2) = ( P4(I2) - P5(I2) ) * ( P1(I2) * XC(7) * HJD2 +
     &       S1(I2) * ( HJE2 - HJF2 ) )
      DER10(3) = ( P4(I1) - P5(I1) ) * ( P2(I1) * XC(8) * HJD1 +
     &      S2(I1) * ( HJE1 - HJF1 ) )
      DER10(4) = ( P4(I2) - P5(I2) ) * ( P2(I2) * XC(8) * HJD2 +
     &      S2(I2) * ( HJE2 - HJF2 ) )
      DER10(5) = ( P4(I1) - P5(I1) ) * ( P3(I1) * XC(9) * HJD1 +
     &      S3(I1) * ( HJE1 - HJF1 ) )
      DER10(6) = ( P4(I2) - P5(I2) ) * ( P3(I2) * XC(9) * HJD2 +
     &      S3(I2) * ( HJE2 - HJF2 ) )
      DER10(7) = - P4(I1) * HJC1 + ( P4(I1) - P5(I1) ) *
     &      ( P4(I1) * XC(10) * HJD1 + S4(I1) * ( HJE1 - HJF1 ) )
      DER10(8) = - P4(I2) * HJC2 + ( P4(I2) - P5(I2) ) *
     &      ( P4(I2) * XC(10) * HJD2 + S4(I2) * ( HJE2 - HJF2 ) )
C
C     TRANSFORM THE DERIVATIVES OF THE RESIDUALS.
C
      CALL TRANSOME(BETA, DER1, TDER1)
      CALL TRANSOME(BETA, DER2, TDER2)
      CALL TRANSOME(BETA, DER3, TDER3)
      CALL TRANSOME(BETA, DER4, TDER4)
      CALL TRANSOME(BETA, DER5, TDER5)
      CALL TRANSOME(BETA, DER6, TDER6)
      CALL TRANSOME(BETA, DER7, TDER7)
      CALL TRANSOME(BETA, DER8, TDER8)
      CALL TRANSOME(BETA, DER9, TDER9)
      CALL TRANSOME(BETA, DER10, TDER10)
C
      SUMG1 = SUMG1
     &      + ETA1(I1)*TDER1(1) + ETA1(I2)*TDER1(2)
     &      + ETA2(I1)*TDER1(3) + ETA2(I2)*TDER1(4)
     &      + ETA3(I1)*TDER1(5) + ETA3(I2)*TDER1(6)
     &      + ETA4(I1)*TDER1(7) + ETA4(I2)*TDER1(8)
C
      SUMG2 = SUMG2
     &      + ETA1(I1)*TDER2(1) + ETA1(I2)*TDER2(2)
     &      + ETA2(I1)*TDER2(3) + ETA2(I2)*TDER2(4)
     &      + ETA3(I1)*TDER2(5) + ETA3(I2)*TDER2(6)
     &      + ETA4(I1)*TDER2(7) + ETA4(I2)*TDER2(8)
C
      SUMG3 = SUMG3
     &      + ETA1(I1)*TDER3(1) + ETA1(I2)*TDER3(2)
     &      + ETA2(I1)*TDER3(3) + ETA2(I2)*TDER3(4)
     &      + ETA3(I1)*TDER3(5) + ETA3(I2)*TDER3(6)
     &      + ETA4(I1)*TDER3(7) + ETA4(I2)*TDER3(8)
C
      SUMG4 = SUMG4
     &      + ETA1(I1)*TDER4(1) + ETA1(I2)*TDER4(2)
     &      + ETA2(I1)*TDER4(3) + ETA2(I2)*TDER4(4)
     &      + ETA3(I1)*TDER4(5) + ETA3(I2)*TDER4(6)
     &      + ETA4(I1)*TDER4(7) + ETA4(I2)*TDER4(8)
C
```

```
      SUMG5 = SUMG5
     &        + ETA1(I1)*TDER5(1) + ETA1(I2)*TDER5(2)
     &        + ETA2(I1)*TDER5(3) + ETA2(I2)*TDER5(4)
     &        + ETA3(I1)*TDER5(5) + ETA3(I2)*TDER5(6)
     &        + ETA4(I1)*TDER5(7) + ETA4(I2)*TDER5(8)
C
      SUMG6 = SUMG6
     &        + ETA1(I1)*TDER6(1) + ETA1(I2)*TDER6(2)
     &        + ETA2(I1)*TDER6(3) + ETA2(I2)*TDER6(4)
     &        + ETA3(I1)*TDER6(5) + ETA3(I2)*TDER6(6)
     &        + ETA4(I1)*TDER6(7) + ETA4(I2)*TDER6(8)
C
      SUMG7 = SUMG7
     &        + ETA1(I1)*TDER7(1) + ETA1(I2)*TDER7(2)
     &        + ETA2(I1)*TDER7(3) + ETA2(I2)*TDER7(4)
     &        + ETA3(I1)*TDER7(5) + ETA3(I2)*TDER7(6)
     &        + ETA4(I1)*TDER7(7) + ETA4(I2)*TDER7(8)
C
      SUMG8 = SUMG8
     &        + ETA1(I1)*TDER8(1) + ETA1(I2)*TDER8(2)
     &        + ETA2(I1)*TDER8(3) + ETA2(I2)*TDER8(4)
     &        + ETA3(I1)*TDER8(5) + ETA3(I2)*TDER8(6)
     &        + ETA4(I1)*TDER8(7) + ETA4(I2)*TDER8(8)
C
      SUMG9 = SUMG9
     &        + ETA1(I1)*TDER9(1) + ETA1(I2)*TDER9(2)
     &        + ETA2(I1)*TDER9(3) + ETA2(I2)*TDER9(4)
     &        + ETA3(I1)*TDER9(5) + ETA3(I2)*TDER9(6)
     &        + ETA4(I1)*TDER9(7) + ETA4(I2)*TDER9(8)
C
      SUMG10 = SUMG10
     &          + ETA1(I1)*TDER10(1) + ETA1(I2)*TDER10(2)
     &          + ETA2(I1)*TDER10(3) + ETA2(I2)*TDER10(4)
     &          + ETA3(I1)*TDER10(5) + ETA3(I2)*TDER10(6)
     &          + ETA4(I1)*TDER10(7) + ETA4(I2)*TDER10(8)
C
 1000 CONTINUE
C
      GC(1) = 2.0D0 * SUMG1
      GC(2) = 2.0D0 * SUMG2
      GC(3) = 2.0D0 * SUMG3
      GC(4) = 2.0D0 * SUMG4
      GC(5) = 2.0D0 * SUMG5
      GC(6) = 2.0D0 * SUMG6
      GC(7) = 2.0D0 * SUMG7
      GC(8) = 2.0D0 * SUMG8
      GC(9) = 2.0D0 * SUMG9
      GC(10) = 2.0D0 * SUMG10
C
 99   FORMAT (1H0,"HELP IS NEGATIVE. HELP =",1PD20.5)
 98   FORMAT(/,3X,4D16.6)
C
      RETURN
      END
C
C
C         SUBROUTINE MONIT
C         ================
C
C         PURPOSE:
C         ========
C         THE SUBROUTINE MONITORS THE PROGRESS OF THE MINIMIZATION
C         PERFORMED IN NAG-SUBROUTINE E04KBF.
C
      SUBROUTINE MONIT(NVAR, XC, FC, GC, ISTATE, GPJNRM, COND,
     &                 POSDEF, NITER, NF, IW, LIW, W, LW)
C
C         LOCAL INTEGER VARIABLES.
C
      INTEGER
     &        NVAR, NITER, NF, LIW, LW, NOUT, ISTATE(NVAR), IW(LIW)
```

```
C
C NVAR    -    NO. OF COEFFICIENTS TO BE ESTIMATED.
C NITER   -    NO. OF ITERATIONS PERFORMED .
C NF      -    NO. OF FUNCTION EVALUATIONS.
C LIW     -    DEFINED IN MAIN PROGRAM.
C LW      -    DEFINED IN MAIN PROGRAM.
C IW      -    DEFINED IN MAIN PROGRAM.
C ISTATE  -    CONTAINS INFORMATION ABOUT WHICH VARIABLES HAVE CURRENTLY
C              THEIR BOUNDARY VALUES AND WHICH ARE FREE.
C NOUT    -    LOGICAL UNIT NUMBER FOR PRINTER.
C
C          LOCAL DOUBLE PRECISION VARIABLES.
C
      DOUBLE PRECISION FC, GPJNRM, COND, XC(NVAR), GC(NVAR), W(LW)
C
C XC      -    CURRENT VALUES OF THE UNKNOWN COEFFICIENTS.
C FC      -    FUNCTION VALUE AT XC.
C GC      -    GRADIENT VALUE AT XC.
C GPJNRM  -    THE EUCLIDIAN NORM OF THE (PROJECTED) GRADIENT.
C COND    -    ESTIMATED VALUE OF THE CONDITION NUMBER OF THE (PROJECTED)
C              HESSIAN MATRIX
C
C          LOCAL LOGICAL VARIABLES
C
      LOGICAL POSDEF
C
C POSDEF  -    LOGICAL VARIABEL.(ALWAYS EQUAL TO "TRUE" IN THIS PROGRAM.)
C
      DATA NOUT /6/
C
      WRITE(NOUT,99) NITER, NF, FC, GPJNRM, COND
      WRITE(NOUT,98)
      DO 100 J = 1, NVAR
         WRITE(NOUT,97) J, XC(J), GC(J), ISTATE(J)
 100  CONTINUE
C
 99   FORMAT(/,1H0,"NITER  NF          FC       GPJNRM        COND",
     &         /,I4,I5,3(1PD13.5))
 98   FORMAT(/,3H0  ,1X,"J        XC(J)        GC(J) ISTATE(J)",/)
 97   FORMAT(1H ,I4,1P2D13.5,I10)
C
      RETURN
      END
C
C
C         SUBROUTINE SER
C         ==============
C
C         PURPOSE:
C         ========
C         INVERSION OF A CHOLESKY FACTORIZED MATRIX.
C
C
      SUBROUTINE SER(N, IL, HL, HD, XL, SE)
      DOUBLE PRECISION XL(IL), HL(IL), HD(N), SE(N), SS
      INTEGER N, IL
C
C   N      -    NO. OF UNKNOWN COEFFICIENTS.
C   IL     -    DIMENSION OF HESSIAN MATRIX.
C   HL     -    HESL (SEE MAIN PROGRAM).
C   HD     -    HESD (SEE MAIN PROGRAM).
C   SE     -    VECTOR CONTAINING THE DIAGONAL ELEMENTS OF THE INVERSE
C               OF THE HESSIAN.
C   XL     -    VECTOR CONTAINING THE INVERSE OF HESL,STACKED ANALOGOUSLY.
```

```
C
C        FIRST, WE ESTABLISH THE VECTOR XL.
C
      DO 1 I = 2, N
         II = I - 1
            DO 2 J = 1, II
               IND = (I-1) * (I-2) / 2 + J
               SS = - HL(IND)
               IF (J .EQ. II)  GO TO 2
               IXI = I - J - 1
               DO 3 K = 1, IXI
                  INL = (I - 1) * (I - 2) / 2 + K + J
                  INX = (K + J - 1) * (K + J - 2) / 2 + J
 3             CONTINUE
               SS = SS - HL(INL) * XL(INX)
 2          XL(IND) = SS
 1    CONTINUE
C
C       SECOND, WE COMPUTE THE DIAGONAL ELEMENTS OF THE INVERSE OF
C       THE HESSIAN.
C
      DO 4 I = 1, N
         SS = 1.0 / HD(I)
         IF(I .EQ. N)  GO TO 6
            II = I + 1
            DO 5 J = II, N
               IND = (J - 1) * (J - 2) / 2 + I
               SS = SS + XL(IND) * XL(IND) / HD(J)
 5          CONTINUE
 6       CONTINUE
         SE(I) = SS
 4    CONTINUE
      RETURN
      END
```

REFERENCES

Amemiya, T. (1971): The Estimation of the Variances in a Variance-Components Model. *International Economic Review*, 12 (1971), 1-13.

Anderson, T.W. and C. Hsiao (1981): Estimation of Dynamic Models with Error Components. *Journal of the American Statistical Association*, 76 (1981), 598-606.

Avery, R.B. (1977): Error Components and Seemingly Unrelated Regressions. *Econometrica*, 45 (1977), 199-209.

Balestra, P. (1975): *La dérivation matricielle. Technique et résultats pour économistes.* (Paris: Sirey, 1975.)

Balestra, P., and M. Nerlove (1966): Pooling Cross Section and Time Series Data in the Estimation of A Dynamic Model: The Demand for Natural Gas. *Econometrica*, 34 (1966), 585-612.

Baltagi, B.H. (1980): On Seemingly Unrelated Regressions with Error Components. *Econometrica*, 48 (1980), 1547-1551.

Barten, A.P. (1969): Maximum Likelihood Estimation of a Complete System of Demand Equations. *European Economic Review*, 1 (1969-1970), 7-73.

Barten, A.P. (1977): The Systems of Consumer Demand Functions Approach: A Review. *Econometrica*, 45 (1977), 23-51.

Berzeg, K. (1979): The Error Components Model: Conditions for the Existence of the Maximum Likelihood Estimates. *Journal of Econometrics*, 10 (1979), 99-102.

Biørn, E. (1974): Estimating the Flexibility of the Marginal Utility of Money: An Errors-in-Variables Approach. *European Economic Review* 5 (1974), 177-185.

Biørn, E. (1977): Two Notes on the Stochastic Specification of a Complete Set of Consumer Demand Functions. Working Papers from the Central Bureau of Statistics of Norway 77/17.

Biørn, E. (1978): *Comparing Consumer Expenditure Functions Estimated from Household Budget Data from the Years 1967 and 1973.* Articles from the Central Bureau of Statistics of Norway, no. 108. (Oslo: Central Bureau of Statistics, 1978.)

Biørn, E. (1981a): Estimating Economic Relations from Incomplete Cross-Section/Time-Series Data. *Journal of Econometrics*, 16 (1981), 221-236.

Biørn, E. (1981b): *Estimating Seemingly Unrelated Regression Models from Incomplete Cross-Section/Time-Series Data.* Reports from the Central Bureau of Statistics of Norway, 81/33. (Oslo: Central Bureau of Statistics, 1981.)

Biørn, E., and E.S. Jansen (1980): *Consumer Demand in Norwegian Households 1973-1977. A Data Base for Micro-Econometrics.* Reports from the Central Bureau of Statistics of Norway 80/4. (Oslo: Central Bureau of Statistics, 1980.)

Bjerkholt, O., and S. Longva (1980): *MODIS IV A Model for Economic Analysis and National Planning.* Samfunnsøkonomiske studier no. 43. (Oslo: Central Bureau of Statistics, 1980.)

Bojer, H. (1977): The Effect on Consumption of Household Size and Composition. *European Economic Review*, 9 (1977), 169-193.

Brown, A., and A. Deaton (1972): Models of Consumer Behaviour: A Survey. *Economic Journal*, 82 (1972), 1145-1206.

Carlevaro, F. (1976): A Generalization of the Linear Expenditure System. In L. Solari, and J.-N. Du Pasquier (eds.): *Private and Enlarged Consumption.* (Amsterdam: North-Holland Publishing Company, 1976.)

Chamberlain, G., and Z. Griliches (1975): Unobservables with a Variance-Components Structure: Ability, Schooling and the Economic Success of Brothers. *International Economic Review*, 16 (1975), 422-450.

Christensen, L.R., D.W. Jorgenson, and L.J. Lau (1975): Transcendental Logarithmic Utility Functions. *American Economic Review*, 65 (1975), 367-383.

Deaton, A.S. (1974): The Analysis of Consumer Demand in the United Kingdom. *Econometrica*, 42 (1974), 341-367.

Deaton, A.S. (1975a): *Models and Projections of Demand in Post-War Britain.* (London: Chapman and Hall, 1975.)

Deaton, A.S. (1975b): The Measurement of Income and Price Elasticities. *European Economic Review*, 6 (1975), 261-273.

Deaton, A., and J. Muellbauer (1980a): An Almost Ideal Demand System. *American Economic Review*, 70 (1980), 312-326.

Deaton, A., and J. Muellbauer (1980b): *Economics and Consumer Behavior.* (Cambridge: Cambridge University Press, 1980.)

Diewert, W.E. (1974): Applications of Duality Theory. In M.D. Intriligator and D.A. Kendrick (eds.): *Frontiers in Quantitative Economics, Vol. II.* (Amsterdam: North-Holland Publishing Company, 1974.)

Diewert, W.E. (1982): Duality Approaches to Microeconomic Theory. Ch. 12 in K.J. Arrow and M.D. Intriligator (eds.): *Handbook of Mathematical Economics. Vol. II.* (Amsterdam: North-Holland Publishing Company, 1982.)

Fourgeaud, C., and A. Nataf (1959): Consommation en prix et revenue réels et théorie des choix. *Econometrica*, 27 (1959), 329-354.

Frisch, R. (1932): *New Methods of Measuring Marginal Utility.* (Tübingen: Verlag von J.C.B. Mohr (Paul Siebeck), 1932.)

Frisch, R. (1937): Determinateness and Indeterminateness in the Measurement of Money Flexibility. Summarized in E.H. Phelps Brown: Report of the Oxford Meeting, September 25-29, 1936. *Econometrica*, 5 (1937), 361-383.

Frisch, R. (1959): A Complete Scheme for Computing All Direct and Cross Demand Elasticities in a Model with Many Sectors. *Econometrica*, 27 (1959), 177-196.

Fuller, W.A., and G.E. Battese (1974): Estimation of Models with Cross-Error Structure. *Journal of Econometrics*, 2 (1974), 67-78.

Gill, P.E. and W. Murray (1977): The Computation of Lagrange-Multiplier Estimates for Constrained Optimisation. *NPL Report NAC 77*. (Teddington, Middlesex: National Physical Laboratory, 1977.)

Goldberger, A.S. (1967): Functional Form and Utility: A Review of Consumer Demand Theory. Social System Research Institute, University of Wisconsin. Systems Formulation, Methodology, and Policy Workshop Paper No. 6703, mimeo.

Goldberger, A.S. and T. Gamaletsos (1970): A Cross-Country Comparison of Consumer Expenditure Patterns. *European Economic Review*, 1 (1969-1970), 357-400.

Goldfeld, S. and R.E. Quandt (1972): *Nonlinear Methods in Econometrics*. (Amsterdam: North-Holland Publishing Company, 1972.)

Gorman, W.M. (1961): On a Class of Preference Fields. *Metroeconomica*, 13 (1961), 53-56.

Griliches, Z., B.H. Hall, and J.A. Hausman (1978): Missing Data and Self-Selection in Large Panels. *Annales de l'INSEE*, 30-31 (1978), 137-176.

Hasenkamp, G. (1978): Economic and Atomic Index Numbers: Contrasts and Similarities. In W. Eichhorn, R. Henn, O. Opitz, and R.W. Shephard, (eds.): *Theory and Application of Economic Indices*. (Würzburg: Physica-Verlag, 1978.)

Hausman, J.A. and D.A. Wise (1977): Social Experimentation, Truncated Distributions, and Efficient Estimation. *Econometrica*, 45 (1977), 919-938.

Hausman, J.A. and W.E. Taylor (1981): Panel Data and Unobservable Individual Effects. *Econometrica*, 49 (1981), 1377-1398.

Houthakker, H.S. (1957): An International Comparison of Household Expenditure Patterns Commemorating the Centenary of Engel's Law. *Econometrica*, 25 (1957), 532-551.

Houthakker, H.S. (1960): Additive Preferences. *Econometrica*, 28 (1960), 248-257.

Jansen, E.S. (1978): *On Duality in Consumer Theory. An Empirical Test of the Quadratic Indirect Utility Function.* (In Norwegian.) Memorandum from the Institute of Economics, University of Oslo, 13 April 1978.

Johansen, L. (1969): On the Relationships between Some Systems of Demand Functions. *Liiketaloudellinen Aikakauskirja*, 1 (1969), 30-41.

Johansen, L. (1981): Suggestions Towards Freeing Systems of Demand Functions from a Strait-Jacket. In A. Deaton (ed.): *Essays in the Theory and Measurement of Consumer Behaviour*. (Cambridge: Cambridge University Press, 1981.)

Kelejian, H.H. (1974): Random Parameters in a Simultaneous Equation Framework: Identification and Estimation. *Econometrica*, 42 (1974), 517-527.

Kendall, M.G. and A. Stuart (1973): *The Advanced Theory of Statistics. Vol. 2: Inference and Relationship*. (London: Charles Griffin, 1973.)

Klein, L.R. (1974): *A Textbook of Econometrics*, 2. edition. (Englewood Cliffs, N.J.: Prentice - Hall, 1974.)

Koopmans, T.C. (1953): Identification Problems in Economic Model Construction. Chapter II in W.C. Hood and T.C. Koopmans (eds.): *Studies in Econometric Method*. (New York: John Wiley & Sons, 1953.)

Lau, L.J. (1969): Duality and the Structure of Utility Functions. *Journal of Economic Theory*, 1 (1969), 374-396.

Lau, L.J. (1977): Complete Systems of Consumer Demand Functions Through Duality. In M.D. Intriligator (ed.): *Frontiers in Quantitative Economics, Vol. III A*. (Amsterdam: North-Holland Publishing Company, 1977.)

Lau, L.J. (1978): Testing and Imposing Monotonicity, Convexity, and Quasi-Convexity Constraints. Appendix A.4 in M. Fuss and D. McFadden (eds.): *Production Economics: A Dual Approach to Theory and Application*. (Amsterdam: North-Holland Publishing Company, 1978.)

Leser, C.E.V. (1963): Forms of Engel Functions. *Econometrica*, 31 (1963), 694-703.

Lluch, C. and A. Powell (1975): International Comparisons of Expenditure Patterns. *European Economic Review*, 5 (1975), 275-303.

Madansky, A. (1976): *Foundations of Econometrics*. (Amsterdam: North-Holland Publishing Company, 1976.)

Maddala, G.S. (1971): The Use of Variance Components Models in Pooling Cross Section and Time Series Data. *Econometrica*, 39 (1971), 341-358.

Maddala, G.S. (1978): Selectivity Problems in Longitudinal Data. *Annales de l'INSEE*, 30-31 (1978), 423-450.

Malinvaud, E. (1966): *Statistical Methods of Econometrics*. (Amsterdam: North-Holland Publishing Company, 1966.)

Malinvaud, E. (1970): The Consistency of Non-Linear Regressions. *Annals of Mathematical Statistics*, 41 (1970), 956-969.

Marquandt, D.W. (1963): An Algorithm for Least-Squares Estimation of Non-Linear Parameters. *Journal of the Society for Industrial and Applied Mathematics*, 11 (1963), 431-444.

Mazodier, P. (1971): L'estimation des modèles à erreurs composées. *Annales de l'INSEE*, 7 (1971), 43-74.

Mundlak, Y. (1978): On the Pooling of Time Series and Cross Section Data. *Econometrica*, 46 (1978), 69-85.

NAG (1978): *NAG Library Manual, NAG Library Computer Programs Mark 7.* (Oxford: Numerical Algorithm Group, 1978.)

Nasse, P. (1973): Un système complet de fonctions de demande: les équations de Fourgeaud et Nataf. *Econometrica*, 41 (1973), 1137-1158.

Nerlove, M. (1971): A Note on Error Components Models. *Econometrica*, 39 (1971), 383-396.

Oberhofer, W. and J. Kmenta (1974): A General Procedure for Obtaining Maximum Likelihood Estimates in Generalized Regression Models. *Econometrica*, 42 (1974), 579-590.

Parks, R.W. (1967): Efficient Estimation of a System of Regression Equations when Disturbances are Both Serially and Contemporaneously Correlated. *Journal of the American Statistical Association*, 62 (1967), 500-509.

Pollak, R.A. (1971): Additive Utility Functions and Linear Engel Curves. *Review of Economic Studies*, 38 (1971), 401-414.

Pollak, R.A. (1972): Generalized Separability. *Econometrica*, 40 (1972), 431-453.

Pollak, R.A. and T.J. Wales (1969): Estimation of the Linear Expenditure System. *Econometrica*, 37 (1969), 611-628.

Pollak, R.A. and T.J. Wales (1978): Estimation of Complete Demand Systems from Household Budget Data: The Linear and Quadratic Expenditure Systems. *American Economic Review*, 68 (1978), 348-359.

Pollak, R.A. and T.J. Wales (1980): Comparison of the Quadratic Expenditure System and Translog Demand Systems with Alternative Specifications of Demographic Effects. *Econometrica*, 48 (1980), 595-612.

Pollak, R.A. and T.J. Wales (1981): Demographic Variables in Demand Analysis. *Econometrica*, 49 (1981), 1533-1551.

Praag, B. van (1971): The Welfare Function of Income in Belgium: An Empirical Investigation. *European Economic Review*, 2 (1971), 337-369.

Prais, S.J. and H.S. Houthakker (1955): *The Analysis of Family Budgets*. (Cambridge: Cambridge University Press, 1955.)

Roy, R. (1942): *De l'Utilité: Contribution à la Théorie des Choix.* (Paris: Hermann, 1942.)

Salvas-Bronsard, L. (1978): Estimating Systems of Demand Equations from French Time-Series of Cross-Section Data. *Annales de l'INSEE*, 30-31 (1978), 543-564.

Samuelson, P.A. (1965): Using Full Duality to Show that Simultaneously Additive Direct and Indirect Utilities Implies Unitary Price Elasticity of Demand. *Econometrica*, 33 (1965), 781-796.

Sargan, J.D. (1964): Wages and Prices in the United Kingdom: A Study in Econometric Methodology. In P.E. Hart, G. Mills, and J.K. Whitaker, (eds.): *Econometric Analysis for National Economic Planning*. Colston Economic Papers. (London: Butterworth, 1964.)

Somermeyer, W.H. (1974): Delimitation of the Class of Budget-Constrained Utility Maximizing Partially Linear Consumer Expenditure Functions: An Alternative Approach. *Zeitschrift für Nationalökonomie*, 34 (1974), 309-326.

Stone, R. (1954): Linear Expenditure Systems and Demand Analysis: An Application to the Pattern of British Demand. *Economic Journal*, 64 (1954), 511-527.

Swamy, P.A.V.B. (1970): Efficient Inference in a Random Coefficient Regression Model. *Econometrica*, 38 (1970), 311-323.

Swamy, P.A.V.B. (1974): Linear Models with Random Coefficients. In P. Zarembka (ed.): *Frontiers in Econometrics*. (New York: Academic Press, 1974.)

Taylor, W.E. (1980): Small Sample Considerations in Estimation from Panel Data. *Journal of Econometrics*, 13 (1980), 203-223.

Theil, H. (1971): *Principles of Econometrics*. (New York: John Wiley & Sons, 1971.)

Theil, H. (1975): *Theory and Measurement of Consumer Demand. Vol. 1.* (Amsterdam: North-Holland Publishing Company, 1975.)

Theil, H. (1976): *Theory and Measurement of Consumer Demand. Vol. 2.* (Amsterdam: North-Holland Publishing Company, 1976.)

Theil, H. and R.B. Brooks (1970-71): How Does the Marginal Utility of Income Change when Real Income Changes? *European Economic Review*, 2 (1970-71), 218-240.

Wallace, T.D. and A. Hussain (1969): The Use of Error Components Models in Combining Cross Section with Time Series Data. *Econometrica*, 37 (1969), 55-72.

Wansbeek, T. and A. Kapteyn (1981a): Estimators of the Covariance Structure of a Model for Longitudinal Data. In E.G. Charatsis (ed.): *Proceedings of the Econometric Society European Meeting 1979*. (Amsterdam: North-Holland Publishing Company, 1981.)

Wansbeek, T. and A. Kapteyn (1981b): Maximum Likelihood Estimation in Incomplete Panels. Unpublished paper, Centraal Bureau voor de Statistiek, Voorburg, The Netherlands, August 1981.

Wolberg, J.R. (1967): *Prediction Analysis*. (Princeton: D. van Nostrand, 1967.)

Woodland, A.D. (1979): Stochastic Specification and the Estimation of Share Equations. *Journal of Econometrics*, 10 (1979), 361-383.

Working, H. (1943): Statistical Laws of Family Expenditures. *Journal of the American Statistical Association*, 38 (1943), 43-56.

Zellner, A. (1962): An Efficient Method of Estimating Seemingly Unrelated Regressions and Tests for Aggregation Bias. *Journal of the American Statistical Association*, 57 (1962), 348-368.

Utkommet i serien Samfunnsøkonomiske studier (SØS)
Issued in the series Social Economic Studies (SES)

* Utsolgt Out of sale

Nr. 1 Det norske skattesystems virkninger på den personlige inntektsfordeling The Effects of the Norwegian Tax System on the Personal Income Distribution 1954 Sidetall 103 Pris kr 3,00

- 2* Skatt på personleg inntekt og midel Tax on Personal Income and Capital 1954 Sidetall 120 Pris kr 3,00

- 3 Økonomisk utsyn 1900 - 1950 Economic Survey 1955 Sidetall 217 Pris kr 4,00

- 4* Nasjonalregnskap. Teoretiske prinsipper National Accounts. Theoretical Principles 1955 Sidetall 123 Pris kr 3,00

- 5* Avskrivning og skattlegging Depreciation and Taxation 1956 Sidetall 85 Pris kr 3,00

- 6* Bedriftsskatter i Danmark, Norge og Sverige Corporate Taxes in Denmark, Norway and Sweden 1958 Sidetall 101 Pris kr 4,00

- 7* Det norske skattesystemet 1958 The Norwegian System of Taxation 1958 Sidetall 159 Pris kr 6,50

- 8* Produksjonsstruktur, import og sysselsetting Structure of Production, Imports and Employment 1959 Sidetall 129 Pris kr 5,50

- 9 Kryssløpsanalyse av produksjon og innsats i norske næringer 1954 Input-Output Analysis of Norwegian Industries 1960 Sidetall 614 Pris kr 10,00

- 10 Dødeligheten og dens årsaker i Norge 1856 - 1955 Trend of Mortality and Causes of Death in Norway 1962 Sidetall 246 Pris kr 8,50

- 11 Kriminalitet og sosial bakgrunn Crimes and Social Background 1962 Sidetall 194 Pris kr 7,00

- 12 Norges økonomi etter krigen The Norwegian Post-War Economy 1965 Sidetall 437 Pris kr 15,00

- 13 Ekteskap, fødsler og vandringer i Norge 1856 - 1960 Marriages, Births and Migrations in Norway 1965 Sidetall 221 Pris kr 9,00

- 14* Foreign Ownership in Norwegian Enterprises Utenlandske eierinteresser i norske bedrifter 1965 Sidetall 213 Pris kr 12,00

- 15 Progressiviteten i skattesystemet 1960 Statistical Tax Incidence Investigation 1966 Sidetall 95 Pris kr 7,00

- 16* Langtidslinjer i norsk økonomi 1955 - 1960 Trends in Norwegian Economy 1966 Sidetall 150 Pris kr 8,00

- 17 Dødelighet blant spedbarn i Norge 1901 - 1963 Infant Mortality in Norway 1966 Sidetall 74 Pris kr 7,00

- 18* Storbyutvikling og arbeidsreiser En undersøkelse av pendling, befolkningsutvikling, næringsliv og urbanisering i Osloområdet Metropolitan Growth, Commuting and Urbanization in the Oslo Area 1966 Sidetall 298 Pris kr 12,00

Nr. 19 Det noske kredittmarked siden 1900 The Norwegian Credit Market since 1900 Sidetall 395 Pris kr 11,00

- 20 Det norske skattesystemet 1967 The Norwegian System of Taxation 1968 Sidetall 146 Pris kr 9,00

- 21 Estimating Production Functions and Technical Change from Micro Data. An Exploratory Study of Individual Establishment Time-Series from Norwegian Mining and Manufacturing 1959 - 1967 Estimering av produktfunksjoner og tekniske endringer fra mikro data. Analyser på grunnlag av tidsrekker for individuelle bedrifter fra norsk bergverk og industri 1971 Sidetall 226 Pris kr 9,00 ISBN 82-537-0014-8

- 22 Forsvarets virkninger på norsk økonomi The Impact of the Defence on the Norwegian Economy 1972 Sidetall 141 Pris kr 9,00 ISBN 82-537-0149-7

- 23 Prisutvikling og prisatferd i 1960-årene En presentasjon og analyse av nasjonalregnskapets prisdata 1961 - 1969 The Development and Behaviour of Prices in the 1960's Presentation and Analysis of the Price-Data of the Norwegian National Accounts 1974 Sidetall 478 Pris kr 15,00 ISBN 82-537-0279-5

- 24 Det norske skattesystemet I Direkte skatter 1974 The Norwegian System of Taxation I Direct Taxes 1974 Sidetall 139 Pris kr 9,00 ISBN 82-537-0399-6

- 25* Friluftsliv, idrett og mosjon Outdoor Recreation, Sport and Exercise 1975 Sidetall 114 Pris kr 8,00 ISBN 82-537-0469-0

- 26 Nasjonalregnskap, modeller og analyse En artikkelsamling til Odd Aukrusts 60-årsdag National Accounts, Models and Analysis to Odd Aukrust in Honour of his Sixtieth Birthday 1975 Sidetall 320 Pris kr 13,00 ISBN 82-537-0530-1

- 27 Den repræsentative undersøgelsesmethode The Representative Method of Statistical Surveys 1976 Sidetall 64 Pris kr 8,00 ISBN 82-537-0538-7

- 28 Statistisk Sentralbyrå 100 år 1876 - 1976 Central Bureau of Statistics 100 Years 1976 Sidetall 128 Pris kr 9,00 ISBN 82-537-0557-3

- 29 Statistisk Sentralbyrås 100-årsjubileum Prolog og taler ved festmøtet i Universitetets aula 11. juni 1976 Central Bureau of Statistics Prologue and Addresses at the Centenary Celebration, University Hall 1976 Sidetall 32 Pris kr 7,00 ISBN 82-537-0637-5

- 30 Inntekts- og forbruksbeskatning fra et fordelingssynspunkt - En modell for empirisk analyse Taxation of Income and Consumption from a Distributional Point of View - A Model for Empirical Analysis 1976 Sidetall 148 Pris kr 9,00 ISBN 82-537-0647-2

- 31 Det norske skattesystemet II Indirekte skatter og offentlige trygdeordninger 1976 The Norwegian System of Taxation II Indirect Taxes and Social Security Schemes 1977 Sidetall 124 Pris kr 13,00 ISBN 82-537-0713-4

- 32 Inntekt og forbruk for funksjonshemma Income and Consumer Expenditure of Disabled Persons 1977 Sidetall 166 Pris kr 13,00 ISBN 82-537-0732-0

Nr. 33 Prinsipper og metoder for Statistisk Sentralbyrås utvalgsundersøkelser Sampling Methods Applied by the Central Bureau of Statistics of Norway 1977 Sidetall 105 Pris kr 11,00 ISBN 82-537-0771-1

- 35 Flyttemotivundersøkelsen 1972 Survey of Migration Motives 1978 Sidetall 233 Pris kr 15,00 ISBN 82-537-0783-5

- 36 Konjunkturbølger fra utlandet i norsk økonomi International Cycles in Norwegian Economy 1979 Sidetall 141 Pris kr 13,00 ISBN 82-537-0910-2

- 37 Norske lytter- og seervaner Radio Listening and Television Viewing in Norway 1979 Sidetall 216 Pris kr 13,00 ISBN 82-537-0931-5

- 38 Analyse av investeringsatferd Problemer, metoder og resultater Analysing Investment Behaviour Problems, Methods and Results 1979 Sidetall 91 Pris kr 13,00 ISBN 82-537-0952-8

- 39 Kvinners yrkesdeltaking i Norge Female Labour Activity in Norway 1979 Sidetall 162 Pris kr 13,00 ISBN 82-537-0961-7

- 40 Framskriving av befolkningens utdanning til år 2000 Projections of the Education Characteristics of the Population to the Year 2000 1979 Sidetall 112 Pris kr 13,00 ISBN 82-537-0998-6

- 41 Nordmenns feriereiser Holiday Trips by Norwegians 1979 Sidetall 222 Pris kr 13,00 ISBN 82-537-0999-4

- 42 Analyse av sammenhengen mellom forbruk, inntekt og formue i norske husholdninger Analysing the Relationship between Consumption, Income and Wealth in Norwegian Households 1980 Sidetall 95 Pris kr 13,00 ISBN 82-537-1012-7

- 43 MODIS IV A Model for Economic Analysis and National Planning MODIS IV Modell for økonomisk analyse og nasjonal planlegging 1980 Sidetall 189 Pris kr 13,00 ISBN 82-537-1014-3

- 44 Holdninger og atferd på arbeidsmarkedet Attitudes and Behaviour in the Labour Market 1980 Sidetall 223 Pris kr 15,00 ISBN 82-537-1186-7 ISSN 0085-4344

- 45 Nasjonalregnskapet i Norge System og beregningsmetoder Nationa Accounts of Norway System and Methods of Estimation 1980 Sidetall 313 Pris kr 18,00 ISBN 82-537-1191-3 ISSN 0085-4344

- 46 Inntektsfordeling og levekår Income Distribution and Level of l Living 1980 Sidetall 263 Pris kr 15,00 ISBN 82-537-1195-6 ISSN 0085-4344

- 47 Fruktbarhetsutvikling og fruktbarhetsteorier Norge i et internasjonal perspektiv Trends and Theories in Fertility Norway in an International Context 1981 Sidetall 120 Pris kr 15,00 ISBN 82-537-1236-7

- 48 Framskriving av arbeidsstyrken 1979 - 2000 Labour Force Projections 1981 Sidetall 109 Pris kr 15,00 ISBN 82-537-1556-0 ISSN 0085-4344

- 49 Fruktbarhet blant norske kvinner Resultater fra Fruktbarhetsundersøkelsen 1977 Fertility among Norwegian Women Results from the Fertility Survey 1981 Sidetall 349 Pris kr 20,00 ISBN 82-537-1621-4

- 50 Flyttemønstre Norge 1971 - 1974 Patterns of Migration Norway 1971 - 1974 1982 Sidetall 238 Pris kr 20,00 ISBN 82-537-1709-1 ISSN 0085-4344

Nr. 51 Utdanning og sosial bakgrunn Education and Social Background
1982 Sidetall 210 Pris kr 15,00 ISBN 82-537-1759-8
ISSN 0085-4344

- 52 Econometrics of Incomplete Cross-Section/Time-Series Data:
Consumer Demand in Norwegian Households 1975 - 1977
Økonometrisk analyse av ufullstendige tverrsnitts/tidsserie
data: Konsumetterspørselen i norske husholdninger 1982
Sidetall 307 Pris kr 20,00 ISBN 82-537-1782-2 ISSN 0085-4344

Publikasjonen utgis i kommisjon hos
H. Aschehoug & Co. og Universitetsforlaget, Oslo,
og er til salgs hos alle bokhandlere.
Pris kr 20,00

ISBN 82-537-1782-2
ISSN 0085-4344
Engers Boktrykkeri A/S - Otta